Acta Math. Univ. Comenianae
Vol. LXXX, 1 (2011), pp. 119–132

PERTURBATION RESULTS FOR WEYL TYPE THEOREMS

M. BERKANI AND H. ZARIOUH

ABSTRACT. In [**12**] we introduced and studied properties (gab) and (gaw), which are extensions to the context of B-Fredholm theory, of properties (ab) and (aw) respectively, introduced also in [**12**]. In this paper we continue the study of these properties and we consider their stability under commuting finite rank, compact and nilpotent perturbations. Among other results, we prove that if T is a bounded linear operator acting on a Banach space X, then T possesses property (gaw) if and only if T satisfies generalized Weyl's theorem and $E(T) = E_a(T)$.

We also prove that if T possesses property (ab) or property (aw) or property (gaw), respectively, and N is a nilpotent operator commuting with T, then $T + N$ possesses property (ab) or property (aw) or property (gaw) respectively. The same result holds for property (gab) in the case of a-polaroid operators.

1. INTRODUCTION

Throughout this paper, let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. For $T \in \mathcal{L}(X)$, let $N(T)$, $R(T), \sigma(T)$ and $\sigma_a(T)$ denote the null space, the range, the spectrum and the approximate point spectrum of T, respectively. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by $\alpha(T) = \dim N(T)$ and $\beta(T) = \operatorname{codim} R(T)$. Recall that an operator $T \in \mathcal{L}(X)$ is called an upper semi-Fredholm if $\alpha(T) < \infty$ and $R(T)$ is closed, while $T \in \mathcal{L}(X)$ is called a lower semi-Fredholm if $\beta(T) < \infty$. Let $SF_+(X)$ denote the class of all upper semi-Fredholm operators. If $T \in \mathcal{L}(X)$ is an upper or lower semi-Fredholm operator, then T is called a semi-Fredholm operator, and the index of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a Fredholm operator. An operator $T \in \mathcal{L}(X)$ is called a Weyl operator if it is a Fredholm operator of index 0. Define

$$SF_+^-(X) = \{T \in SF_+(X) : \operatorname{ind}(T) \leq 0\}.$$

The classes of operators defined above generate the following spectra: the Weyl spectrum $\sigma_W(T)$ of $T \in \mathcal{L}(X)$ is defined by

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl operator}\},$$

Received June 17, 2010.
2001 *Mathematics Subject Classification.* Primary 47A53, 47A10, 47A11.
Key words and phrases. property (ab), property (gab), property (aw), property (gaw), B-Weyl operators.
Supported by Protars D11/16 and PGR- UMP.

while the Weyl essential approximate spectrum $\sigma_{SF_+^-}(T)$ of T is defined by

$$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(X)\}.$$

For $T \in \mathcal{L}(X)$, let $\Delta(T) = \sigma(T) \backslash \sigma_W(T)$ and $\Delta_a(T) = \sigma_a(T) \backslash \sigma_{SF_+^-}(T)$. Following Coburn [**16**], we say that Weyl's theorem holds for $T \in \mathcal{L}(X)$ if $\Delta(T) = E^0(T)$, where $E^0(T) = \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(T - \lambda I) < \infty\}$. Here and elsewhere in this paper, for $A \subset \mathbb{C}$, $\operatorname{iso} A$ is the set of all isolated points of A, and $\operatorname{acc} A$ denote the set of all points of accumulation of A.

According to Rakočević [**25**], an operator $T \in \mathcal{L}(X)$ is said to satisfy a-Weyl's theorem if $\Delta_a(T) = E_a^0(T)$, where $E_a^0(T) = \{\lambda \in \operatorname{iso} \sigma_a(T) : 0 < \alpha(T - \lambda I) < \infty\}$. It is known [**25**] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

Recall that the ascent $a(T)$, of an operator T, is defined by

$$a(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$$

and the descent $\delta(T)$ of T is defined by

$$\delta(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$$

with $\inf \emptyset = \infty$. An operator $T \in \mathcal{L}(X)$ is called Drazin invertible if it has a finite ascent and descent. The Drazin spectrum $\sigma_D(T)$ of an operator T is defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}.$$

An operator $T \in \mathcal{L}(X)$ is called Browder if it is Fredholm of finite ascent and descent and is called upper semi-Browder if it is upper semi-Fredholm of finite ascent. The Browder spectrum $\sigma_b(T)$ of T is defined by

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}$$

and the upper semi-Browder spectrum $\sigma_{ub}(T)$ of T is defined by

$$\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Browder}\}$$

(see [**15**] and [**24**]).

Define also the set $LD(X)$ by

$$LD(X) = \{T \in \mathcal{L}(X) : a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed }\}$$

and

$$\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin LD(X)\}.$$

Following [**10**], an operator $T \in \mathcal{L}(X)$ is said to be left Drazin invertible if $T \in LD(X)$. We say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I \in LD(X)$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda I) < \infty$. Let $\Pi_a(T)$ denote the set of all left poles of T and let $\Pi_a^0(T)$ denotes the set of all left poles of T of finite rank.

Let $\Pi(T)$ be the set of all poles of the resolvent of T and let $\Pi^0(T)$ be the set of all poles of the resolvent of T of finite rank, that is $\Pi^0(T) = \{\lambda \in \Pi(T) : \alpha(T - \lambda I) < \infty\}$. According to [**19**], a complex number λ is a pole of the resolvent of T if and only if $0 < \max\,(a(T - \lambda I), \delta(T - \lambda I)) < \infty$. Moreover, if this is true then $a(T - \lambda I) = \delta(T - \lambda I)$. According also to [**19**], the space $R((T - \lambda I)^{a(T-\lambda I)+1})$

is closed for each $\lambda \in \Pi(T)$. Hence we have always $\Pi(T) \subset \Pi_a(T)$ and $\Pi^0(T) \subset \Pi_a^0(T)$.

For $T \in \mathcal{L}(X)$ and a nonnegative integer n define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range space $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then T is called an upper (resp. a lower) semi-B-Fredholm operator. In this case the index of T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [**11**]. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a B-Fredholm operator, see [**5**]. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. An operator T is said to be a B-Weyl operator [**6**, Definition 1.1] if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\},$$

and the B-Fredholm spectrum $\sigma_{BF}(T)$ of T is defined by

$$\sigma_{BF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Fredholm operator}\}.$$

For $T \in \mathcal{L}(X)$, let $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T)$. According to [**10**], an operator $T \in \mathcal{L}(X)$ is said to satisfy generalized Weyl's theorem if $\Delta^g(T) = E(T)$, where $E(T) = \{\lambda \in \operatorname{iso} \sigma(T) : \alpha(T - \lambda I) > 0\}$. According also to [**10**] we say that generalized Browder's theorem holds for $T \in \mathcal{L}(X)$ if $\Delta^g(T) = \Pi(T)$, and that Browder's theorem holds for $T \in \mathcal{L}(X)$ if $\Delta(T) = \Pi^0(T)$. It is proved in [**4**, Theorem 2.1] that generalized Browder's theorem is equivalent to Browder's theorem.

Let $SBF_+(X)$ be the class of all upper semi-B-Fredholm operators,

$$SBF_+^-(X) = \{T \in SBF_+(X) \ : \ \operatorname{ind}(T) \le 0\}.$$

The upper B-Weyl spectrum $\sigma_{SBF_+^-}(T)$ of T is defined by

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \notin SBF_+^-(X)\}.$$

Let $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. We say that a-Browder's theorem holds for $T \in \mathcal{L}(X)$ if $\Delta_a(T) = \Pi_a^0(T)$, and that generalized a-Browder's theorem holds for $T \in \mathcal{L}(X)$ if $\Delta_a^g(T) = \Pi_a(T)$. It is proved in [**4**, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem. According to [**10**], an operator $T \in \mathcal{L}(X)$ is said to satisfy generalized a-Weyl's theorem if $\Delta_a^g(T) = E_a(T)$, where $E_a(T) = \{\lambda \in \operatorname{iso} \sigma_a(T) : \alpha(T - \lambda I) > 0\}$. It is known [**10**] that an operator obeying generalized a-Weyl's theorem obeys generalized Weyl's theorem, but the converse is not true in general.

Definition 1.1. An operator $T \in \mathcal{L}(X)$ is called a-polaroid (resp. isoloid) if all isolated points of the approximate point spectrum are left poles of T, i.e. $\operatorname{iso} \sigma_a(T) = \Pi_a(T)$ (resp. all isolated points of the spectrum are eigenvalues of T, i.e. $\operatorname{iso} \sigma(T) = E(T)$).

In [**12**], we introduced and studied the new properties (gab), (ab), (gaw) and (aw) (see Definition 2.1). Properties (gab) and (gaw) extend properties (ab) and (aw) respectively to the context of B-Fredholm theory. In this paper we study the preservation of these properties under perturbations by finite rank, compact and nilpotent operators. In the second section in a first step we give an equivalence condition for properties (gaw) and (aw) and we prove that under the assumption $\Pi(T) = E_a(T)$, the two properties are equivalent. We show in Theorem 2.3 that if $T \in \mathcal{L}(X)$ possesses property (gaw), then T obeys generalized Weyl's theorem, but the converse does not hold in general as shown by Example 2.4.

In the third section, in Theorem 3.1 we prove that if $T \in \mathcal{L}(X)$ possesses property (ab) and $N \in \mathcal{L}(X)$ is a nilpotent operator commuting with T, then $T + N$ possesses property (ab), and in Theorem 3.2 we prove a similar result for property (gab) in the case of a-polaroid operators. We also prove in Theorem 3.6 that if $T \in \mathcal{L}(X)$ possesses property (gaw) and $N \in \mathcal{L}(X)$ is a nilpotent operator commuting with T, then $T + N$ possesses property (gaw), and in Theorem 3.5 we prove a similar result for property (aw).

In the last part, we provide certain conditions under which the new properties are preserved under commuting compact and finite rank perturbations. Thus, we prove in Theorem 4.5 that if $T \in \mathcal{L}(X)$ is an operator possessing property (gab) and $F \in \mathcal{L}(X)$ is a finite rank operator commuting with T such that $\Pi_a(T + F) \subset \sigma_a(T)$, then $T + F$ possesses property (gab). Similarly, we prove in Theorem 4.3 that if $T \in \mathcal{L}(X)$ is an operator possessing property (ab) and $K \in \mathcal{L}(X)$ is a compact operator commuting with T such that $\Pi_a^0(T + K) \subset \sigma_a(T)$, then $T + K$ possesses property (ab). We end this section by some illustrating examples.

2. Property (gaw) and generalized Weyl's theorem

Definition 2.1. [**12**] Let $T \in \mathcal{L}(X)$. We will say that:
- (i) T possesses property (ab) if $\Delta(T) = \Pi_a^0(T)$.
- (ii) T possesses property (gab) if $\Delta^g(T) = \Pi_a(T)$.
- (iii) T possesses property (aw) if $\Delta(T) = E_a^0(T)$.
- (iv) T possesses property (gaw) if $\Delta^g(T) = E_a(T)$.

In a first step we give an equivalence condition for properties (gaw) and (aw). In [**12**, Theorem 3.3], it is proved that if $T \in \mathcal{L}(X)$ possesses property (gaw) then T possesses property (aw) and the converse is not true in general. But under the assumption $\Pi(T) = E_a(T)$, the following result proves that the two properties are equivalent.

Theorem 2.2. *Let X be a Banach space and let $T \in \mathcal{L}(X)$. Then T possesses property (gaw) if and only if T possesses property (aw) and $\Pi(T) = E_a(T)$.*

Proof. Assume that T possesses property (gaw), then $\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$. From [**12**, Theorem 3.3], T possesses property (aw). By Theorem 3.5 and Corollary 2.6 of [**12**], T satisfies generalized Browder's theorem, that is $\sigma(T)\setminus\sigma_{BW}(T) = \Pi(T)$. Hence $\Pi(T) = E_a(T)$.
Conversely, assume that T possesses property (aw) and $\Pi(T) = E_a(T)$. If $\lambda \in \Delta^g(T)$, we can assume without loss of generality that $\lambda = 0$. Then T is a B-Weyl operator. In particular T is an operator of topological uniform descent [**11**].

We show that 0 is a pole of the resolvent of T. Since T is B-Weyl, from [**11**, Corollary 3.2], there exists $\varepsilon > 0$ such that $T - \mu I$ is Weyl for every μ such that $0 < |\mu| < \varepsilon$. Let $|\mu| < \varepsilon$ and $\mu \notin \sigma(T)$, then $a(T - \mu I) = \delta(T - \mu I) = 0$. In the second case $\mu \in \sigma(T)$, then $\mu \in \sigma(T) \setminus \sigma_W(T) = E_a^0(T)$ since T possesses property (aw). Therefore $\mu \in \Pi^0(T)$ and $a(T - \mu I) = \delta(T - \mu I) < \infty$. From [**18**, Corollary 4.8] we conclude that $a(T) = \delta(T) < \infty$. As $0 \in \sigma(T)$, then $0 \in \Pi(T) = E_a(T)$.

On the other hand, if $\lambda \in E_a(T)$, then $\lambda \in \Pi(T)$. Therefore $T - \lambda I$ is a B-Fredholm operator of index 0. Thus $\lambda \in \Delta^g(T)$. Hence $\Delta^g(T) = E_a(T)$ and T possesses property (gaw). □

Theorem 2.3. *Let X be a Banach space and let $T \in \mathcal{L}(X)$. Then T possesses property (gaw) if and only if T satisfies generalized Weyl's theorem and $E(T) = E_a(T)$.*

Proof. Assume that T possesses property (gaw), then $\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$. If $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, then $\lambda \in E_a(T)$. Since T possesses property (gaw), it follows that $E_a(T) = \Pi(T)$. Therefore $\lambda \in \Pi(T)$. As $\Pi(T) \subset E(T)$ is always true, then $\sigma(T) \setminus \sigma_{BW}(T) \subset E(T)$. Now if $\lambda \in E(T)$, as we have always $E(T) \subset E_a(T)$, then $\lambda \in E_a(T) = \sigma(T) \setminus \sigma_{BW}(T)$. Hence $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$, i.e. T satisfies generalized Weyl's theorem and $E(T) = E_a(T)$.
Conversely, assume that T satisfies generalized Weyl's theorem and $E(T) = E_a(T)$. Then $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$ and $E(T) = E_a(T)$. So $\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$ and T possesses property (gaw). □

The following example shows that there is an operator obeying generalized a-Weyl's theorem and generalized Weyl's theorem but not the property (gaw).

Example 2.4. Let $R \in \mathcal{L}(\ell^2(\mathbb{N}))$ be the unilateral right shift and $S \in \mathcal{L}(\ell^2(\mathbb{N}))$ the operator defined by $S(x_1, x_2, x_3, \ldots) = (0, x_2, x_3, x_4, \ldots)$.

Consider the operator T defined on the Banach space $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = R \oplus S$, then $\sigma(T) = D(0,1)$ is the closed unit disc in $\mathbb{C}$, $\operatorname{iso}\sigma(T) = \emptyset$ and $\sigma_a(T) = C(0,1) \cup \{0\}$, where $C(0,1)$ is the unit circle of $\mathbb{C}$. Moreover, we have $\sigma_{SBF_+^-}(T) = C(0,1)$ and $E_a(T) = \{0\}$. Hence $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$, i.e. T obeys generalized a-Weyl's theorem and so T obeys generalized Weyl's theorem. On the other hand, $\sigma_{BW}(T) = D(0,1)$. Then $\sigma(T) \setminus \sigma_{BW}(T) \neq E_a(T)$ and T does not possess property (gaw).

Similarly to Theorem 2.3, we have the following result in the case of property (aw).

Theorem 2.5. *Let X be a Banach space and let $T \in \mathcal{L}(X)$. Then T possesses property (aw) if and only if T satisfies Weyl's theorem and $E^0(T) = E_a^0(T)$.*

Proof. Suppose that T possesses property (aw), then $\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$. From Theorem 3.6 and Theorem 2.4 of [**12**], T satisfies Browder's theorem, that is $\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$. Since we have always $\Pi^0(T) \subset E^0(T)$, then $\sigma(T) \setminus \sigma_W(T) \subset E^0(T)$. Now let us consider $\lambda \in E^0(T)$, then $\lambda \in E_a^0(T) = \sigma(T) \setminus \sigma_W(T)$.

Hence $\sigma(T)\setminus\sigma_W(T) = E^0(T)$, i.e. T satisfies Weyl's theorem and $E^0(T) = E^0_a(T)$. Conversely, assume that Weyl's theorem holds for T and $E^0(T) = E^0_a(T)$. Then $\sigma(T) \setminus \sigma_W(T) = E^0(T)$ and $E^0(T) = E^0_a(T)$. So $\sigma(T) \setminus \sigma_W(T) = E^0_a(T)$ and T possesses property (aw). □

Generally, a-Weyl's theorem and Weyl's theorem do not imply property (aw). Indeed, if we consider the operator T as in Example 2.4, then $\sigma_{SF_+^-}(T) = C(0, 1)$ and $E^0_a(T) = \{0\}$. Hence $\sigma_a(T)\setminus\sigma_{SF_+^-}(T) = E^0_a(T)$, i.e. T obeys a-Weyl's theorem. So T obeys Weyl's theorem. On the other hand, $\sigma_W(T) = D(0, 1)$. Consequently, $\sigma(T) \setminus \sigma_W(T) \neq E^0_a(T)$ and T does not possess property (aw).

3. Nilpotent perturbations

Theorem 3.1. *Let X be a Banach space and let $T \in \mathcal{L}(X)$. If $N \in \mathcal{L}(X)$ is a nilpotent operator commuting with T, then T possesses property (ab) if and only if $T + N$ possesses property (ab).*

Proof. As N is nilpotent and commutes with T, we know that $\sigma_a(T) = \sigma_a(T + N)$, and $\sigma(T) = \sigma(T + N)$. Moreover, from [**22**, Lemma 2], we know that $\sigma_W(T) = \sigma_W(T + N)$. If $\lambda \in \sigma(T + N) \setminus \sigma_W(T + N)$, then $\lambda \in \sigma(T) \setminus \sigma_W(T) = \Pi^0_a(T)$, since T possesses property (ab). Therefore $\lambda \in \operatorname{iso}\sigma_a(T+N)$. As $T+N-\lambda I$ is an upper semi-Fredholm with $\operatorname{ind}(T + N - \lambda I) \leq 0$, by [**10**, Theorem 2.8] we have $\lambda \in \Pi^0_a(T + N)$. Hence $\sigma(T + N) \setminus \sigma_W(T + N) \subset \Pi^0_a(T + N)$. On the other hand, if $\lambda \in \Pi^0_a(T + N)$, then $T + N - \lambda I$ is an upper semi-Fredholm such that $\operatorname{ind}(T+N-\lambda I) \leq 0$. From [**17**, Theorem 2.13], $T-\lambda I$ is an upper semi-Fredholm of index less or equal than zero. As $\lambda \in \operatorname{iso}\sigma_a(T)$, then $\lambda \in \Pi^0_a(T)$ which implies that $\lambda \in \sigma(T+N)\setminus\sigma_W(T+N)$. Finally, we have $\sigma(T+N)\setminus\sigma_W(T+N) = \Pi^0_a(T+N)$ and $T + N$ possesses property (ab). Conversely, assume that $T + N$ possesses property (ab). By symmetry, we have $T = (T + N) - N$ possesses property (ab). □

Theorem 3.2. *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be an a-polaroid operator. If T possesses property (gab) and $N \in \mathcal{L}(X)$ is a nilpotent operator commuting with T, then $T + N$ possesses property (gab).*

Proof. It is well known that $\sigma(T) = \sigma(T + N)$. By virtue of [**12**, Corollary 2.7], we know that if T possesses property (gab), then $\sigma_{BW}(T) = \sigma_D(T)$ and $\Pi(T) = \Pi_a(T)$. Let $\lambda \in \sigma(T+N)\setminus\sigma_{BW}(T+N)$. There is no loss of generality if we assume that $\lambda = 0$. Then $T + N$ is a B-Weyl operator. We show that $T + N$ has ascent $a(T + N)$ finite. Since $T + N$ is B-Weyl, there exists $\varepsilon > 0$ such that $T + N - \mu I$ is Weyl for every μ such that $0 < |\mu| < \varepsilon$. Therefore $T - \mu I$ is Weyl. Let $|\mu| < \varepsilon$ and $\mu \notin \sigma(T) = \sigma(T + N)$, then $a(T + N - \mu I) = 0$. The second possibility is that $\mu \in \sigma(T)$, then $\mu \in \sigma(T) \setminus \sigma_W(T)$. Since T possesses property (gab), then from [**12**, Theorem 2.2], T possesses property (ab). So $\mu \in \sigma(T) \setminus \sigma_W(T) = \Pi^0_a(T)$. Thus $\mu \in \operatorname{iso}\sigma_a(T) = \operatorname{iso}\sigma_a(T + N)$. As $T + N - \mu I$ is an upper semi-Fredholm operator, then by Theorem 3.23 and Theorem 3.16 of [**1**], we deduce that the ascent $a(T + N - \mu I) < \infty$. From [**18**, Corollary 4.8] we conclude that $a(T + N) < \infty$. Since $T + N$ is B-Weyl, it is also an operator of topological uniform descent, and for n large enough, $R((T+N)^n)$ is closed. By [**21**, Lemma 12], we then deduce that $R((T+N)^{a(T+N)+1})$ is closed. Clearly, $0 \in \sigma_a(T+N)$, since $T+N$ is B-Weyl. Hence $0 \in \Pi_a(T+N)$.

To show the opposite inclusion, let us consider $\lambda \in \Pi_a(T+N)$. Then $\lambda \in \text{iso}\sigma_a(T+N) = \text{iso}\sigma_a(T)$. Since T is a-polaroid, then $\lambda \in \Pi_a(T) = \Pi(T)$. From [**13**, Lemma 2.2] we know that $\Pi(T) = \Pi(T+N)$. Thus $T+N-\lambda I$ is Drazin invertible, hence B-Weyl, so that $\lambda \in \sigma(T+N) \setminus \sigma_{BW}(T+N)$. Hence $\sigma(T+N) \setminus \sigma_{BW}(T+N) = \Pi_a(T+N)$ and $T+N$ possesses property (gab). □

In [**14**] the authors asked the following question: let $T \in \mathcal{L}(X)$ and let $N \in \mathcal{L}(X)$ be a nilpotent operator commuting with T. Under which conditions $\Pi_a(T+N) = \Pi_a(T)$? The next corollary answers positively this question, in the case of a-polaroid operators possessing property (gab).

Corollary 3.3. *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be an a-polaroid operator possessing property (gab). If $N \in \mathcal{L}(X)$ is a nilpotent operator commuting with T, then $\Pi_a(T+N) = \Pi_a(T)$.*

Proof. We already have that $\sigma(T+N) = \sigma(T)$, $\Pi(T) = \Pi(T+N)$. Since T possesses property (gab), T satisfies generalized Browder's theorem which implies by [**13**, Theorem 2.3] that $T+N$ satisfies generalized Browder's theorem. So $\sigma(T+N) \setminus \sigma_{BW}(T+N) = \Pi(T+N)$, $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$. Hence $\sigma_{BW}(T+N) = \sigma_{BW}(T)$. On the other hand, as both T and $T+N$ possess property (gab), then $\sigma(T+N) \setminus \sigma_{BW}(T+N) = \Pi_a(T+N)$, $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a(T)$. Hence $\Pi_a(T+N) = \Pi_a(T)$. □

In the next theorem we consider an operator T possessing property (gab) and a nilpotent operator N commuting with T, and we give necessary and sufficient conditions for $T+N$ to possess property (gab).

Theorem 3.4. *Let X be a Banach space and let $T \in \mathcal{L}(X)$ and $N \in \mathcal{L}(X)$ be a nilpotent operator commuting with T. If T possesses property (gab), then the following statements are equivalent.*

(i) $T+N$ *possesses property* (gab),

(ii) $\Pi(T) = \Pi_a(T+N)$,

(iii) $\Pi_a(T) = \Pi_a(T+N)$.

Proof. *(i)* $\Longleftrightarrow$ *(ii)* If $T+N$ possesses property (gab), then from [**12**, Corollary 2.7] we have $\Pi(T+N) = \Pi_a(T+N)$. So $\Pi(T) = \Pi_a(T+N)$. Conversely, if $\Pi(T) = \Pi_a(T+N)$, since T possesses property (gab), then from [**12**, Corollary 2.6], T satisfies generalized Browder's theorem. From [**13**, Theorem 2.3], $T+N$ satisfies generalized Browder's theorem, that is $\sigma(T+N) \setminus \sigma_{BW}(T+N) = \Pi(T+N)$. As by hypothesis $\Pi(T) = \Pi_a(T+N)$, then $\sigma(T+N) \setminus \sigma_{BW}(T+N) = \Pi_a(T+N)$ and $T+N$ possesses property (gab).

Since T possesses property (gab), then $\Pi(T) = \Pi_a(T)$. This makes *(ii)* $\Longleftrightarrow$ *(iii)*. □

Theorem 3.5. *Let X be a Banach space and let $T \in \mathcal{L}(X)$. If $N \in \mathcal{L}(X)$ is a nilpotent operator commuting with T, then T possesses property (aw) if and only if $T + N$ possesses property (aw).*

Proof. We already have that $\sigma(T + N) = \sigma(T)$ and $\sigma_W(T + N) = \sigma_W(T)$. We prove that $E_a^0(T + N) = E_a^0(T)$. Let $\lambda \in E_a(T)$ be arbitrary. We may assume that $\lambda = 0$. As $\sigma_a(T + N) = \sigma_a(T)$, then $0 \in \text{iso}\,\sigma_a(T + N)$. Let $m \in \mathbb{N}$ be such that $N^m = 0$. If $x \in N(T)$, then $(T + N)^m(x) = \sum_{k=0}^{m} C_m^k T^k N^{m-k}(x) = 0$. So $N(T) \subset N(T + N)^m$. As $\alpha(T) > 0$, it follows that $\alpha((T + N)^m) > 0$ and this implies that $\alpha(T+N) > 0$. Hence $0 \in E_a(T+N)$. Therefore $E_a(T) \subset E_a(T+N)$. By symmetry, we have $E_a(T) \supset E_a(T+N)$. Hence $E_a(T+N) = E_a(T)$. It remains only to show that $\alpha(T) < \infty$ if and only if $\alpha(T+N) < \infty$. If $\alpha(T+N) < \infty$, then from [**26**, Lemma 3.3, (a)] we have $\alpha((T+N)^m) < \infty$. As $N(T) \subset N(T+N)^m$, then $\alpha(T) < \infty$. By symmetry, we prove the reverse implication. Hence $\Delta(T) = E_a^0(T)$ if and only if $\Delta(T + N) = E_a^0(T + N)$, as desired. □

In the next theorem, we prove a similar perturbation result for property (gaw).

Theorem 3.6. *Let X be a Banach space and let $T \in \mathcal{L}(X)$. If $N \in \mathcal{L}(X)$ is a nilpotent operator commuting with T, then T possesses property (gaw) if and only if $T + N$ possesses property (gaw).*

Proof. If T possesses property (gaw), then from Theorem 2.2, $\Pi(T) = E_a(T)$. Let $\lambda \in \sigma(T + N) \setminus \sigma_{BW}(T + N)$. We may assume that $\lambda = 0$. Then $T + N$ is B-Weyl. Therefore there exists an $\varepsilon > 0$ such that $T+N-\mu I$ is Weyl for any μ such that $0 < |\mu| < \varepsilon$. From classical Fredholm theory we know that $T - \mu I$ is Weyl. Let $|\mu| < \varepsilon$ and $\mu \notin \sigma(T) = \sigma(T + N)$. Then $a(T + N - \mu I) = \delta(T + N - \mu I) = 0$. In the second case $\mu \in \sigma(T)$, then $\mu \in \sigma(T) \setminus \sigma_W(T) = E_a^0(T)$ since T possesses property (aw). Hence $\mu \in \Pi^0(T)$ which implies that $\mu \in \text{iso}\,\sigma(T) = \text{iso}\,\sigma(T + N)$. By [**1**, Theorem 3.77], it then follows that $a(T+N-\mu I) = \delta(T+N-\mu I) < \infty$. In the two cases, we have $a(T+N-\mu I) = \delta(T+N-\mu I) < \infty$. By [**18**, Corollary 4.8] we then deduce that $a(T + N) = \delta(T + N) < \infty$. As $0 \in \sigma(T + N)$, then 0 is a pole of the resolvent of $T + N$, in particular an isolated point of the approximate point spectrum of $T + N$. Clearly, $\alpha(T + N) > 0$, since $T + N$ is B-Weyl, so that $0 \in E_a(T + N)$. To prove the opposite inclusion, let us consider $\lambda \in E_a(T + N)$. Then $\lambda \in E_a(T) = \Pi(T) = \Pi(T + N)$. Hence $T + N - \lambda I$ is B-Weyl, so that $\lambda \in \sigma(T+N)\setminus\sigma_{BW}(T+N)$. Finally, we have $\sigma(T+N)\setminus\sigma_{BW}(T+N) = E_a(T+N)$ and $T+N$ possesses property (gaw). Conversely, if $T+N$ possesses property (gaw), then by symmetry we have $T = (T + N) - N$ possesses property (gaw). □

Remark 3.7. (1) The following example shows that Theorem 3.5 and Theorem 3.6 do not hold if we do not assume that the nilpotent operator N commutes with T. Let $X = \ell^2(\mathbb{N})$, and let T and N be defined by

$$T(x_1, x_2, x_3, \ldots) = (0, x_1/2, x_2/3, \ldots)\ ,\ N(x_1, x_2, x_3, \ldots) = (0, -x_1/2, 0, 0, \ldots).$$

Clearly N is a nilpotent operator which does not commute with T. Moreover, we have $\sigma(T) = \{0\}$, $\sigma_{BW}(T) = \{0\}$ and $E_a(T) = \emptyset$. So $\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$ and T possesses property (gaw). Hence T possesses also property (aw). On the other hand, $\sigma(T+N)=\{0\}$, $\sigma_W(T+N)=\{0\}$, $\sigma_{BW}(T+N)=\{0\}$, $E_a(T+N)=\{0\}$ and $E_a^0(T+N)=\{0\}$. Consequently, $\sigma(T+N)\setminus\sigma_W(T+N)\neq E_a^0(T+N)$ and $\sigma(T+N)\setminus\sigma_{BW}(T+N)\neq E_a(T+N)$. So $T+N$ does not possess property (aw) and property (gaw).

(2) Generally, Theorem 3.5 and Theorem 3.6 do not extend to commuting quasinilpotent perturbations. Indeed, on the Hilbert space $\ell^2(\mathbb{N})$ let T and the quasinilpotent operator Q be defined by

$$T=0 \quad \text{and} \quad Q(x_1,x_2,x_3,\ldots)=(x_2/2,x_3/3,x_4/4,\ldots).$$

Then $TQ=QT=0$, $\sigma(T)=\{0\}$, $\sigma_W(T)=\{0\}$, $\sigma_{BW}(T)=\emptyset$ and $E_a^0(T)=\emptyset$. Moreover, we have $E_a(T)=\{0\}$. Thus $\sigma(T)\setminus\sigma_W(T)=E_a^0(T)$ and $\sigma(T)\setminus\sigma_{BW}(T)=E_a(T)$. So T possesses property (gaw) and property (aw). But, since $\sigma(T+Q)=\{0\}$, $\sigma_{BW}(T+Q)=\{0\}$, $E_a(T+Q)=\{0\}$, $E_a^0(T+Q)=\{0\}$ and $\sigma_W(T+Q)=\{0\}$, then $\sigma(T+Q)\setminus\sigma_W(T+Q)\neq E_a^0(T+Q)$ and $\sigma(T+Q)\setminus\sigma_{BW}(T+Q)\neq E_a(T+Q)$. So $T+Q$ does not possess property (gaw) and property (aw).

Recall that an operator $T\in\mathcal{L}(X)$ is said to possess property (gw) [**3**, Definition 2.1] if $\Delta_a^g(T)=E(T)$. In the next theorem we consider an operator T possessing property (gw) and a nilpotent operator N commuting with T, and we give necessary and sufficient conditions for $T+N$ to possess property (gw).

Theorem 3.8. *Let X be a Banach space and let $T\in\mathcal{L}(X)$ and $N\in\mathcal{L}(X)$ be a nilpotent operators commuting with T. If T possesses property (gw), then the following statements are equivalent.*
(i) $T+N$ possesses property (gw);
(ii) $\sigma_{SBF_+^-}(T)=\sigma_{SBF_+^-}(T+N)$;
(iii) $E(T)=\Pi_a(T+N)$.

Proof. *(i)* $\Longleftrightarrow$ *(iii)* If $T+N$ possesses property (gw), then from [**3**, Theorem 2.6], we have $E(T+N)=\Pi_a(T+N)$. As we know that $E(T)=E(T+N)$, then $E(T)=\Pi_a(T+N)$. Conversely, assume that $E(T)=\Pi_a(T+N)$, since T possesses property (gw), again by [**3**, Theorem 2.6], T satisfies generalized a-Browder's theorem. As we know that generalized a-Browder's theorem is equivalent to a-Browder's theorem, then T satisfies a-Browder's theorem. So $\sigma_{SF_+^-}(T)=\sigma_{ub}(T)$. As N is nilpotent and commutes with T, we know from [**1**, Theorem 3.65] that $\sigma_{ub}(T)=\sigma_{ub}(T+N)$ and as it had already been mentioned we have $\sigma_{SF_+^-}(T)=\sigma_{SF_+^-}(T+N)$. Therefore $\sigma_{SF_+^-}(T+N)=\sigma_{ub}(T+N)$. Hence $T+N$ satisfies a-Browder's theorem, so it satisfies generalized a-Browder's theorem, that is $\sigma_a(T+N)\setminus\sigma_{SBF_+^-}(T+N)=\Pi_a(T+N)$. Since $E(T)=\Pi_a(T+N)$, then $\sigma_a(T+N)\setminus\sigma_{SBF_+^-}(T+N)=E(T)=E(T+N)$ and $T+N$ possesses property (gw).
(i) $\Longleftrightarrow$ *(ii)* If $T+N$ possesses property (gw), then $\sigma_a(T+N)\setminus\sigma_{SBF_+^-}(T+N)=E(T+N)$. Since T possesses property (gw), $\sigma_a(T)\setminus\sigma_{SBF_+^-}(T)=E(T)$. As $\sigma_a(T)=\sigma_a(T+N)$ and $E(T)=E(T+N)$, it then follows that $\sigma_{SBF_+^-}(T)=$

$\sigma_{SBF_+^-}(T+N)$. Conversely, if $\sigma_{SBF_+^-}(T) = \sigma_{SBF_+^-}(T+N)$, then $\sigma_a(T+N) \setminus \sigma_{SBF_+^-}(T+N) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T) = E(T+N)$ and $T+N$ possesses property (gw). □

Remark 3.9. The hypothesis of commutativity in the previous theorem is crucial. The following example shows that if we do not assume that N commutes with T, then the result may fail. Let $X = \ell^2(\mathbb{N})$ and let T and N be as in part (1) of Remark 3.7. Clearly, $\sigma_a(T) = \{0\}$, $\sigma_{SBF_+^-}(T) = \{0\}$ and $E(T) = \emptyset$. So $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$ and T possesses property (gw). On the other hand, we have $\sigma_a(T+N) = \{0\}$, $\sigma_{SBF_+^-}(T+N) = \{0\}$ and $E(T+N) = \{0\}$. So $\sigma_a(T+N) \setminus \sigma_{SBF_+^-}(T+N) \neq E(T+N)$ and $T+N$ does not possess property (gw). Though we have $E(T) = \Pi_a(T+N) = \emptyset$.

We finish this section by posing the following two questions.

Open questions: The proof of Corollary 3.3 suggests the following questions:

1. Let $T \in \mathcal{L}(X)$, and let $N \in \mathcal{L}(X)$ be a nilpotent operator commuting with T. Do we always have $\sigma_{BW}(T+N) = \sigma_{BW}(T)$?
2. Let $T \in \mathcal{L}(X)$, and let $N \in \mathcal{L}(X)$ be a nilpotent operator commuting with T. Under which conditions $\sigma_{BF}(T+N) = \sigma_{BF}(T)$?

4. Finite rank and compact perturbations

Theorem 4.1. *Let X be a Banach space and let $T \in \mathcal{L}(X)$. If $K \in \mathcal{L}(X)$ is a compact operator commuting with T and if T possesses property (ab), then $T+K$ possesses property (ab) if and only if $\Pi^0(T+K) = \Pi_a^0(T+K)$.*

Proof. Assume that $T+K$ possesses property (ab), then from [**12**, Corollary 2.6], we have $\Pi^0(T+K) = \Pi_a^0(T+K)$. Conversely, assume that $\Pi^0(T+K) = \Pi_a^0(T+K)$. Since T possesses property (ab), then from [**12**, Theorem 2.4], T satisfies Browder's theorem. So $\sigma_b(T) = \sigma_W(T)$. Since K commutes with T, then from [**1**, Corollary 3.49], we have $\sigma_b(T) = \sigma_b(T+K)$, and by [**1**, Corollary 3.41], we have $\sigma_W(T) = \sigma_W(T+K)$. Therefore $\sigma_b(T+K) = \sigma_W(T+K)$ which implies that $T+K$ satisfies Browder's theorem, that is $\sigma(T+K) \setminus \sigma_W(T+K) = \Pi^0(T+K)$. Since $\Pi^0(T+K) = \Pi_a^0(T+K)$, then $\Delta(T+K) = \Pi_a^0(T+K)$ and $T+K$ possesses property (ab). □

Theorem 4.2. *Let X be a Banach space and let $T \in \mathcal{L}(X)$. If $K \in \mathcal{L}(X)$ is a compact operator commuting with T and if T possesses property (gab), then $T+K$ possesses property (gab) if and only if $\Pi(T+K) = \Pi_a(T+K)$.*

Proof. If $T+K$ possesses property (gab), then from [**12**, Corollary 2.7], we have $\Pi(T+K) = \Pi_a(T+K)$. Conversely, if $\Pi(T+K) = \Pi_a(T+K)$, as T possesses property (gab), by virtue of [**12**, Corollary 2.6], T satisfies generalized Browder's theorem. Since we know that Browder's theorem is equivalent to generalized Browder's theorem, it follows that $\sigma(T+K) \setminus \sigma_{BW}(T+K) = \Pi(T+K)$. As $\Pi(T+K) = \Pi_a(T+K)$, then $\sigma(T+K) \setminus \sigma_{BW}(T+K) = \Pi_a(T+K)$ and $T+K$ possesses property (gab). □

Theorem 4.3. *Let X be a Banach space and let $T \in \mathcal{L}(X)$ and $K \in \mathcal{L}(X)$ be a compact operator commuting with T. If T possesses property (ab), and if $\Pi_a^0(T+K) \subset \sigma_a(T)$, then $T+K$ possesses property (ab).*

Proof. We only have to show, by Theorem 4.1, that $\Pi_a^0(T+K) = \Pi^0(T+K)$. Let $\lambda \in \Pi_a^0(T+K)$, then $\lambda \notin \sigma_{ub}(T+K)$. Since K commutes with T, then from [**1**, Corollary 3.45], we have $\sigma_{ub}(T+K) = \sigma_{ub}(T)$. So $\lambda \notin \sigma_{ub}(T)$, and since by hypothesis $\lambda \in \sigma_a(T)$, then $\lambda \in \sigma_a(T) \setminus \sigma_{ub}(T) = \Pi_a^0(T)$. Since T possesses property (ab), then $\lambda \notin \sigma_W(T)$. As $\sigma_W(T+K) = \sigma_W(T)$, then $\lambda \notin \sigma_W(T+K)$ and $\text{ind}(T+K-\lambda I) = 0$. Since $T+K-\lambda I$ has ascent $a(T+K-\lambda I)$ finite, then $\delta(T+K-\lambda I) < \infty$ and $T+K-\lambda I$ is Drazin invertible. Since $\lambda \in \sigma(T+K)$, then λ is a pole of the resolvent of $T+K$. Therefore $\lambda \in \Pi^0(T+K)$. Hence $\Pi_a^0(T+K) \subset \Pi^0(T+K)$ and since the opposite inclusion holds for every operator, it then follows that $\Pi_a^0(T+K) = \Pi^0(T+K)$, as desired. □

Corollary 4.4. *Let X be a Banach space and let $T \in \mathcal{L}(X)$ and $F \in \mathcal{L}(X)$ be a finite rank operator commuting with T. If $\text{iso}\,\sigma_a(T) = \emptyset$, then T possesses property (ab) if and only if $T+F$ possesses property (ab).*

Proof. Assume that T possesses property (ab). Since F is a finite rank operator commuting with T, and since $\text{iso}\,\sigma_a(T) = \emptyset$, then from [**2**, Lemma 2.6], we have $\sigma_a(T) = \sigma_a(T+F)$. Hence $\Pi_a^0(T+F) \subset \sigma_a(T)$. As T possesses property (ab), then from Theorem 4.3, $T+F$ possesses property (ab). Conversely, assume that $T+F$ possesses property (ab). As $\text{iso}\sigma_a(T+F) = \emptyset$, then by symmetry, $T = (T+F) - F$ possesses property (ab). □

Theorem 4.5. *Let X be a Banach space and let $T \in \mathcal{L}(X)$ and $F \in \mathcal{L}(X)$ be a finite rank operator commuting with T. If T possesses property (gab), and if $\Pi_a(T+F) \subset \sigma_a(T)$, then $T+F$ possesses property (gab).*

Proof. We only have to show, by Theorem 4.2, that $\Pi(T+F) = \Pi_a(T+F)$. If $\lambda \in \Pi_a(T+F)$, then $\lambda \notin \sigma_{LD}(T+F)$. Since F commutes with T, then from [**14**, Theorem 2.1], we have $\sigma_{LD}(T+F) = \sigma_{LD}(T)$, and so $\lambda \notin \sigma_{LD}(T)$. Since by the assumption $\lambda \in \sigma_a(T)$, then $\lambda \in \sigma_a(T) \setminus \sigma_{LD}(T) = \Pi_a(T)$. Since T possesses property (gab), then $T - \lambda I$ is a B-Weyl operator. As F is a finite rank operator, then from [**7**, Theorem 4.3] it follows that $T+F-\lambda I$ is also a B-Fredholm operator and $\text{ind}(T+F-\lambda I) = 0$. As $a(T+F-\lambda I)$ is finite and $\lambda \in \sigma(T+F)$, then λ is a pole of the resolvent of $T+F$ and $\lambda \in \Pi(T+F)$. Hence $\Pi_a(T+F) \subset \Pi(T+F)$. As we always have $\Pi_a(T+F) \supset \Pi(T+F)$, then $\Pi(T+F) = \Pi_a(T+F)$. Hence $T+F$ possesses property (gab). □

Corollary 4.6. *Let X be a Banach space and let $T \in \mathcal{L}(X)$ and $F \in \mathcal{L}(X)$ be a finite rank operator commuting with T. Ifiso $\sigma_a(T) = \emptyset$, then T possesses property (gab) if and only if $T+F$ possesses property (gab).*

Proof. Since F is a finite rank operator commuting with T and since $\text{iso}\sigma_a(T) = \emptyset$, then from [**2**, Lemma 2.6], we have $\text{iso}\sigma_a(T+F) = \emptyset$. Hence $\Pi_a(T+F) = \Pi(T+F) = \emptyset$. As T possesses property (gab), then from Theorem 4.2, $T+F$ possesses property (gab). Conversely, assume that $T+F$ possesses property (gab). Since $\mathrm{iso}\sigma_a(T+F)=\emptyset$, then by symmetry we have T possesses property (gab). □

Theorem 4.7. *Let $T \in \mathcal{L}(X)$ and let $K \in \mathcal{L}(X)$ be a compact operator commuting with T. If T possesses property (aw), then $T+K$ possesses property (aw) if and only if $\Pi^0(T+K)=E^0_a(T+K)$.*

Proof. If $T+K$ possesses property (aw), then from [**12**, Theorem 3.6], $T+K$ possesses property (ab). So $\sigma(T+K)\setminus\sigma_W(T+K)=E^0_a(T+K)$ and $\sigma(T+K)\setminus\sigma_W(T+K)=\Pi^0_a(T+K)$. Thus $\Pi^0_a(T+K)=E^0_a(T+K)$. On the other hand, since $T+K$ possesses property (ab), by Theorem 4.1 we have $\Pi^0(T+K)=\Pi^0_a(T+K)$. Hence $\Pi^0(T+K)=E^0_a(T+K)$. Conversely, assume that $\Pi^0(T+K)=E^0_a(T+K)$. Since T possesses property (aw), then T satisfies Browder's theorem. Hence $T+K$ satisfies Browder's theorem, that is $\sigma(T+K)\setminus\sigma_W(T+K)=\Pi^0(T+K)$. As $\Pi^0(T+K)=E^0_a(T+K)$, then $\sigma(T+K)\setminus\sigma_W(T+K)=E^0_a(T+K)$ and $T+K$ possesses property (aw). □

Theorem 4.8. *Let $T \in \mathcal{L}(X)$ and let $K \in \mathcal{L}(X)$ be a compact operator commuting with T. If T possesses property (gaw), then $T+K$ possesses property (gaw) if and only if $\Pi(T+K)=E_a(T+K)$.*

Proof. If $T+K$ possesses property (gaw), then from Theorem 2.2, we have $\Pi(T+K)=E_a(T+K)$. Conversely, assume that $\Pi(T+K)=E_a(T+K)$. Since T possesses property (gaw), then from [**12**, Theorem 3.5], T possesses property (gab). Therefore T satisfies generalized Browder's theorem. Hence $T+K$ satisfies generalized Browder's theorem, that is $\sigma(T+K)\setminus\sigma_{BW}(T+K)=\Pi(T+K)$. As $\Pi(T+K)=E_a(T+K)$, then $\sigma(T+K)\setminus\sigma_{BW}(T+K)=E_a(T+K)$ and $T+K$ possesses property (gaw). □

There exist quasinilpotent operators which do not possess property (gaw). For example, if we consider the operator T defined on $\ell^2(\mathbb{N})$ by $T(x_1,x_2,x_3,\ldots)=(x_3/3,x_4/4,x_5/5\ldots)$, then T is quasinilpotent, but property (gaw) fails for T, since $\sigma(T)=\sigma_{BW}(T)=\{0\}$ and $E_a(T)=\{0\}$. But if a quasinilpotent operator possesses property (gaw), then the following perturbation result holds.

Theorem 4.9. *Let $T \in \mathcal{L}(X)$ be a quasinilpotent operator and let $F \in \mathcal{L}(X)$ be a finite rank operator commuting with T. If T possesses property (gaw), then $T+F$ possesses property (gaw).*

Proof. As $\mathrm{iso}\,\sigma(T)=\sigma(T)=\{0\}$, then $\mathrm{acc}\,\sigma(T)=\emptyset$. By [**20**, Lemma 2.1] it then follows that $\mathrm{acc}\,\sigma(T+F)=\emptyset$.

If 0 is an eigenvalue of T, then T is isoloid. If $\lambda\in E_a(T+F)$, then $\lambda\in\mathrm{iso}\,\sigma(T+F)$. Thus $\lambda\in E(T+F)$. As T possesses property (gaw), then from Theorem 2.3, T satisfies generalized Weyl's theorem and since T is isoloid, it then follows from [**8**, Theorem 2.6] that $T+F$ satisfies generalized Weyl's theorem. From [**9**, Theorem 3.2], we conclude that $E(T+F)=\Pi(T+F)$. Hence $E_a(T+F)\subset\Pi(T+F)$ and since the opposite inclusion holds for every operator, it then follows that $E_a(T+F)=\Pi(T+F)$. By Theorem 4.8, $T+F$ possesses property (gaw).

If 0 is not an eigenvalue of T, this means that T is injective. Since F commutes with a quasinilpotent operator T, TF is a finite rank quasinilpotent operator. Hence TF is nilpotent. As T is injective, then F is nilpotent. From Theorem 3.6, $T+F$ possesses property (gaw). □

Remark 4.10. The hypothesis of commutativity in Theorem 4.9 is crucial. Indeed, if we consider the Hilbert space $H=\ell^2(\mathbb{N})$, and the operators T and F defined on H by:

$$T(x_1,x_2,x_3,\ldots)=(0,x_1/2,x_2/3,\ldots),\quad F(x_1,x_2,x_3,\ldots)=(0,-x_1/2,0,0,\ldots).$$

Then T is quasinilpotent, F is a finite rank operator which does not commute with T. Moreover, we have $\sigma(T)=\sigma_{BW}(T)=\{0\}$ and $E_a(T)=\emptyset$. Hence T possesses property (gaw). But $T+F$ does not possess property (gaw) because $\sigma(T+F)=\sigma_{BW}(T+F)=\{0\}$ and $E_a(T+F)=\{0\}$.

We conclude this section by some examples:

Examples 4.11. 1. Let R be the unilateral right shift operator defined on the Hilbert space $\ell^2(\mathbb{N})$. It is well known from [**23**, Theorem 3.1] that $\sigma(R)=D(0,1)$ is the closed unit disc in $\mathbb{C}$, $\sigma_a(R)=C(0,1)$ is the unit circle of $\mathbb{C}$ and R has an empty eigenvalues set. Moreover, $\sigma_W(R)=D(0,1)$ and $\Pi_a^0(R)=\emptyset$. Define T on the Banach space $X=\ell^2(\mathbb{N})\oplus\ell^2(\mathbb{N})$ by $T=0\oplus R$. Then $\sigma(T)=D(0,1)$, $N(T)=\ell^2(\mathbb{N})\oplus\{0\}$, $\sigma_a(T)=\{0\}\cup C(0,1)$, $\sigma_W(T)=D(0,1)$, $\sigma_{BW}(T)=D(0,1)$, $\Pi_a(T)=\{0\}$ and $\Pi_a^0(T)=\emptyset$. Hence $\sigma(T)\setminus\sigma_W(T)=\Pi_a^0(T)$ and $\sigma(T)\setminus\sigma_{BW}(T)\neq\Pi_a(T)$. Consequently, T possesses property (ab), but it does not possess property (gab).

2. Let T be the operator defined on the Banach space $X=\ell^2(\mathbb{N})\oplus\ell^2(\mathbb{N})$ by $T(x_1,x_2,x_3,...)=0\oplus(0,x_1/2,x_2/3,x_3/4,...)$. Then $\sigma(T)=\{0\}$, $\sigma_W(T)=\{0\}$, $\sigma_{BW}(T)=\{0\}$, $E_a^0(T)=\emptyset$ and $E_a(T)=\{0\}$. Therefore $\sigma(T)\setminus\sigma_W(T)=E_a^0(T)$ and $\sigma(T)\setminus\sigma_{BW}(T)\neq E_a(T)$. So T possesses property (aw), but it does not possess property (gaw).

3. Let R the unilateral right shift operator defined on the Hilbert space $\ell^2(\mathbb{N})$, then $\sigma(R)=D(0,1)$, $\sigma_{BW}(R)=D(0,1)$ and $E_a(R)=\emptyset$. Therefore $\sigma(R)\setminus\sigma_{BW}(R)=E_a(R)$ and R possesses property (gaw). Moreover, we have $\operatorname{iso}\sigma_a(R)=\emptyset$. Hence if $F\in\mathcal{L}(X)$ is a finite rank operator commuting with R, then $R+F$ possesses property (gaw).

4. Let $T\in\mathcal{L}(X)$ be an injective quasinilpotent operator. Then $\sigma(T)=\sigma_{BW}(T)=\{0\}$ and $E_a(T)=\Pi_a(T)=\emptyset$. Hence T possesses property (gaw). If $F\in\mathcal{L}(X)$ is a finite rank operator commuting with T, then TF is a finite rank quasinilpotent operator, therefore TF is a nilpotent operator. As T is injective, then F is nilpotent. Hence $T+F$ possesses property (gaw).

References

1. Aiena P., *Fredholm and Local Spectral Theory, with Application to Multipliers*, Kluwer Academic, 2004.

2. ———, *Property (w) and perturbations II*, J. Math. Anal. Appl. **342** (2008), 830–837.

3. Amouch M. and Berkani M., *On the property (gw)*, Mediterr. J. Math. **5(3)** (2008), 371–378.

4. Amouch M. and Zguitti H., *On the equivalence of Browder's and generalized Browder's theorem*, Glasgow Math. J., **48** (2006), 179–185.
5. Berkani M., *On a class of quasi-Fredholm operators.* Integr. Equ. and Oper. Theory **34(2)** (1999), 244–249.
6. ______, *B-Weyl spectrum and poles of the resolvent*, J. Math. Anal. Applications, **272(2)** (2002), 596–603.
7. ______, *Index of B-Fredholm operators and generalization of a-Weyl's theorem*, Proc. Amer. Math. Soc., **130** (2002), 1717–1723.
8. Berkani M. and Amouch M., *Preservation of property (gw) under perturbations*, Acta. Sci. Math. (Szeged), **74** (2008), 767–779.
9. Berkani M. and Arroud A., Generalized Weyl's theorem and hyponormal operators, J. Aust. Math. Soc. **76** (2004), 291–302.
10. Berkani M. and Koliha J. J., *Weyl type theorems for bounded linear operators*, Acta Sci. Math. (Szeged) **69** (2003), 359–376.
11. Berkani M. and Sarih M., *On semi B-Fredholm operators*, Glasgow Math. J. **43** (2001), 457–465.
12. Berkani M. and Zariouh H., *New extended Weyl type theorems*, Matematički Vesnik, **62(2)** (2010), 145–154.
13. ______, *Extended Weyl type theorems and perturbations*, Mathematical Proceedings of the Royal Irish Academy, **110** (2010), 73–82.
14. ______, *Generalized a-Weyl's theorem and perturbations*, Journal Functional Analysis, Approximation and Computation, to appear.
15. Barnes B. A., *Riesz points and Weyl's theorem*, Integral Equations and Operator Theory, **34** (1999), 187–196.
16. Coburn L. A., *Weyl's theorem for nonnormal operators*, Michigan Math. J., **13** (1966), 285–288.
17. Djordjević, D. S. and Djordjević, S. V., *On a-Weyl's theorem*, Rev. Roumaine Math. Pures Appl. **44** (1999), 361–369.
18. Grabiner S., *Uniform ascent and descent of bounded operators*, J. Math. Soc. Japan, **34** (1982), 317–337.
19. Heuser H., *Functional Analysis*, John Wiley & Sons Inc, New York, 1982.
20. Lee S. H. and Lee W. Y., *On Weyl's theorem II*, Math. Japon., **43** (1996), 549–553.
21. Mbekhta M. and Müller V., *On the axiomatic theory of the spectrum, II*, Studia Math., **119** (1996), 129–147.
22. Oberai K. K., *On the Weyl spectrum II,* Illinois J. Math., **21** (1977), 84–90.
23. Radjavi H. and Rosenthal P., *Invariant subspaces,* Springer Verlag, Berlin, 1973.
24. Rakočević. V, *Semi-Fredholm operators with finite ascent or descent and perturbations,* Proc. Amer. Math. Soc., **123(12)** (1995), 3823–3825.
25. ______, *Operators obeying a-Weyl's theorem*, Roumaine Math. Pures Appl. **34** (1989), 915–919.
26. Taylor A. E., *Theorems on ascent, descent, nullity and defect of linear operators*, Math. Ann. **163** (1966), 18–49.

M. Berkani, Department of mathematics, Science faculty of Oujda, University Mohammed I, Operator Theory Team, Maroc, *e-mail*: berkanimo@aim.com

H. Zariouh, Department of mathematics, Science faculty of Meknes, University Moulay Ismail, Maroc, *e-mail*: h.zariouh@yahoo.fr