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CURVES WHOSE SECANT DEGREE IS ONE IN POSITIVE CHARACTERISTIC

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ABSTRACT. Here we study (in positive characteristic) integral curves $X \subset \mathbb{P}^r$ with secant degree one, i.e., for which a general $P \in \mathrm{Sec}^{k-1}(X)$ is in a unique k-secant $(k-1)$-dimensional linear subspace.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed base field. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate closed subvariety. For each $x \in \{0, \dots, r\}$, let $G(x, r)$ denote the Grassmannian of all x-dimensional linear subspaces of $\mathbb{P}^r$. For each integer $k \geq 1$ let $\sigma_k(X)$ denote the closure in $\mathbb{P}^r$ of the union of all $A \in G(k-1, r)$ spanned by k points of X (the variety $\sigma_k(X)$ is sometimes called the $(k-1)$-secant variety of X and written $\mathrm{Sec}^{k-1}(X)$, but we prefer to call it the k-secant variety of X). The integral variety $\sigma_k(X)$ may be obtained in the following way. Assume that X is non-degenerate. For any closed subscheme $E \subseteq \mathbb{P}^r$ let $\langle E \rangle$ denote its linear span. Let $V(X, k) \subseteq G(k-1, r)$ denote the closure in $G(k-1, r)$ of the set of all $A \in G(k-1, r)$ spanned by k-points of X. Set

$$S[X, k] := \{(P, A) \in \mathbb{P}^r \times G(k-1, r) \colon P \in A, A \in V(X, k)\}.$$

Let $p_1 \colon \mathbb{P}^r \times G(k-1, r) \to \mathbb{P}^r$ denote the projection onto the first factor. We have $\sigma_k(X) = p_1(S[X, k])$. Set $m_{X,k} := p_{1|S[X,k]}$. If $\sigma_k(X)$ has the expected dimension $k \cdot (\dim(X) + 1) - 1$ (i.e., if $m_{X,k}$ is generically finite), then we write $i_k(X)$ for the inseparable degree of $m_{X,k}$ and $s_k(X)$ for its separable degree. For any $P \in X_{\mathrm{reg}}$, let $T_P X \subset \mathbb{P}^r$ denote the tangent space to X at P. If $k \geq 2$, we say that X is *k-unconstrained* if

$$\dim(\langle T_{P_1} X \cup \cdots \cup T_{P_k} X \rangle) = \dim(\sigma_k(X))$$

for a general $(P_1, \dots, P_k) \in X^k$. Terracini's lemma says that

$$\dim(\langle T_{P_1} X \cup \cdots \cup T_{P_k} X \rangle) \leq \dim(\sigma_k(X)))$$

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and that in characteristic zero equality always holds ([**1**, §1] or [**3**, §2]). The case $k = 2$ of this notion was introduced in [**3**]. A non-degenerate curve $Y \subset \mathbb{P}^r$ is 2-unconstrained if and only if either $r = 2$ or Y is not strange [**3**, Example (e1) at page 333]. From now on we assume $\dim(X) = 1$. We first prove the following result.

Theorem 1. *Fix integers $r \geq 2k \geq 4$. Let $X \subset \mathbb{P}^r$ be an integral, non-degenerate and k-unconstrained curve. Then $s_k(X) = 1$.*

For each integer i such that $2 \leq 2i \leq r$ we define the integer $e_i(X)$ in the following way. Fix a general $(P_1, \dots, P_i) \in X^i$. Thus $P_j \in X_{\mathrm{reg}}$ for all j. Set $V := \langle T_{P_1}X \cup \dots \cup T_{P_i}X \rangle$. Notice that $(V \cap X)_{\mathrm{red}} \supseteq \{P_1, \dots, P_i\}$ and the scheme $V \cap X$ is zero-dimensional. Varying $(P_1, \dots, P_i)$ in X^i we see that each P_j appears with the same multiplicity in the zero-dimensional scheme $V \cap X$. We call $e_i(X)$ this multiplicity. In characteristic zero we always have $e_i(X) = 2$. The integer $e_1(X)$ is the intersection multiplicity of X with its general tangent line at its contact point. Hence if $\mathrm{char}(\mathbb{K})$ is odd the curve X is reflexive if and only if $e_1(X) = 2$ ([**4**, 3.5]). In the general case we have $e_1(X) \geq 2$ and $e_i(X) \leq e_{i+1}(X)$. For any $P \in X_{\mathrm{reg}}$ and any integer $t \in \{1, \dots, r\}$, let $O(X, P, t) \in G(t, r)$ denote the t-dimensional osculating plane of X at P. Thus $O(X, P, 1) = T_P X$. Fix integers $i \geq 1$, and $j_h \geq 0$, $1 \leq h \leq i$. We only need the case $2i + \sum_{h=1}^{i} j_h \leq r$. Fix a general $(P_1, \dots, P_i) \in X^i$ and set $V := \langle \cup_{h=1}^{i} O(X, P_h, 1 + j_h) \rangle$. For any $h \in \{1, \dots, i\}$, let $E(X; i; j_1, \dots j_i; h)$ be the multiplicity of P_h in the scheme $V \cap X$. We will only use the case $j_1 = 1$ and $j_h = 0$ for all $h \neq 1$. If either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > \deg(X)$, then $E(X; i; j_1, \dots j_i; h) = 2 + j_h$ (Lemma 9). Here we prove the following result.

Theorem 2. *Let $X \subset \mathbb{P}^{2k-1}$, $k \geq 2$, be an integral, non-degenerate and k-unconstrained curve. Set $j_1 := 1$ and $j_h := 0$ for all $h \in \{2, \dots, k-1\}$.*

(a) *If $s_k(X) = 1$ and $E(X; k-1; j_1, \dots, j_{k-1}; 1) = e_{k-1}(X) + 1$, then X is smooth and rational and $\deg(X) = (k-1)e_{k-1}(X) + 1$.*
(b) *X is a rational normal curve if and only if $s_k(X) = 1$, $e_{k-1}(X) = 2$ and $E(X; k-1; j_1, \dots, j_{k-1}; 1) = 3$.*

We do not know if in the statement of Theorem 2 we may drop the conditions "$e_{k-1}(X) = 2$" and "$E(X; k-1; j_1, \dots, j_{k-1}; 1) = 3$". We are able to prove that we may drop the first one in the case $k = 2$, i.e., we prove the following result.

Proposition 1. *Let $X \subset \mathbb{P}^3$ be an integral and non-degenerate curve. The following conditions are equivalent:*

(a) *X is not strange, $s_2(X) = 1$ and $E(X; 1; 1; 1) = e_1(X) + 1$;*
(b) *$i_2(X) = s_2(X) = 1$ and $E(X; 1; 1; 1) = e_1(X) + 1$;*
(c) *X is a rational normal curve.*

The picture is very easy if $\mathrm{char}(\mathbb{K}) > \deg(X)$. As a byproduct of Theorem 2 we give the following result.

Theorem 3. *Let $X \subset \mathbb{P}^{2k-1}$ be an integral and non-degenerate curve. Assume $\mathrm{char}(\mathbb{K}) > \deg(X)$. X is a rational normal curve if and only if $s_k(X) = 1$.*

2. The proofs

Remark 1. Assume X of arbitrary dimension and that

$$\dim(\sigma_k(X)) = k(\dim(X)+1)-1.$$

As in [**3**] (the case $k=2$) X is k-unconstrained if and only if $i_k(X)=1$.

Lemma 1. *Fix integers $c>0$, $s>y\geq 2$ and $r\geq s(c+1)-1$. Let $X\subset\mathbb{P}^r$ be an integral and non-degenerate c-dimensional subvariety such that $\dim(\sigma_s(X)) = s(c+1)-1$. If X is s-unconstrained, then X is y-unconstrained.*

Proof. Since $\dim(\sigma_s(X)) = s(c+1)-1$ and X is s-unconstrained, we have

$$\dim(\langle T_{P_1}X\cup\cdots\cup T_{P_s}X\rangle = s(c+1)-1$$

for a general $(P_1,\dots,P_s)\in X^s$. Hence $\dim(\langle T_{P_1}X\cup\cdots\cup T_{P_y}(X)\rangle = y(c+1)-1$. Hence X is y-unconstrained. □

We recall the following very useful result ([**1**, §1]).

Lemma 2. *Let $X\subset\mathbb{P}^r$ be an integral and non-degenerate curve. Then X is non-defective, i.e., $\dim(\sigma_a(X)) = \min\{r,2a-1\}$ for all integers $a\geq 2$.*

From Lemmas 1 and 2 we get the following result.

Lemma 3. *Fix integers $s>y\geq 2$ and $r\geq 2s-1$. Let $X\subset\mathbb{P}^r$ be an integral and non-degenerate curve. If X is s-unconstrained, then X is y-unconstrained and not strange.*

We recall that a finite set $S\subset\mathbb{P}^x$ is said to be *in linearly general position* if $\dim(\langle S'\rangle) = \min\{x,\sharp(S')-1\}$ for every $S'\subseteq S$. The general hyperplane section of a non-degenerate curve $X\subset\mathbb{P}^r$ is in linearly general position if X is not strange ([**6**, Lemma 1.1]). Hence Lemma 3 implies the following result.

Lemma 4. *Fix integers r,s such that $r\geq 2s-1\geq 3$. Let $X\subset\mathbb{P}^r$ be an integral and non-degenerate curve. Assume that X is s-unconstrained. Then X is not strange and a general hyperplane section of X is in linearly general position.*

Proof of Theorem 1. We extend the proof of the case $k=2$ given in [**3**, §4]. By Lemma 4 a general $(k-1)$-dimensional k-secant plane of X meets X at exactly k points. Fix a general $(P_1,\dots,P_k)\in X^k$ and set $V:=\langle T_{P_1}X\cup\cdots\cup T_{P_k}\rangle$. Since X is k-unconstrained, we have $\dim(V)=2k-1$. Since $2k-1<r$ and X is non-degenerate, the set $S:=(V\cap X)_{\mathrm{red}}$ is finite. Fix a general $P\in\langle\{P_1,\dots,P_k\}\rangle$. Assume $s_k(X)\geq 2$. Since a general hyperplane section of X is in linearly general position (Lemma 4), the integer $s_k(X)$ is the number of different k-ples of points of X such that a general point of $\sigma_k(X)$ is in their linear span. Since P may be considered as a general point of $\sigma_k(X)$ and $s_k(X)\geq 2$, there is $(Q_1,\dots,Q_k)\in X^k$ such that $P\in\langle\{Q_1,\dots,Q_k\}\rangle$ and $\{P_1,\dots,P_k\}\neq\{Q_1,\dots,Q_k\}$. For general P we may also assume that $(Q_1,\dots,Q_k)$ is general in X^k. Hence each P_i and each Q_j is a smooth point of X. Terracini's lemma gives $\langle T_{P_1}X\cup\cdots\cup T_{P_k}X\rangle\subseteq T_P\sigma_k(X)$ and $\langle T_{Q_1}X\cup\cdots\cup T_{Q_k}X\rangle\subseteq T_P\sigma_k(X)$. Since X is k-unconstrained and both $(P_1,\dots,P_k)$

and $(Q_1, \dots, Q_k)$ are general in X^k, we have $\langle T_{P_1}X \cup \cdots \cup T_{P_k} \rangle = T_P\sigma_k(X)$ and $\langle T_{Q_1}X \cup \cdots \cup T_{Q_k}X \rangle = T_P\sigma_k(X)$. Hence $\{Q_1, \dots, Q_k\} \subseteq S$. Since S is finite, the union of the linear spans of all $S' \subseteq S$ with $\sharp(S') = k$ is a finite number of linear subspaces of dimension at most $k-1$ and $\langle S' \rangle = \langle \{P_1, \dots, P_k\} \rangle$ if and only if $S' = \{P_1, \dots, P_k\}$, because $\langle \{P_1, \dots, P_k\} \rangle \cap X = \{P_1, \dots, P_k\}$. Hence $\dim(\langle S' \rangle \cap \langle \{P_1, \dots, P_k\} \rangle) \leq k-2$ for all $S' \neq \{P_1, \dots, P_k\}$. Varying $P \in \langle \{P_1, \dots, P_k\} \rangle \cong \mathbb{P}^{k-1}$, we get a contradiction. □

Lemma 5. *Let $X \subset \mathbb{P}^r$, $r \geq 2k-1 \geq 5$, be an integral, non-degenerate and k-unconstrained curve. Fix an integer s such that $1 \leq s \leq k-2$. Fix a general $(A_1, \dots, A_s) \in X^s$ and set $W := \langle T_{A_1}X \cup \cdots \cup T_{A_s}X \rangle$. Then $\dim(W) = 2s-1$. Let $\ell_W \colon \mathbb{P}^r \setminus W \to \mathbb{P}^{r-2s}$ denote the linear projection from W. Let $Y \subset \mathbb{P}^{r-2s}$ denote the closure of $\ell_W(Y \setminus Y \cap W)$. Then Y is $(k-s)$-unconstrained and it is not strage.*

Proof. Fix a general $A_{s+1}, \dots, A_k \in X^{k-s}$. Notice that $(\ell_W(A_{s+1}), \dots, \ell_W(A_k))$ is general in Y^{k-s} and

$$\ell_W(\langle W \cup T_{A_{s+1}}X \cup \cdots \cup T_{A_k}X \rangle \setminus W) = \langle T_{\ell_W(A_{s+1})}Y \cup \cdots \cup T_{\ell_W(A_k))}Y \rangle.$$

Hence the latter space has dimension $2k-2s-1$. Hence Y is $(k-s)$-unconstrained. Since $k-s \geq 2$, Y is not strange. □

Lemma 6. *Fix integers $c > 0$, $k \geq 2$ and $r \geq (c+1)k-1$. Let $X \subset \mathbb{P}^r$ be a k-unconstrained c-dimensional variety such that $\dim(\sigma_k(X)) = (c+1)k-1$. Fix an integer $s \in \{1, \dots, k-1\}$ and a general $(P_1, \dots, P_s) \in X^s$. Set $V := \langle T_{P_1}X \cup \cdots \cup T_{P_s}X \rangle$. Then $\dim(V) = (c+1)s-1$ and the restriction to X of the linear projection $\ell_V \colon \mathbb{P}^r \setminus V \to \mathbb{P}^{r-(c+1)s}$ is a generically finite separable morphism.*

Proof. Since $s+1 \leq k$ and $\dim(\sigma_k(X)) = (c+1)k-1$, we have $\dim(\sigma_s(X)) = (c+1)s-1$. Lemma 1 gives that X is s-unconstrained. Since X is $(s+1)$-unconstrained and $\dim(\sigma_{s+1}(X)) = (c+1)(s+1)-1$, we have

$$\dim(\langle V \cup T_PX \rangle) = \dim(V) + \dim(T_PX) + 1$$

for a general $P \in X$, i.e., $V \cap T_PX = \emptyset$ for a general $P \in X$. Thus $\ell_V|(X \setminus V)$ has differential with rank c, i.e., it is separable and generically finite. □

Proof of Theorem 2. If X is a rational normal curve, then it is k-unconstrained, $s_k(X) = 1$ ([**2**, First 4 lines of page 128]) and $i_k(X) = 1$ (Remark 1).

Now assume $s_k(X) = 1$. In step (c) we will use the assumption $E(X; k-1; 1, 0, \dots, 0; 1) = e_{k-1}(X) + 1$. We need to adapt a part of the characteristic zero proof given in [**2**] to the positive characteristic case. We will follow [**2**] as much as possible. Fix a general $(P_1, \dots, P_{k-1}) \in X^{k-1}$ and set $V := \langle T_{P_1}X \cup \cdots \cup T_{P_{k-1}}X \rangle$. Since X is k-unconstrained, we have $\dim(V) = 2k-3$. Since X is non-degenerate, the set $S := (V \cap X)_{\mathrm{red}}$ is finite.

(a) Here we check that $S \subset X_{\mathrm{reg}}$. If $k = 2$, then for a general P_1 we have $T_{P_1}X \cap \mathrm{Sing}(X) = \emptyset$, because X is not strange by [**3**, Example (e1) at page 333]. Now assume $k \geq 3$. Since X is not strange (use Lemma 1), for general $P_1 \in X$, we have $T_{P_1}X \cap \mathrm{Sing}(X) = \emptyset$. Then by induction on i we check using a linear projection from $T_{P_i}X$ as in Lemma 5 that $\langle T_{P_1}X \cup \cdots \cup T_{P_i}X\rangle \cap \mathrm{Sing}(X) = \emptyset$ (more precisely, for any finite set $\Sigma \subset X$ we check by induction on i that $\langle T_{P_1}X \cup \cdots \cup T_{P_i}X\rangle \cap \Sigma = \emptyset$ for a general $(P_1, \dots, P_i) \in X^i$). For $i = k-1$ we get $S \subset X_{\mathrm{reg}}$.

(b) Obviously $\{P_1, \dots, P_{k-1}\} \subseteq S$. Here we check that $S = \{P_1, \dots, P_{k-1}\}$. Assume for the moment the existence of $Q \in S \setminus \{P_1, \dots, P_{k-1}\}$. Since X is not strange, it is not very strange, i.e., a general hyperplane section of X is in linearly general position ([**6**, Lemma 1.1]). Since $(P_1, \dots, P_{k-1})$ is general in X^{k-1}, we get $\langle\{P_1, \dots, P_{k-1}\}\rangle \cap X = \{P_1, \dots, P_{k-1}\}$. Thus $\dim(\langle\{P_1, \dots, P_{k-1}, Q\}\rangle) = k-1$. Fix a general $z \in \langle\{P_1, \dots, P_{k-1}, Q\}\rangle$. We have

$$\mathbb{P}^{2k-1} = T_z\sigma_k(X) \supseteq \langle T_{P_1}X \cup \cdots \cup T_{P_{k-1}}X \cup T_QX\rangle$$

(Terracini's lemma ([**3**, §2] or [**1**, Proposition 1.9]). The additive map giving Terracini's lemma for joins in the proof of [**1**, Proposition 1.9], shows that the map $m_{X,k}$ has non-invertible differential over the point z. Since $\mathbb{P}^{2k-1}$ is smooth and $m_{X,k}$ is separable, we get that $m_{X,k}$ is not finite of degree 1 near z. Since $s_k(X) = 1$, $m_{X,k}$ contracts a curve over z. Since z lies in infinitely many $(k-1)$-dimensional k-secant subspaces, we get that $\dim(\sigma_k(X)) \le 2k-2$, contradicting Lemma 2. The contradiction proves $S = \{P_1, \dots, P_{k-1}\}$.

(c) Step (b) means that $\{P_1, \dots, P_{k-1}\}$ is the reduction of the scheme--theoretically intersection $X \cap V$. Let Z_i denote the connected component of the scheme $X \cap V$ supported by P_i. Set $e := \deg(Z_1)$. Since $T_{P_1}X \subseteq V$, we have $e \ge 2$. Varying $(P_1, \dots, P_{k-1})$ in X^{k-1} we get $\deg(Z_i) = e$ for all i. The definition of the integer $e_{k-1}(X)$ gives $e = e_{k-1}(X)$. Set $\phi := \ell_V|(X \setminus V \cap X)$. Since $X \cap V \subset X_{\mathrm{reg}}$, ϕ is dominant and X_{reg} is a smooth curve, ϕ induces a finite morphism $\psi\colon X \to \mathbb{P}^1$. Bezout's theorem gives $\deg(X) = (k-1)e + \deg(\psi)$. Lemma 6 gives that ψ is separable. Hence $\deg(\psi)$ is the separable degree of ψ. Assume $\deg(\psi) \ge 2$. Since $\mathbb{P}^1$ is algebraically simply connected, there is $Q \in X$ at which ψ ramifies.

First assume $Q \in X_{\mathrm{reg}}$. Since $E(X; k-1; 1, 0, \dots, 0; 1) = e_{k-1}(X) + 1$, ψ is not ramified at P_1. Moving $P_1, \dots, P_{k-1}$ we get $Q \notin \{P_1, \dots, P_{k-1}\}$. The definition of ϕ gives $\dim(V \cup T_QX) \le \dim(V) + 1$. Hence the additive map giving Terracini's lemma for joins in the proof of [**1**, Proposition 1.9], shows that the map $m_{X,k}$ has non-invertible differential over the general point $z \in \langle\{P_1, \dots, P_{k-1}, Q\}\rangle$. As in step (b) we get a contradiction.

Now assume $Q \in \mathrm{Sing}(X)$. Let $u\colon C \to X$ denote the normalization map. Since we assumed $\deg(\psi) \ge 2$, we have $\deg(\psi \circ u) \ge 2$. Since $\mathbb{P}^1$ is algebraically simply connected, there is $Q' \in C$ such that $\psi \circ u$ is ramified at Q'. We repeat the construction of joins and secant variety starting from the non-embedded curve C and get a contradiction using Q' instead of Q. Thus $\deg(\psi) = 1$, i.e.

$$\deg(X) = (k-1)e_{k-1}(X) + 1,$$

and X is rational.

X is a rational normal curve if and only if $\deg(X) = 2k-1$, i.e., if and only if $e = 2$. Take any $P \in \mathrm{Sing}(X)$ (if any). Set $H := \langle\{P\} \cup V\rangle$. Since X is singular at P, we have $\deg(H \cap X) \geq 2 + (k-1)e > \deg(X)$, that is contradiction. Thus X is smooth. □

Proof of Proposition 1. We have $i_2(X) = 1$ if and only if X is 2-unconstrained ([**3**] or Remark 1). Obviously X is 2-unconstrained. Hence it is sufficient to prove that if X is 2-unconstrained, $s_2(X) = 1$, and $E(X;1;1;1) = e_1(X) + 1$, then X is a rational normal curve. Theorem 2 says that X is smooth and rational and $\deg(X) = e_1(X) + 1$. Thus it is sufficient to prove $e_1(X) = 2$. Assume $e_1(X) \geq 3$. Since $\deg(X) = e_1(X) + 1$, Bezout's theorem says that any two different tangent lines are disjoint. Let $TX \subset \mathbb{P}^3$ denote the tangent developable of X. Fix a general $P \in \mathbb{P}^3$ and let $\ell_P \colon \mathbb{P}^3 \setminus \{P\} \to \mathbb{P}^2$ be the linear projection from P. Set $\ell := \ell_P|X$. Since $P \notin TX$, ℓ is unramified. Since X is smooth, $s_2(X) = 1$ and P is general, the map ℓ is birational onto its image and the curve $\ell(X)$ has a unique singular point (the point $\ell(P_1) = \ell(P_2)$ with $P \in \langle \{P_1, P_2\} \rangle$ and $(P_1, P_2) \in X^2$). We have $p_a(\ell(X)) = e_1(X)(e_1(X)-1)/2 \geq 2$. Since $P \notin TX$, we have $P \notin T_{P_i}X$, $i = 1, 2$. Since $T_{P_1}X \cap T_{P_2}(X) = \emptyset$, the line $T_{P_2}X$ is not contained in the plane $\langle \{P\} \cup T_{P_1}X \rangle$. Thus $\ell_P(T_{P_1}X) \neq \ell_P(T_{P_2}X)$. Thus $\ell(P_1)$ is an ordinary double point of $\ell(X)$. Hence $\ell(X)$ has geometric genus $p_a(X) - 1 > 0$, thath is contradiction. □

Lemma 7. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve. Assume char$(\mathbb{K}) > \deg(X)$. Then $e_i(X) = 2$ for all positive integers i such that $2i \leq r$.*

Proof. We have $e_1(X) = 2$, because in large characteristic the Hermite sequence of X at its general point is the classical one ([**5**, Theorem 15]). The case $i \geq 2$ is obtained by induction on i taking instead of X its image by the linear projection from $T_{P_i}X$, P_i general in X. □

Lemma 8. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve. Assume char$(\mathbb{K}) > \deg(X)$. Then X is i-unconstrained for all integers $i \geq 2$.*

Proof. Fix a linear subspace $V \subset \mathbb{P}^r$ such that $v := \dim(V) \leq r - 2$. Let $\ell_V \colon \mathbb{P}^r \setminus V \to \mathbb{P}^{r-v-1}$ denote the linear projection from V. Since char$(\mathbb{K}) > \deg(X)$, the restriction of ℓ_V to X is separable. Hence $T_{P_i}X \cap V = \emptyset$ for a general $P_i \in X$. Take $V = \langle T_{P_1}X \cup \cdots \cup T_{P_{i-1}}X \rangle$ with $(P_1, \ldots, P_{i-1})$ general in X^{i-1} and use induction on i. □

Lemma 9. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve. Assume* char$(\mathbb{K}) > \deg(X)$. *Then $E(X; i; j_1, \ldots, j_i; h) = 2 + j_h$ for all $i, j_1, \ldots, j_i$ such that*

$$2i + \sum_{x=1}^{i} j_x \leq r$$

and for a general $(P_1, \ldots, P_i) \in X^i$, the linear span of the osculating spaces

$$O(X, P_x, 1 + j_x)\ , 1 \leq x \leq i,$$

has dimension $2i - 1 + \sum_{x=1}^{i} j_x$.

Proof. The case $i = 1$ is true by [**5**, Theorem 15]. Hence we may assume $i \geq 2$. Fix an index $c \in \{1, \dots, i\} \setminus \{h\}$. For a general $P_c \in X$, the point P_c appears with multiplicity exactly $j_c + 2$ in the scheme $O(X, P_c, j_c + 1)$ ([**5**, Theorem 15]). Since $\mathrm{char}(\mathbb{K}) > \deg(X)$, the rational map ℓ obtained restricting to X the linear projection from $O(X, P_c, 1 + j_c)$ is separable. Call Y the closure in $\mathbb{P}^{r-j_c-2}$ of $\ell(X \setminus O(X, P, 1 + j_c) \cap X)$. Take P_x, $x \neq c$, such that $(P_1, \dots, P_i)$ is general in X^c and write $Q_x := \ell(P_x)$ for all $x \neq c$. Let V be the linear span of the osculating spaces $O(X, P_x, 1 + j_x)$, $1 \leq x \leq i$, U the linear span of the osculating spaces $O(X, P_x, 1 + j_x)$, $x \neq c$, and W the linear span of the osculating spaces $O(Y, Q_x, 1 + j_x)$, $x \neq c$. By the inductive assumption U and W have dimension $2i - 3 + \sum_{x \neq c} j_x$. Hence $\ell(U) = W$ and $\dim(V) = 2i - 1 + \sum_{x=1}^{i} j_x$. Since the points Q_i are general and ℓ is separable, the scheme $\ell^{-1}((2 + j_x)Q_x))$, $x \neq c$, is a divisor of X whose connected component supported by P_x has degree $2 + j_x$. Use the inductive assumption on Y to get $E(X; i; j_1, \dots, j_i; h) = 2 + j_h$. □

Proof of Theorem 3. Apply Theorem 2 and Lemmas 7, 8 and 9. □

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