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ON THE COMPUTATION OF MULTIPLICITY BY THE REDUCTION OF DIMENSION

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Abstrakt. In this short note we describe one method for the computation of the Samuel multiplicity of the polynomial ideals and prove a formula for the multiplicity of the ideal $(\alpha_i x_i^{a_i} - \beta_{i+1} x_{i+1}^{b_{i+1}};\ i = 1, \dots, n) \cdot R$ in R (with the convention $x_{n+1} = x_1$, $\beta_{n+1} = \beta_1$, $b_{n+1} = b_1$), where $(R, m) = k\,[x_1, x_2, \dots, x_n]_{(x_1, x_2, \dots, x_n)}$ is a local polynomial ring over an algebraic closed field k.

Let (A, m) be a Noetherian local ring with $\dim A = d$. For any m-primary ideal Q in A the A-module A/Q^n is of the finite length for all $n \in \mathbb{N}$. For large n this length function becomes a polynomial (Hilbert-Samuel polynomial) which can be written as

$$L(A/Q^n) = e_0(Q, A)\frac{n^d}{d!} + \text{terms of lower degree.}$$

The coefficient $e_0(Q, A)$ is called the Samuel multiplicity (or simply) multiplicity of Q in A. We present one method how to count this multiplicity when Q is generated by a system of parameters in a local polynomial ring.

Let $P = k[x_1, \dots, x_n]$ be a polynomial ring over an algebraic closed field k. Let $f_1, \dots, f_{n-r}$ denote a system of polynomials in P such that algebraic variety $V(f_1, \dots, f_{n-r})$ is of dimension r, $0 \le r < n$. We say that the set of polynomials $\{u_i(s_1, \dots, s_r) \in k[s_1, \dots, s_r],\ i = 1, \dots, n\}$ represents the polynomial parametrization of W if the image of the map

$$k^r \to E^n$$

given by

$$(a_1, a_2, \dots, a_r) \longmapsto (u_1(a_1, \dots, a_r), \dots, u_n(a_1, \dots, a_r))$$

is $V(f_1, \dots, f_{n-r})$.

Now we can formulate the main theorem of this note.

Theorem 1. *Let $P = k[x_1, \dots, x_n]$ be a polynomial ring over an algebraic closed field k and $(R, m) = k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$ the localization of P with respect to maximal ideal $(x_1, \dots, x_n) \cdot P$. Let $f_1, \dots, f_n$ denote a system of polynomials*

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in P such that $(f_1,\dots,f_n)\cdot R$ is an m-primary ideal in R. Let W be an algebraic variety in E^n defined by the equations $f_1(x_1,\dots,x_n)=\ldots=f_{n-r}(x_1,\dots,x_n)=0$ with $\dim W=r$ and the polynomial parametrization $\{u_i(s_1,\dots,s_r)\in k[s_1,\dots,s_r],\ i=1,\dots,n\}$. Suppose that the polynomial ring $k[s_1,\dots,s_r]$ is a finite $k[u_1,\dots,u_n]$-module. Let d denote the dimension of the field $k(s_1,\dots,s_r)$ as a vector space over the field $k(u_1,\dots,u_n)$. With this hypothesis we have

$$e_0((f_1,\dots,f_n)\cdot R,R)\cdot d=e_0((F_{n-r+1},\dots,F_n)\cdot S,S)$$

where $F_i=f_i(u_1(s_1,\dots,s_r),\dots,u_n(s_1,\dots,s_r))$ for $i=n-r+1,\dots,n$ and $S=k[s_1,\dots,s_r]_{(s_1,\dots,s_r)}$.

Proof. From our construction we have the monomorphism

$$k[x_1,\dots,x_n]/(f_1,\dots,f_{n-r})\cdot k[x_1,\dots,x_n]\cong k[u_1\dots,u_n]\hookrightarrow k[s_1,\dots,s_r]$$

and hence the local monomorphism

$$R/(f_1,\dots,f_{n-r})\cdot R\cong k[u_1,\dots,u_n]_{(u_1,\dots,u_n)}\hookrightarrow k[s_1\dots,s_r]_{(s_1,\dots,s_r)}.$$

As the module $k[s_1,\dots,s_r]_{(s_1,\dots,s_r)}$ is finite over the ring $k[u_1,\dots,u_n]_{(u_1,\dots,u_n)}$, the additivity formula applied to the multiplicity $e_0((f_1,\dots,f_n)\cdot R,R)$ provides the equality

$$\begin{aligned}&e_0((f_1,\dots,f_n)\cdot R/(f_1,\dots,f_{n-r})\cdot R,R/(f_1,\dots,f_{n-r})\cdot R)\cdot d\\&=e_0((F_{n-r+1},\dots,F_n)\cdot S,S)\end{aligned}$$

(cf. [**3**, Theorem 14.7]). As the ideal $(f_1,\dots,f_n)\cdot R$ is generated by a system of parameters, we have

$$e_0((f_1,\dots,f_n)\cdot R,R)\cdot d=e_0((F_{n-r+1},\dots,F_n)\cdot S,S),$$

(cf. [**4**, Chap.7, Theorem 18]) which completes the proof. □

Let us shift to the ideal $(\alpha_i x_i^{a_i}-\beta_{i+1}x_{i+1}^{b_{i+1}};i=1,\dots,n)\cdot R$ in the local polynomial ring $(R,m)=k[x_1,x_2,\dots,x_n]_{(x_1,x_2,\dots,x_n)}$. As the mentioned ideal satisfies the condition of the above formulated Theorem 1, we can prove the formula for its multiplicity. We start with $n=2$.

Lemma 2. *Let $(\alpha x^a-\beta y^b,\gamma y^c-\delta x^d)\cdot A$ be a parameter ideal in the local ring $(A,m)=k[x,y]_{(x,y)}$ ($a,b,c,d\in\mathbb{N};\ \alpha,\beta,\gamma,\delta\in k$). Then*

$$e_0\ ((\alpha x^a-\beta y^b,\gamma y^c-\delta x^d)\cdot A,A)=\min\{ac,bd\}.$$

Proof. After dividing the polynomials of the basis by α resp. γ, we can assume that $\alpha=\gamma=1$. If $\gcd(a,b)=r$, $a=\overline{a}r$, $b=\overline{b}r$, then

$$x^a-\beta y^b=\prod_{i=1}^{r}(x^{\overline{a}}-\xi_i y^{\overline{b}})$$

for certain $\xi_i\in k$ (k being algebraically closed). As

$$e_0\ ((x^a-\beta y^b,y^c-\delta x^d)\cdot A,A)=\sum_{i=1}^{r}e_0\ ((x^{\overline{a}}-\xi_i y^{\overline{b}},y^c-\delta x^d)\cdot A,A)$$

(see [**4**, Chap. VII, Theorem 7]), we can assume that a, b are relatively prime with $k \cdot a - l \cdot b = 1$ for certain $k, l \in N$. Then the equations

$$\begin{aligned} x &= \beta^k s^b \\ y &= \beta^l s^a \end{aligned}$$

represent the polynomial parametrization of the curve V given by $x^a - \beta y^b = 0$. In addition, $k(\beta^k s^b, \beta^l s^a) = k(s)$. Now Theorem 1 provides the following equalities

$$\begin{aligned} e_0\ \left((x^a - \beta y^b, y^c - \delta x^d) \cdot A, A\right) &= e_0((\beta^{l \cdot c} s^{a \cdot c} - \delta \beta^{k \cdot d} s^{b \cdot d}) \cdot k\,[s]_{(s)}\,, k\,[s]_{(s)}) \\ &= \min\{ac, bd\} \end{aligned}$$

which completes the proof. □

And now we formulate the general result.

Theorem 3. *Let* $I = (\alpha_i x_i^{a_i} - \beta_{i+1} x_{i+1}^{b_{i+1}}; i = 1, \ldots, n) \cdot R$ *be a parameter ideal in* R *(with the convention* $x_{n+1} = x_1$, $\beta_{n+1} = \beta_1$, $b_{n+1} = b_1$*), where* $(R, m) = k\,[x_1, x_2, \ldots, x_n]_{(x_1, x_2, \ldots, x_n)}$ *is a local polynomial ring over an algebraic closed field* k*. Then*

$$e_0(I, R) = \min\left\{\prod_{i=1}^n a_i, \prod_{i=1}^n b_i\right\}.$$

Proof. We use induction on $n \geq 2$. For $n = 2$ the assertion is the above Lemma 2. Let now

$$I = (\alpha_i x_i^{a_i} - \beta_{i+1} x_{i+1}^{b_{i+1}}; i = 1, \ldots, n) \cdot k\,[x_1, x_2, \ldots, x_n]_{(x_1, x_2, \ldots, x_n)}\,, \qquad n > 2.$$

As in Lemma 2 we can assume that the first polynomial is of the form $x_1^{a_1} - \beta_2 x_2^{b_2}$ with a_1, b_1 being relatively prime with $k \cdot a_1 - l \cdot b_2 = 1$ for certain $k, l \in \mathbb{N}$. So the polynomial parametrization of the hypersurface $V(x_1^{a_1} - \beta_2 x_2^{b_2})$ in $\mathbb{E}^n$ has the following form

$$\begin{aligned} x_1 &= \beta_2^k s_1^{b_2} \\ x_2 &= \beta_2^l s_1^{a_1} \\ x_i &= s_{i-1} \qquad \text{for } i = 3, \ldots, n. \end{aligned}$$

As $k(\beta_2^k s_1^{b_2}, \beta_2^l s_1^{a_1}, s_2, \ldots, s_{n-1}) = k(s_1, \ldots, s_{n-1})$, the induction hypothesis and the Theorem 1 imply

$$\begin{aligned} e_0(I, R) = e_0((&\alpha_2 \beta_2^{l \cdot a_2} s_1^{a_1 \cdot a_2} - \beta_3 s_2^{b_3}, \alpha_3 s_2^{a_3} - \beta_4 s_3^{b_4}, \ldots\ldots, \alpha_{n-1} s_{n-2}^{a_{n-1}} - \beta_n s_{n-1}^{b_n}, \\ &\ldots, \alpha_n s_{n-1}^{a_n} - \beta_1 \beta_2^{k \cdot b_1} s_1^{b_2 \cdot b_1}) \cdot k\,[s_1 \ldots, s_{n-1}]_{(s_1, \ldots, s_{n-1})}\,, k\,[s_1 \ldots, s_{n-1}]_{(s_1, \ldots, s_{n-1})}) \\ = \min\{&a_1 \cdot a_2 \ldots a_n, b_1 \cdot b_2 \ldots b_n\}, \end{aligned}$$

which completes the proof. □

Finally, we illustrate the previous results by an example.

Example 4. Let $I = (x^3 - y^4, x^5 - z^7, y^6 - z^8) \cdot C\,[x, y, z]\,_{(x,y,z)}$ be a parameter ideal in the ring $C\,[x, y, z]_{(x,y,z)}$. As $\gcd(3, 4) = \gcd(5, 7) = 1$, we can take the curve W given by the equations

$$x^3 - y^4 = x^5 - z^7 = 0$$

and the parametrization

$$x = s^{28}$$
$$y = s^{21}$$
$$z = s^{20}.$$

Then the Theorem 1 applied to our ideal I and the variety W provides the equality

$$e_0(x^3 - y^4, x^5 - z^7, y^6 - z^8) \cdot C\,[x, y, z]_{(x,y,z)}\,, C\,[x, y, z]_{(x,y,z)}$$
$$= e_0((s^{6\cdot 21} - s^{8\cdot 20}) \cdot C\,[s]_{(s)}\,, C\,[s]_{(s)}) = 126.$$

On the other hand, we can take the polynomial

$$y^6 - z^8 = (y^3)^2 - (z^4)^2 = (y^3 - z^4)(y^3 + z^4)$$

and the surface V given by $y^3 - z^4 = 0$, resp. parametrically

$$x = s$$
$$y = t^4$$
$$z = t^3$$

and compute

$$e_0((x^3 - y^4, x^5 - z^7, y^6 - z^8) \cdot C\,[x, y, z]_{(x,y,z)}\,, C\,[x, y, z]_{(x,y,z)})$$
$$= 2 \cdot e_0((y^3 - z^4, x^3 - y^4, x^5 - z^7) \cdot C\,[x, y, z]_{(x,y,z)}\,, C\,[x, y, z]_{(x,y,z)})$$
$$= 2 \cdot e_0((s^3 - t^{16}, s^5 - t^{21}) \cdot C\,[s, t]_{(s,t)}\,, C\,[s, t]_{(s,t)}) = 2 \cdot \min\{3 \cdot 21, 5 \cdot 16\} = 126.$$

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