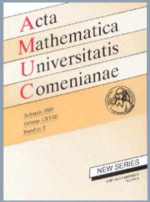




# ON SMALL INJECTIVE, SIMPLE-INJECTIVE AND QUASI-FROBENIUS RINGS

LE VAN THUYET AND TRUONG CONG QUYNH

ABSTRACT. Let $R$ be a ring. A right ideal $I$ of $R$ is called small in $R$ if $I+K \neq R$ for every proper right ideal $K$ of $R$. A ring $R$ is called *right small finitely injective* (briefly, *SF-injective*) (resp., *right small principally injective* (briefly, *SP-injective*) if every homomorphism from a small and finitely generated right ideal (resp., a small and principally right ideal) to $R_R$ can be extended to an endomorphism of $R_R$. The class of right SF-injective and SP-injective rings are broader than that of right small injective rings (in [**15**]). Properties of right SF-injective rings and SP-injective rings are studied and we give some characterizations of a QF-ring via right SF-injectivity with ACC on right annihilators. Furthermore, we answer a question of Chen and Ding.

## 1. INTRODUCTION

Throughout the paper $R$ represents an associative ring with identity $1 \neq 0$ and all modules are unitary $R$-module. We write $M_R$ (resp. ${}_RM$) to indicate that $M$ is a right (resp. left) $R$-module. We use $J$ (resp. $Z_r$, $S_r$) for the Jacobson radical (resp. the right singular ideal, the right socle of $R$) and $E(M_R)$ for the injective hull of $M_R$. If $X$ is a subset of $R$, the right (resp. left) annihilator of $X$ in $R$ is denoted by $r_R(X)$ (resp. $l_R(X)$) or simply $r(X)$ (resp. $l(X)$) if no confusion appears. If

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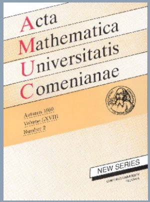
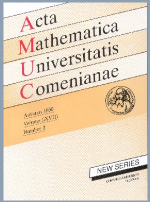



$N$ is a submodule of $M$ (resp. proper submodule) we denote by $N \leq M$ (resp. $N < M$). Moreover, we write $N \leq^{\mathrm{e}} M$, $N \ll M$, $N \leq^{\oplus} M$ and $N \leq^{\max} M$ to indicate that $N$ is an essential submodule, a small submodule, a direct summand and a maximal submodule of $M$, respectively. A module $M$ is called *uniform* if $M \neq 0$ and every non-zero submodule of $M$ is essential in $M$. $M$ is *finite dimensional* (or has *finite rank*) if $E(M)$ is a finite direct sum of indecomposable submodules; or equivalently, if $M$ has an essential submodule which is a finite direct sum of uniform submodules.

A module $M_R$ is called *F-injective* (resp., *P-injective*) if every right homomorphism from a finitely generated (resp., principal) right ideal to $M_R$ can be extended to an $R$-homomorphism from $R_R$ to $M_R$. A ring $R$ is called right F-injective (resp., right P-injective) if $R_R$ is F-injective (resp., P-injective). $R$ is called *right mininjective* if every right $R$-homomorphism from a minimal right ideal to $R$ can be extended to an endomorphism of $R_R$. A ring $R$ is said to be a right PF-ring if the right $R_R$ is an injective cogenerator in the category of right $R$-modules. A ring $R$ is called QF-ring if it is right (or left) Artinian and right (or left) self-injective.

In [**15**], a module $M_R$ is called *small injective* if every homomorphism from a small right ideal to $M_R$ can be extended to an $R$-homomorphism from $R_R$ to $M_R$. A ring $R$ is called right small injective if $R_R$ is small injective. Under small injective condition, Shen and Chen ([**15**]) gave some new characterizations of QF rings and right PF rings. In [**18**], authors showed some characterizations of Jacobson radical $J$ via small injectivity. They proved that $J$ is Noetherian as a right $R$-module if and only if every direct sum of small injective right $R$-modules is small injective if and only if $E^{(\mathbb{N})}$ is small injective for every small injective module $E_R$.

In 1966, Faith proved that $R$ is QF if and only if $R$ is right self-injective and satisfies ACC on right annihilators. Then around 1970, Björk proved that $R$ is QF if and only if $R$ is right F-injective and satisfies ACC on right annihilators. In this paper, we show that $R$ is QF if and only if $R$ is a semiregular and right SF-injective ring with ACC on right annihilators if and only if $R$ is a semilocal and right SF-injective ring with ACC on right annihilators if and only if $R$ is a right SF-injective ring with ACC on right annihilators in which $S_r \leq^{\mathrm{e}} R_R$. We also give some




characterizations of rings whose $R$-homomorphism from a small, finitely generated right ideal to $R$ with a simple image, can be extended to an endomorphism of $R_R$. Furthermore, we prove that if $R$ is a right perfect, right simple-injective and left pseudo-coherent ring, then $R$ is QF. Some known results are obtained as corollaries.

A general background material can be found in [1], [7], [19].

## 2. On SP(SF)-injective rings

**Definition 2.1.** A module $M_R$ is called *small principally injective* (briefly, *SP-injective*) if every homomorphism from a small and principal right ideal to $M_R$ can be extended to an $R$-homomorphism from $R_R$ to $M_R$. A module $M_R$ is called *small finitely injective* (briefly, *SF-injective*) if every homomorphism from a small and finitely generated right ideal to $M_R$ can be extended to an $R$-homomorphism from $R_R$ to $M_R$. A ring $R$ is called right SP-injective (resp., right SF-injective) if $R_R$ is SP-injective (resp., SF-injective).

The following implications are obvious:

$$\begin{array}{ccccc} & \nearrow & \textit{small injective} & \searrow & & \\ \textit{injective} & & & & SF-\textit{injective} & \longrightarrow SP-\textit{injective} \\ & \searrow & F-\textit{injective} & \nearrow & & \end{array}$$

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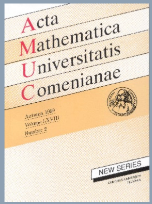



**Lemma 2.2.** *The following conditions are equivalent for a ring $R$:*

(1) *$R$ is right SP-injective.*

(2) *$lr(a) = Ra$ for all $a \in J$.*

(3) *$r(a) \leq r(b)$, where $a \in J$, $b \in R$, implies $Rb \leq Ra$.*

(4) *$l(bR \cap r(a)) = l(b) + Ra$ for all $a \in J$ and $b \in R$.*

(5) *If $\gamma : aR \to R$, $a \in J$, is an $R$-homomorphism, then $\gamma(a) \in Ra$.*

*Proof.* A similar proving to [10, Lemma 5.1]. $\square$

We also have:

**Lemma 2.3.** *A ring $R$ is right SF-injective if and only if it satisfies the following two conditions:*

(1) *$l(T \cap T') = l(T) + l(T')$ for all small, finitely generated right ideals $T$ and $T'$.*

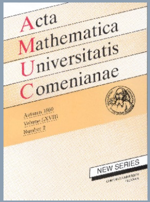
(2) *$R$ is right SP-injective.*

*Proof.* $(\Rightarrow)$ : Assume that $R$ is right SF-injective. If $T$ and $T'$ are small, finitely generated right ideals, then $T + T'$ is a small finitely generated right ideal. Let $b \in l(T \cap T')$ and then we define $\alpha : T + T' \to R$ via $\alpha(t + t') = bt$, for all $t \in T$ and $t' \in T'$, so $\alpha = a.$, for some $a \in R$ by hypothesis. Then $b - a \in l(T)$ and $a \in l(T')$. Hence $b \in l(T) + l(T')$. Thus (1) holds. (2) is clear.

$(\Leftarrow)$ : We can prove it by induction on the number of generators of $T$ and $T'$. $\square$

**Corollary 2.4.** *Let $R$ be a right SP-injective ring such that $l(T \cap T') = l(T) + l(T')$ for all right ideals $T$ and $T'$ of $R$ where $T$ is small, finitely generated. Then every $R$-homomorphism $\varphi : I \to R$ extends to $R \to R$ where $I$ is a small right ideal and the image $\varphi(I)$ is finitely generated.*

**Proposition 2.5.** *A direct product $R = \prod_{i \in I} R_i$ of rings $R_i$ is right SF-injective (resp., right SP-injective) if and only if $R_i$ is right SF-injective (resp., right SP-injective) for each $i \in I$.*



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*Proof.* Assume that $R = \prod_{i\in I} R_i$ is right SF-injective. For each $i \in I$, we take any $a_i \in J(R_i)$ and $b_i \in R_i$ such that $r_{R_i}(a_i) \le r_{R_i}(b_i)$. Let $a = (a_j)_{j\in I}$, $b = (b_j)_{j\in I}$, where $a_j = 0, b_j = 0$, $\forall j \ne i$ and $a_j = a_i, b_j = b_i$ if $j = i$. Then $a \in J(R), b \in R$ and $r_R(a) \le r_R(b)$. So $b \in Ra$ since $R$ is right SP-injective. Therefore $b_i \in R_i a_i$. Thus $R_i$ is right SP-injective. On the other hand, for all small, finitely generated right ideals $T_i$ and $T'_i$ of $R_i$, $\iota_i(T_i)$, $\iota_i(T'_i)$ are small, finitely generated right ideals of $R$, where $\iota_i : R_i \hookrightarrow R$ is the inclusion for each $i \in I$. By hypothesis, $l_R(\iota_i(T_i) \cap \iota_i(T'_i)) = l_R(\iota_i(T_i)) + l_R(\iota_i(T'_i))$. This implies that $l_{R_i}(T_i \cap T'_i) = l_{R_i}(T_i) + l_{R_i}(T'_i)$. Thus $R_i$ is right SF-injective by Lemma 2.3.

Conversely, $R = \prod_{i\in I} R_i$, where $R_i$ is right SF-injective. For each $a = (a_i)_{i\in I} \in J(R)$ and $b = (b_i)_{i\in I} \in R$ such that $r_R(a) \le r_R(b)$, then for each $i \in I$, $a_i \in J(R_i)$ and $r_{R_i}(a_i) \le r_{R_i}(b_i)$. Since $R_i$ is right SF-injective, $b_i \in R_i a_i$. Hence $b \in Ra$. If $T$ and $T'$ are small, finitely generated right ideals of $R$, then we can prove that $l_R(T \cap T') = l_R(T) + l_R(T')$. Thus $R$ is right SF-injective. □

A ring $R$ is called *left minannihilator* if $lr(K) = K$ for every minimal left ideal $K$ of $R$.

**Proposition 2.6.** *Let $R$ be a right SP-injective ring. Then:*

(1) *$R$ is right mininjective and left minannihilator.*

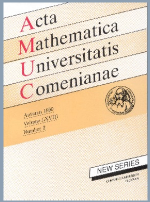
(2) *$J \le Z_r$.*

*Proof.* (1) Since every minimal one-sided ideal of $R$ is either nilpotent or a one-sided direct summand of $R$, each right SP-injective ring is right mininjective and left minannihilator.

(2) If $a \in J$ we will show that $r(a) \le^{\mathrm{e}} R_R$. In fact, let $b \in R$ such that $bR \cap r(a) = 0$. By Lemma 2.2, $R = l(b) + Ra$, so $l(b) = R$ because $a \in J$. Hence $b = 0$. This proves that $a \in Z_r$. □

A ring $R$ is called *right Kasch* if every simple right $R$-module embeds in $R_R$.

**Proposition 2.7.** *Let $R$ be a right Kasch ring. Then:*




(1) *If $R$ is right SP-injective, then:*
   a) *The map $\psi : T \mapsto l(T)$ from the set of maximal right ideals $T$ of $R$ to the set of minimal left ideals of $R$ is a bijection. And the inverse map is given by $K \mapsto r(K)$, where $K$ is a minimal left ideal of $R$.*
   b) *For $k \in R$, $Rk$ is minimal iff $kR$ is minimal, in particular $S_r = S_l$.*

(2) *If $R$ is right SF-injective, then $\mathrm{rl}(I) = I$ for every small, finitely generated right ideal $I$ of $R$. In particular, $R$ is left SP-injective.*

*Proof.* (1) a): By Proposition 2.6 (1) and [10, Theorem 2.32]. For b), if $Rk$ is minimal, then $r(k)$ is maximal by a). This means $kR$ is minimal. Conversely, by [10, Theorem 2.21].

(2): Firstly, we have $J = \mathrm{rl}(J)$ because $R$ is right Kasch. Let $T$ be a right small, finitely generated ideal of $R$. Therefore, $T \leq \mathrm{rl}(T) \leq \mathrm{rl}(J) = J$. If $b \in \mathrm{rl}(T)\backslash T$, take $I$ such that $T \leq I \leq^{\max} (bR + T)$. Since $R$ is right Kasch, we can find a monomorphism $\sigma : (bR + T)/I \to R$, and then define $\gamma : bR+T \to R$ via $\gamma(x) = \sigma(x+I)$. Since $bR+I$ is a small, right finitely generated ideal of $R$ and $R$ is right SF-injective, it follows that $\gamma = c$, where $c \in R$. Hence $cb = \sigma(b + I) \neq 0$ because $b \notin I$. But if $t \in T$, then $ct = \sigma(t + I) = 0$ because $T \leq I$, so $c \in l(I)$. Since $b \in \mathrm{rl}(T)$ this gives $cb = 0$, a contradiction. Thus $T = \mathrm{rl}(T)$. It is clear that $R$ is left SP-injective. □

Recall that a ring $R$ is called semiregular if $R/J$ is von Neumann regular and idempotents can be lifted modulo $J$. Note that if $R$ is semiregular, then for every finitely generated right ideal $I$ of $R$, $R = H \oplus K$, where $H \leq I$ and $I \cap K \ll R$.

Motivated by [15, Lemma 3.1] we have the following result.

**Lemma 2.8.** *If $R$ is a semiregular ring and $I$ is a right ideal of $R$, then the following conditions are equivalent:*

(1) *Every homomorphism from a finitely generated right ideal to $I$ can be extended to an endomorphism of $R_R$.*

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(2) *Every homomorphism from a small, finitely generated right ideal to $I$ can be extended to an endomorphism of $R_R$.*

*Proof.* (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (1): Let $f : K \to I$ be an $R$-homomorphism, where $K$ is a finitely generated right ideal. Since $R$ is semiregular, then $R = H \oplus L$, where $H \leq K$ and $K \cap L \ll R$. Hence $R = K + L$ and $K = H \oplus (K \cap L)$, $K \cap L$ is a small, finitely generated right ideal of $R$. Thus there exists an endomorphism $g$ of $R_R$ such that $g(x) = f(x)$ for all $x \in K \cap L$. We construct a homomorphism $\varphi : R_R \to R_R$ defined by $\varphi(r) = f(k) + g(l)$ for any $r = k + l$, $k \in K$, $l \in L$. Now we show that $\varphi$ is well defined. Indeed, if $k_1 + l_1 = k_2 + l_2$, where $k_i \in K$, $l_i \in L$, $i = 1, 2$, then $k_1 - k_2 = l_1 - l_2 \in K \cap L$. Hence $f(k_1 - k_2) = g(l_1 - l_2)$, which implies that $\varphi(k_1 + l_1) = \varphi(k_2 + l_2)$. Thus $\varphi$ is an endomorphism of $R_R$ such that $\varphi_{|K} = f$. □

Let $I$ be an ideal of $R$. A ring $R$ is called right $I$-semiregular if for every $a \in I$, $aR = eR \oplus T$, where $e^2 = e$ and $T \leq I_R$.

**Corollary 2.9.** *Let $R$ be a right $Z_r$-semiregular ring. Then $R$ is right SF-injective if and only if $R$ is right F-injective.*

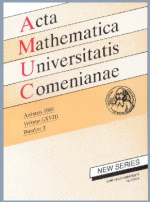
It is well-known if $R$ is semiperfect and right small injective with $S_r \leq^{e} R_R$, then $R$ is right self-injective. This result is proved by Yousif and Zhou (see [20, Theorem 2.11]). In [15, Theorem 3.4], they showed that a semilocal (or semiregular) ring $R$ is right self-injective if and only if $R$ is right small injective. From Lemma 2.8 we also have a similar result.

**Theorem 2.10.** *Let $R$ be a semiregular ring. Then*

(1) *$R$ is right P-injective if and only if $R$ is right SP-injective.*

(2) *$R$ is right F-injective if and only if $R$ is right SF-injective.*

Because a semiperfect ring is semiregular, we have:




**Corollary 2.11.** *Let $R$ be a semiperfect ring. Then*

(1) *$R$ is right P-injective if and only if $R$ is right SP-injective.*

(2) *$R$ is right F-injective if and only if $R$ is right SF-injective.*

Next we obtain some characterizations of QF-ring via right SF-injectivity with ACC on right annihilators. The following theorem extends [15, Theorem 3.8].

**Theorem 2.12.** *For a ring $R$, the following conditions are equivalent:*

(1) *$R$ is QF.*

(2) *$R$ is a semiregular and right SF-injective ring with ACC on right annihilators.*

(3) *$R$ is a semilocal and right SF-injective ring with ACC on right annihilators.*

(4) *$R$ is a right SF-injective ring with ACC on right annihilators in which $S_r \leq^{\mathrm{e}} R_R$.*

*Proof.* It is obvious that $(1) \Rightarrow (2), (3), (4)$.

$(2) \Rightarrow (1)$: By Theorem 2.10, $R$ is right F-injective. Thus $R$ is QF by [3, Theorem 4.1].

$(3) \Rightarrow (1)$: Since $R$ satisfies ACC on right annihilators, $Z_r$ is nilpotent and so $Z_r \leq J$. Therefore, $J = Z_r$ is nilpotent by Proposition 2.6. Hence $R$ is semiprimary.

$(4) \Rightarrow (1)$: By [13, Theorem 2.1] or [14, Lemma 2.11], $R$ is semiprimary. $\square$

**Corollary 2.13.** *Let $R$ be a ring. Then $R$ is QF if and only if $R$ is a semilocal, left and right SP-injective ring with ACC on right annihilators.*

*Remark.* The condition "semilocal" in Theorem 2.12 can not be omitted, since the ring of integers $\mathbb{Z}$ is SP-injective, Noetherian, but $\mathbb{Z}$ is not QF.

The following result extends [11, Theorem 2.2].

**Proposition 2.14.** *If $R$ is right SP-injective and $R/\operatorname{Soc}(R_R)$ has ACC on right annihilators, then $J$ is nilpotent.*

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*Proof.* Here we use a similar argument to the one in [2, Theorem 3]. Suppose that $R/\operatorname{Soc}(R_R)$ has ACC on right annihilators. Let $S = \operatorname{Soc}(R_R)$ and $\overline{R} = R/S$. For any $a_1, a_2, \dots$ in $J$, since

$$r_{\bar{R}}(\bar{a}_1) \leq r_{\bar{R}}(\bar{a}_2\bar{a}_1) \leq \dots,$$

by hypothesis there exists a positive integer $m$ such that

$$r_{\bar{R}}(\bar{a}_m \dots \bar{a}_2\bar{a}_1) = r_{\bar{R}}(\bar{a}_{m+k} \dots \bar{a}_2\bar{a}_1)$$

for $k = 0, 1, 2, \dots$. Now for any positive integer $n$, since $a_{n+1}a_n \dots a_1 \in J \leq Z_r$, $r(a_{n+1}a_n \dots a_1) \leq^{\mathrm{e}} R_R$. Hence $S \leq r(a_{n+1}a_n \dots a_1)$. We claim that

$$r_{\bar{R}}(\bar{a}_n \dots \bar{a}_2\bar{a}_1) \leq r(a_{n+1}a_n \dots a_1)/S \leq r_{\bar{R}}(\bar{a}_{n+1} \dots \bar{a}_2\bar{a}_1).$$

In fact, assume $b + S \in r_{\bar{R}}(\bar{a}_n \dots \bar{a}_2\bar{a}_1)$. Then we have $a_n \dots a_1 b \in S$. But since $S \leq r(a_{n+1})$, we get $a_{n+1}a_n \dots a_1 b = 0$. Thus $b \in r(a_{n+1}a_n \dots a_1)$, and so $b + S \in r(a_{n+1}a_n \dots a_1)/S$. Now the other inclusion $r(a_{n+1}a_n \dots a_1)/S \leq r_{\bar{R}}(\bar{a}_{n+1} \dots \bar{a}_2\bar{a}_1)$ is obvious.

By this fact, it follows that

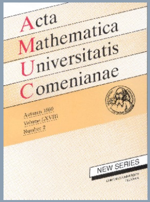
$$r(a_{m+1}a_m \dots a_1)/S = r(a_{m+2}a_{m+1} \dots a_1)/S$$

because $r_{\bar{R}}(\bar{a}_m \dots \bar{a}_2\bar{a}_1) = r_{\bar{R}}(\bar{a}_{m+2} \dots \bar{a}_2\bar{a}_1)$. Therefore

$$r(a_{m+1}a_m \dots a_1) = r(a_{m+2}a_{m+1}a_m \dots a_1),$$

and hence $(a_{m+1}a_m \dots a_1)R \cap r(a_{m+2}) = 0$. But $r(a_{m+2})$ is an essential right ideal of $R$, and so $a_{m+1}a_m \dots a_1 = 0$. Hence $J$ is right T-nilpotent and the ideal $(J + S)/S$ of the ring $\bar{R} = R/S$ is also right T-nilpotent. By [1, Proposition 29.1], $(J + S)/S$ is nilpotent, and so there is a positive integer $t$ such that $J^t \leq S$. Hence $J^{t+1} \leq SJ$. Thus $J$ is nilpotent. □

**Theorem 2.15.** *If $R$ is a semilocal and right SF-injective ring such that $R/S_r$ is right Goldie, then $R$ is QF.*




*Proof.* By Proposition 2.14, $J$ is nilpotent, and hence $R$ is semiprimary. Hence $R$ is right F-injective by Theorem 2.10. This implies that $R$ is right GPF (i.e., $R$ is semiperfect, right P-injective with $S_r \leq^{e} R_R$) and so $R$ is right Kasch by [11, Corollary 2.3]. Therefore $R$ is left P-injective by [3, Proposition 4.1]. Thus $R$ is QF by [10, Theorem 3.38]. □

**Corollary 2.16.** *If $R$ is a semilocal and right SF-injective ring satisfying ACC on essential right ideals, then $R$ is QF.*

Now we consider rings whose small and finitely generated right ideals are projective. We have the following result.

**Theorem 2.17.** *For a ring $R$ the following conditions are equivalent:*

(1) *Every small and finitely generated right ideal of $R$ is projective.*

(2) *Every quotient module of a SF-injective module is SF-injective.*

(3) *Every quotient module of a F-injective module is SF-injective.*

(4) *Every quotient module of a small injective module is SF-injective.*

(5) *Every quotient module of an injective module is SF-injective.*

*Proof.* (2) ⇒ (3) ⇒ (5) and (2) ⇒ (4) ⇒ (5) are obvious.

(1) ⇒ (2): Assume that $E_R$ is SF-injective and $\pi : E \to B$ is an epimorphism. Let $f : I \to B$ be an $R$-homomorphism, where $I$ is a small and finitely generated right ideal of $R$.

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \overset{\iota}{\hookrightarrow} & R \\ & & \overset{f}{\to}\downarrow & & \\ E & \overset{\pi}{\longrightarrow} & B & \longrightarrow & 0 \end{array}$$

where $\iota$ is the inclusion.

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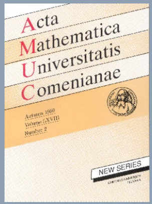



By (1), $I$ is projective. Therefore there exists an $R$-homomorphism $h : I \to E$ such that $\pi h = f$. Now since $E$ is SF-injective, there is an $R$-homomorphism $h' : R \to E$ such that $h'\iota = h$. Let $h'' = \pi h' : R \to B$, then $h''\iota = f$. This means $B_R$ is SF-injective.

(5) ⇒ (1): For every small and finitely generated right ideal $I$ of $R$, we consider the epimorphism $h : A \to B$ and $R$-homomorphism $\alpha : I \to B$.

Since $B = h(A) \overset{\psi}{\cong} A/\operatorname{Ker} h \overset{\iota_1}{\hookrightarrow} E(A)/\operatorname{Ker} h$, where $\iota_1$ is the inclusion and $\psi(h(a)) = a + \operatorname{Ker} h$, for all $a \in A$. Then let $j = \iota_1\psi$. We consider the following diagram:

$$\begin{array}{ccccc} & & I & \overset{\iota}{\hookrightarrow} & R \\ & \overset{\varphi}{\swarrow} & \overset{\alpha}{\to}\downarrow & & \\ E & \overset{h}{\longrightarrow} & B & \longrightarrow & 0 \\ & & \overset{j}{\to}\downarrow & & \\ E(A) & \overset{p}{\longrightarrow} & E(A)/\operatorname{Ker} h & \longrightarrow & 0 \end{array}$$

where $\iota$ is the inclusion and $p$ is the natural epimorphism.

By (5), $E(A)/\operatorname{Ker} h$ is SF-injective and then there exists an $R$-homomorphism $\alpha' : R \to E(A)/\operatorname{Ker} h$ such that $\alpha'\iota = j\alpha$. Since $R_R$ is projective, there is an $R$-homomorphism $\alpha'' : R \to E(A)$ such that $p\alpha'' = \alpha'$. Let $h' = \alpha''\iota : I \to E(A)$. It is easy to see that $h'(I) \le A$, so there exists an $R$-homomorphism $\varphi : I \to A$ such that $\varphi(x) = h'(x)$, for all $x \in I$.

Now we claim that $h\varphi = \alpha$. In fact, for each $x \in I$ we have

$$j(\alpha(x)) = \alpha'(\iota(x)) = \alpha'(x) = p(\alpha''(x)) = p(h'(x)) = p(\varphi(x)).$$

Since $\alpha$ is the epimorphism, $\alpha(x) = h(a)$ for some $a \in A$. Therefore $j(\alpha(x)) = j(h(a)) = a + \operatorname{Ker} h$, and so $a + \operatorname{Ker} h = \varphi(x) + \operatorname{Ker} h$, $h(a - \varphi(x)) = 0$. Hence $h\varphi(x) = h(a) = \alpha(x)$. Thus $I$ is projective. □

**Example 2.18.** i) Let $R = F[x_1, x_2, \ldots]$, where $F$ is a field and $x_i$ are commuting indeterminants satisfying the relations: $x_i^3 = 0$ for all $i$, $x_i x_j = 0$ for all $i \neq j$, and $x_i^2 = x_j^2$ for all $i$ and $j$. Then $R$ is a commutative, semiprimary F-injective ring. But $R$ is not a self-injective ring (see [**10**, Example 5.45]). Thus $R$ is SF-injective, but $R$ is not a small injective ring. Because if $R$ is small injective, then $R$ is self-injective by [**15**, Theorem 3.4], a contradiction.

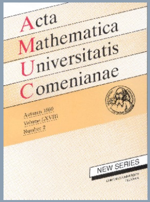
ii) Let $F$ be a field and assume that $a \mapsto \bar{a}$ is an isomorphism $F \to \overline{F} \subseteq F$, where the subfield $\overline{F} \neq F$. Let $R$ denote the left vector space on basis $\{1, t\}$, and make $R$ into an $F$-algebra by defining $t^2 = 0$ and $ta = \bar{a}t$ for all $a \in F$ (see [**10**, Example 2.5]). Then $R$ is a right SP-injective (since $R$ is right P-injective) and semiprimary ring but not a right SF-injective ring. If $R$ is a right SF-injective ring, then $R$ is right F-injective by Theorem 2.10. This is a contradiction by [**10**, Example 5.22]. Moreover, $R$ is not left SP-injective since $R$ is not left mininjective.

iii) The ring of integers $\mathbb{Z}$ is a commutative ring with $J = 0$. So $R$ is small injective, but $R$ is not P-injective.

## 3. On simple-FJ-injective rings

**Definition 3.1.** A ring $R$ is called right simple-FJ-injective if every right $R$-homomorphism from a small, finitely generated right ideal to $R$ with a simple image, can be extended to an endomorphism of $R_R$.

We have the implications $simple-injective \Rightarrow simple-J\text{-}injective \Rightarrow simple-FJ\text{-}injective$. But the converses in general are not true. By Example 2.18(i),$R$ is commutative, semiprimary and simple-FJ-injective. But $R$ is not simple-J-injective. In fact, if $R$ is simple-J-injective then $R$ is simple-injective by [**15**, Corollary 3.6]. Hence $R$ is self-injective by [**10**, Theorem 6.47]. This is a contradiction.




**Lemma 3.2.** *If $R$ is right simple-FJ-injective, then $R$ is right mininjective and a left minannihilator.*

*Proof.* We can prove it as in Proposition 2.6. □

**Lemma 3.3.** *A ring $R$ is right simple-FJ-injective a ring if and only if every $R$-homomorphism $f : I \to R$ extends to $R_R \to R_R$, where $I$ is a small, finitely generated right ideal and $f(I)$ is finitely generated, semisimple.*

*Proof.* Write $f(I) = \bigoplus_{i=1}^{n} S_i$ where $S_i$ is a simple right ideal. Let $\pi_i : \bigoplus_{i=1}^{n} S_i \to S_i$ be the projection for each $i$. Since $R$ is right simple-FJ-injective, $\pi_i f = c_i.$, for some $c_i \in R$ and for each $i$. Thus $f = c.$, with $c = c_1 + \ldots + c_n$. □

**Proposition 3.4.** *Let $R$ be a right simple-FJ-injective and right Kasch ring. Then*

(1) $\mathrm{rl}(I) = I$ *for every small, finitely generated right ideal $I$ of $R$.*

(2) $S_r = S_l$.

*Proof.* By Proposition 2.7. □

In [20], a ring $R$ is called right $(I - K) - m$-injective if for any $m$-generated right ideal $U \leq I$ and any $R$-homomorphism $f : U_R \to K_R$, $f = c.$, for some $c \in R$, where $I, K$ are two right ideals of $R$ and $m \geq 1$.

**Lemma 3.5** ([20], Lemma 2.5)**.** *If $R$ is a right $(J, S_r) - 1$-injective, right Kasch and semiregular ring, then $l(J)$ is an essential left ideal of $_RR$.*

**Lemma 3.6.** *Let $R$ be a right simple-FJ-injective and semiregular ring. Then every $R$-homomorphism $f : K \to R$ extends to $R_R \to R_R$ where $K$ is a finitely generated right ideal and $f(K)$ is simple.*

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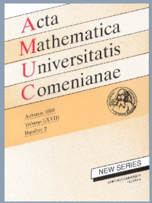



*Proof.* Let $f : K \to I$ be an $R$-homomorphism, where $K$ is a finitely generated right ideal and $f(K)$ is simple. Since $R$ is semiregular, then $K = eR \oplus L$, where $e^2 = e \in R$ and $L \leq J$. So $L$ is a small, finitely generated right ideal of $R$. It is easy to see that $K = eR \oplus (1-e)L$. Therefore $(1-e)L$ is a small, finitely generated right ideal of $R$. By hypothesis, there exists an endomorphism $g$ of $R_R$ such that $g(x) = f(x)$ for all $x \in (1-e)L$. We construct a homomorphism $\varphi : R_R \to R_R$ defined by $\varphi(x) = f(ex) + g((1-e)x)$ for any $x \in R$. Then $\varphi_{|K} = f$. □

**Proposition 3.7.** *Let $R$ be a right simple-FJ-injective ring. Then*

(1) *If $R$ is semiregular and $e$ is a local idempotent of $R$, then* $\mathrm{Soc}(eR)$ *is either* $0$ *or simple and essential in $eR_R$.*

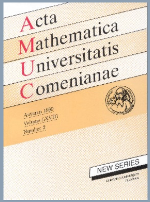
(2) *If $R$ is semiperfect, then the following conditions are equivalent*
   a) $\mathrm{Soc}(eR) \neq 0$ *for each local idempotent $e$.*
   b) *$S_r$ is finitely generated and essential in $R_R$.*

*Proof.* (1) Suppose that $\mathrm{Soc}(eR) \neq 0$ and let $aR$ be a simple right ideal of $eR$. If $0 \neq b \in eR$ such that $aR \cap bR = 0$, then we construct an $R$-homomorphism $\gamma : aR \oplus bR \to eR$ by $\gamma(ax + by) = ax$, for all $x, y \in R$. Therefore $\mathrm{Im}\,\gamma = aR$ is simple. By Lemma 3.6, $\gamma = c.$ for some $c \in R$. Let $c' = ece \in eRe$. So $(e - c')a = ea - eca = 0$. On the other hand, $\mathrm{End}(eR_R) \cong eRe$ is local. It implies that $c'$ is invertible in $eRe$, but $c'b = eceb = ecb = 0$ and so $b = 0$, which is a contradiction. Hence $aR \cap bR \neq 0$, $aR \leq bR$ since $aR$ is simple. Thus $\mathrm{Soc}(eR)$ is simple and essential in $eR_R$.

(2) If $1 = e_1 + \ldots + e_n$, where the $e_i$ are orthogonal local idempotents, then $S_r = \bigoplus_{i=1}^n \mathrm{Soc}(e_iR)$ and $a) \Rightarrow b)$ follows from (1). The converse is clear. □

**Proposition 3.8.** *Let $R$ be a semiperfect, right simple-FJ-injective ring with* $\mathrm{Soc}(eR) \neq 0$ *for each local idempotent $e \in R$. Then:*

(1) $\mathrm{rl}(I) = I$ *for every finitely generated right ideal $I$ of $R$, so $R$ is left P-injective.*




(2) *$R$ is left and right Kasch.*

(3) *$S_r = S_l = r(J) = l(J)$ is essential in ${}_RR$ and in $R_R$.*

(4) *$J = Z_r = Z_l = r(S) = l(S)$, with $S_r = S_l = S$.*

(5) *$R$ is left and right finitely cogenerated.*

*Proof.* (2): by [12, Theorem 3.7] and (1) by Proposition 3.4 and [20, Lemma 1.4].
(3): $S_r = S_l = S$ is essential in ${}_RR$ and in $R_R$ by Proposition 3.4, Lemma 3.5 and Proposition 3.7. $S = r(J) = l(J)$ because $R$ is left and right Kasch.
(4): follows from (2) and (3).
(5): follows from Proposition 3.7 and [10, Theorem 5.31]. □

*Remark.* There exists a semiprimary and right simple-FJ-injective ring, but it can not be right simple-injective. On the other hand, there is a ring $R$ that is right simple-FJ-injective but not right SP-injective (see [20, Example 1.7]).

From the above proposition, we have the following result.

**Proposition 3.9.** *If $R$ is a right simple-FJ-injective ring with ACC on right annihilators in which $S_r \leq^{\mathrm{e}} R_R$, then $R$ is QF.*

*Proof.* By [13, Theorem 2.1] or [14, Lemma 2.11], $R$ is semiprimary. Hence $R$ is left and right mininjective by Proposition 3.8. Thus $R$ is QF. □

**Corollary 3.10** ([14], Theorem 2.15)**.** *If $R$ is a right simple-injective ring with ACC on right annihilators in which $S_r \leq^{\mathrm{e}} R_R$ then $R$ is QF.*

Recall that a ring $R$ is called right pseudo-coherent if $r(S)$ is finitely generated for every finite subset $S$ of $R$ (see [3]). Chen and Ding [5] proved that if $R$ is a left perfect, right simple-injective and right (or left) pseudo-coherent ring, then $R$ is QF. They gave a question: If $R$ is a right simple-injective ring which is also right perfect and right (or left) pseudo-coherent, is $R$ a QF ring? The following results are motivated by this question.

Firstly, we have the following result

**Lemma 3.11** (Osofsky's Lemma). *If $R$ is a left perfect ring in which $J/J^2$ is right finitely generated, then $R$ is right Artinian.*

**Theorem 3.12.** *Assume that $R$ is left perfect, right simple-FJ-injective. If $R$ is right (or left) pseudo-coherent ring, then $R$ is QF.*

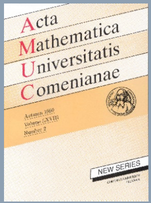
*Proof.* Since $R$ is left perfect, $\mathrm{Soc}(eR) \neq 0$ for each local idempotent $e \in R$. Thus by Proposition 3.8, $J = r(S) = l(S)$ with $S = S_r = S_l = r(J) = l(J)$ is a finitely generated left and right ideal. Hence by hypothesis, $R$ is left (or right) pseudo-coherent, and so $J$ is a finitely generated left (or right) ideal. If $J$ is a finitely generated right $R$-module, then $J/J^2$ is too. Consequently, $R$ is right Artinian by Lemma 3.11. If $J$ is a finitely generated left $R$-module, then $J$ is nilpotent by [10, Lemma 5.64], and so $R$ is semiprimary. Hence $R$ is left Artinian by Lemma 3.11. Thus $R$ is QF. □

**Corollary 3.13** ([5], Theorem 2.6). *Assume that $R$ is left perfect, right simple-injective. If $R$ a is right (or left) pseudo-coherent ring, then $R$ is QF.*

We consider a ring which is right simple-FJ-injective and left pseudo-coherent.

**Theorem 3.14.** *If $R$ is a right perfect, right simple-FJ-injective and left pseudo- -coherent ring then $R$ is QF.*

*Proof.* Since $R$ is right perfect and left pseudo-coherent, $R$ satisfies DCC on finitely generated left ideals. Hence if $A \subseteq R$, $l(A) = l(A_0)$ for some finite subset $A_0$ of $A$. It follows that $R$ satisfies DCC on left annihilators, and hence $R$ has ACC on right annihilators. Therefore $R$ is semiprimary by [6, Proposition 1]. Thus $R$ is QF by Theorem 3.12. □




**Corollary 3.15.** *If $R$ is a right perfect, right simple-injective and left pseudo- -coherent ring, then $R$ is QF.*

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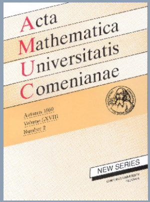
1. Anderson F. W. and Fuller K. R., *Rings and categories of modules*, Springer-Verlag, New York, 1974.
2. Armendariz E. P. and Park J. K., *Self-injective rings with restricted chain conditions*, Arch. Math. **58** (1992), 24–33.
3. Björk J. E., *Rings satisfying certain chain conditions*, J. Reine Angew. Math. **245** (1970), 63–73.
4. Chen J. and Ding N., *On general principally injective rings*, Comm. Algebra, **27(5)** (1999), 2097–2116.
5. ______, *On generalization of injective rings*, In International Symposium on Ring Theory, South Korea, June 28 – July 3, 1999.
6. Faith C., *Ring with ascending condition on annihilators*, Nagoya Math. J., **27** (1966), 179–191.
7. ______, *Algebra II: Ring Theory*, Springer-Verlag, Berlin, 1976.
8. Faith C., and Huynh D. V., *When self-injective rings are QF: A report on a problem*, J. of Algebra and It's Appl. **1(1)** (2002), 75–105.
9. Goodeal K. R. and Warfield R. B., *An introduction to noncommutative Noetherian rings*, Cambridge University Press, 1989.
10. Nicholson W. K. and Yousif M. F., *Quasi-Frobenius Rings*, Cambridge University Press, 2003.
11. ______, *Principally injective rings*, J. Algebra **174** (1995), 77–93.
12. ______, *Mininjective rings*, J. Algebra **187** (1997), 548–578.
13. Quynh T. C. and Thuyet L. V., *On rings with ACC on annihilators and having essential socles*, Proc. of Int. of Math. and Appl. (ICMA, Bangkok 2005)(Organized by Yongwimon Lenbury and Nguyen Van Sanh), East-West J. Math. Spec Vol. (2006), 227–234.
14. Shen L. and Chen J., *A note on Quasi-Frobenius rings*, arXiv: Math., RA/0504068 v.1 (2005).
15. ______, *Small injective rings*, arXiv: Math., RA/0505445 v.21 (2005).




**16.** Thoang L. D. and Thuyet L. V., *On semiperfect mininjective rings with essential socles*, Southeast Asian Bull. Math. **30** (2006), 555–560.
**17.** Thuyet L. V., *On continuous rings with chain conditions*, Vietnam J. Math. **19(1)** (1991), 49–59.
**18.** Thuyet L. V. and Quynh T. C., *On small injective ring and modules*, to appear in Journal of Algebra and it's Application.
**19.** Wisbauer R., *Foundations of module and ring theory*, Gordon and Breach, 1991.
**20.** Yousif M. F. and Zhou Y. Q., *FP-injective, simple-injective and quasi-Frobenius rings*, Comm. Algebra, **32(6)** (2004), 2273–2285.

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