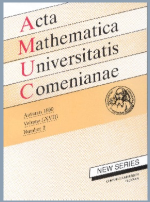

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A NEW EXTENSION OF HILBERT'S INEQUALITY FOR MULTIFUNCTIONS WITH BEST CONSTANT FACTORS

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ABSTRACT. The aim of this paper is to establish a new extension of Hilbert's inequality and Hardy-Hilbert's inequality for multifunctions with best constant factors. Also, we present some applications for Hilbert's inequality which give new integral inequalities.

1. Introduction

Hilbert's inequality has a great interest in analysis and its applications (see [10], [11]). The original Hilbert's inequality can be stated as follows

If $f(x),\ g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x)\mathrm{d}x < \infty$ and $0 < \int_0^\infty g^2(x)\mathrm{d}x < \infty$, then (see [6])

$$\int_0^\infty \int_0^\infty \frac{f(x)\ g(y)}{x+y}\mathrm{d}x\,\mathrm{d}y < \pi \left\{ \int_0^\infty f^2(x)\mathrm{d}x \int_0^\infty g^2(x)\mathrm{d}x \right\}^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible. This inequality was extended by Hardy-Riesz as (see [5]):

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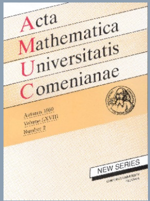


If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x)$, $g(x) \geq 0$, such that $0 < \int_0^\infty f^p(x)\mathrm{d}x < \infty$ and $0 < \int_0^\infty g^q(x)\mathrm{d}x < \infty$, then

$$(2) \qquad \int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{x+y}\mathrm{d}x\,\mathrm{d}y < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \int_0^\infty f^p(x)\mathrm{d}x \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)\mathrm{d}x \right\}^{\frac{1}{q}},$$

where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible.

Hardy-Hilbert's integral inequality is important in analysis and its applications (see[10], [11]). In recent years, the various improvements and extensions on the inequality (1) and (2) appeared in some papers (such as [1]–[4], [7], [9], [12]–[14]) and bibliography therein. They focalize on changing the denominator of the function of the left-hand side of (2). Such as the denominator $(x + y)$ is replaced by $(Ax + By)^\lambda$ in paper [13], the denominator $(x + y)$ is replaced by $(x^t + y^t)$ (t is a parameter which is independent of x and y) in paper [7]. Generally, the denominator $(x+y)$ is replaced by $(xu(x) + yv(y))^\lambda$ in paper [9].

The main objective of this paper is to build some new Hilbert-type integral inequalities with best constant factors which are extensions of above results for multi-functions f, g and h. Moreover the denominator is $(m(x) + n(y) + r(z))$, where m, n and r are arbitrary functions.

2. Main results

We need the formula of the β function as (see [8]):

$$(3) \qquad \beta(u, v) = \int_0^\infty \frac{t^{u-1}}{(1 + t)^{u+v}}\mathrm{d}t = \beta(v, u) \qquad u, v > 0.$$

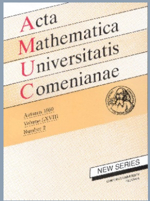
Before stating our results we need the following lemmas.

Lemma 2.1. *Let* $f(x,y,z)\in L^p_{[0,\infty]\times[0,\infty]\times[0,\infty]}$ *and* $g(x,y,z)\in L^q_{[0,\infty]\times[0,\infty]\times[0,\infty]}$, *where* $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_0^\infty |f(x,y,z)\ g(x,y,z)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\le\left(\int_0^\infty\int_0^\infty\int_0^\infty |f(x,y,z)|^p\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{p}}\left(\int_0^\infty\int_0^\infty\int_0^\infty |g(x,y,z)|^q\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{q}}.\end{aligned}\tag{4}$$

Lemma 2.2. *Let* $f(x,y,z)\in L^p_{[0,\infty]\times[0,\infty]\times[0,\infty]}$, $g(x,y,z)\in L^q_{[0,\infty]\times[0,\infty]\times[0,\infty]}$ *and* $h(x,y,z)\in L^k_{[0,\infty]\times[0,\infty]\times[0,\infty]}$ *where* $\dfrac{1}{p}+\dfrac{1}{q}+\dfrac{1}{k}=1$. *Then*

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_0^\infty |f(x,y,z)\ g(x,y,z)h(x,y,z)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\le\left(\int_0^\infty\int_0^\infty\int_0^\infty |f(x,y,z)|^p\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{p}}\times\left(\int_0^\infty\int_0^\infty\int_0^\infty |g(x,y,z)|^q\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{q}}\\ &\times\left(\int_0^\infty\int_0^\infty\int_0^\infty |h(x,y,z)|^r\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{k}}.\end{aligned}\tag{5}$$



Lemma 2.3. *If* $p_1 > 1$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $p > 1$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{k} = 1$ *then for* $0 < \varepsilon < \frac{1}{qk}$, *we have*

$$\int_0^\infty \frac{v^{\frac{1}{p_1 qk} - \frac{\varepsilon}{p_1} - 1}}{(1+v)^{\frac{1}{qk}}} \mathrm{d}v = \beta\left(\frac{1}{p_1 qk}, \frac{1}{q_1 qk}\right) + 0(1) \qquad \varepsilon \to 0^+. \tag{6}$$

Proof. Since

$$\begin{aligned}
&\left| \int_0^\infty \frac{v^{\frac{1}{p_1 qk} - \frac{\varepsilon}{p_1} - 1}}{(1+v)^{\frac{1}{qk}}} \mathrm{d}v - \beta\left(\frac{1}{p_1 qk}, \frac{1}{q_1 qk}\right) \right| \\
&= \left| \int_0^\infty \frac{v^{\frac{1}{p_1 qk} - \frac{\varepsilon}{p_1} - 1} - v^{\frac{1}{p_1 qk} - 1}}{(1+v)^{\frac{1}{qk}}} \mathrm{d}v \right| \\
&\leq \int_0^1 \frac{\left| v^{\frac{1}{p_1 qk} - \frac{\varepsilon}{p_1} - 1} - v^{\frac{1}{p_1 qk} - 1} \right|}{(1+v)^{\frac{1}{qk}}} \mathrm{d}v + \int_1^\infty \frac{\left| v^{\frac{1}{p_1 qk} - \frac{\varepsilon}{p_1} - 1} - v^{\frac{1}{p_1 qk} - 1} \right|}{(1+v)^{\frac{1}{qk}}} \mathrm{d}v \\
&\leq \int_0^1 \left(v^{\frac{1}{p_1 qk} - \frac{\varepsilon}{p_1} - 1} - v^{\frac{1}{p_1 qk} - 1} \right) \mathrm{d}v + \int_1^\infty \frac{v^{\frac{1}{p_1 qk} - 1} - v^{\frac{1}{p_1 qk} - \frac{\varepsilon}{p_1} - 1}}{v^{\frac{1}{qk}}} \mathrm{d}v \\
&= \frac{1}{\frac{1}{p_1 qk} - \frac{\varepsilon}{p_1}} - \frac{1}{\frac{1}{p_1 qk}} + \frac{-1}{\frac{1}{q_1 qk}} + \frac{1}{\frac{1}{q_1 qk} - \frac{\varepsilon}{q_1}} \to 0 \qquad \text{for} \qquad \varepsilon \to 0.
\end{aligned}$$

□

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Lemma 2.4. *If* $p_1 > 1$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $p > 1$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{k} = 1$ *and* $0 < \varepsilon < \frac{1}{qk}$, *setting*

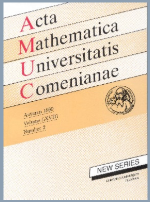
$$J_1 := \int_1^\infty \int_1^\infty \left(\frac{n(y)}{m(x) + n(y)} \right)^{\frac{1}{qk}} (m(x))^{\frac{1}{p_1 qk} - \frac{\varepsilon}{p_1} - 1} \times (n(y))^{-\frac{1}{p_1 qk} - \frac{\varepsilon}{q_1} - 1} \frac{\mathrm{d}m(x)}{\mathrm{d}x} \frac{\mathrm{d}n(y)}{\mathrm{d}y} \mathrm{d}x\, \mathrm{d}y,$$

then we have

$$\frac{1}{\varepsilon}\left(\beta\left(\frac{1}{p_1 qk}, \ \frac{1}{q_1 qk} \right) + o(1) \right) - O(1) \leq J_1 \leq \frac{1}{\varepsilon}\left(\beta\left(\frac{1}{p_1 qk}, \frac{1}{q_1 qk} \right) + o(1) \right), \qquad \varepsilon \to 0^+. \tag{7}$$

Proof. For fixed y, setting $m(x) = n(y)v$, then by (6), we obtain

$$\begin{aligned} J_1 &= \int_1^\infty (n(y))^{-\frac{1}{p_1 qk} - \frac{\varepsilon}{q_1} - 1} \frac{\mathrm{d}n(y)}{\mathrm{d}y} \\ &\quad \times \left[\int_1^\infty \left(\frac{n(y)}{m(x) + n(y)} \right)^{\frac{1}{qk}} (m(x))^{\frac{1}{p_1 qk} - \frac{\varepsilon}{p_1} - 1} \frac{\mathrm{d}m(x)}{\mathrm{d}x} \mathrm{d}x \right] \mathrm{d}y \\ &= \int_1^\infty (n(y))^{-\varepsilon - 1} \frac{\mathrm{d}n(y)}{\mathrm{d}y} \left[\int_{\frac{1}{n(y)}}^\infty \frac{v^{\frac{1}{p_1 qk} - \frac{\varepsilon}{p_1} - 1}}{(1+v)^{\frac{1}{qk}}} \mathrm{d}v \right] \mathrm{d}y \end{aligned}$$



$$
\begin{aligned}
&= \int_1^\infty (n(y))^{-\varepsilon-1} \frac{\mathrm{d}n(y)}{\mathrm{d}y} \left[\int_0^\infty \frac{v^{\frac{1}{p_1 qk}-\frac{\varepsilon}{p_1}-1}}{(1+v)^{\frac{1}{qk}}} \mathrm{d}v \right] \mathrm{d}y \\
&\quad - \int_1^\infty (n(y))^{-\varepsilon-1} \frac{\mathrm{d}n(y)}{\mathrm{d}y} \left[\int_0^{\frac{1}{n(y)}} \frac{v^{\frac{1}{p_1 qk}-\frac{\varepsilon}{p_1}-1}}{(1+v)^{\frac{1}{qk}}} \mathrm{d}v \right] \mathrm{d}y \\
&\geq \frac{1}{\varepsilon}\left(\beta\left(\frac{1}{p_1 qk}, \frac{1}{q_1 qk}\right) + 0(1) \right) - \int_1^\infty (n(y))^{-\varepsilon-1} \frac{\mathrm{d}n(y)}{\mathrm{d}y} \left[\int_0^1 v^{\frac{1}{p_1 qk}-\frac{\varepsilon}{p_1}-1} \mathrm{d}v \right] \mathrm{d}y \\
&= \frac{1}{\varepsilon}\left(\beta\left(\frac{1}{p_1 qk}, \frac{1}{q_1 qk}\right) + 0(1) \right) - \frac{1}{\varepsilon} \frac{1}{\left(\frac{1}{p_1 qk} - \frac{\varepsilon}{p_1} \right)} \\
&= \frac{1}{\varepsilon}\left(\beta\left(\frac{1}{p_1 qk}, \frac{1}{q_1 qk}\right) + o(1) \right) - O(1).
\end{aligned}
$$

By the same way, we have

$$
\begin{aligned}
J_1 &\leq \int_1^\infty \int_0^\infty \left(\frac{n(y)}{m(x)+n(y)} \right)^{\frac{1}{qk}} (m(x))^{\frac{1}{p_1 qk}-\frac{\varepsilon}{p_1}-1} \\
&\qquad \times (n(y))^{-\frac{1}{p_1 qk}-\frac{\varepsilon}{q_1}-1} \frac{\mathrm{d}m(x)}{\mathrm{d}x} \frac{\mathrm{d}n(y)}{\mathrm{d}y} \mathrm{d}x \, \mathrm{d}y, \\
&= \frac{1}{\varepsilon}\left(\beta\left(\frac{1}{p_1 qk}, \frac{1}{q_1 qk}\right) + o(1) \right).
\end{aligned}
$$

The lemma is proved. $\square$

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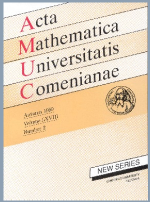


Theorem 2.5. *Assume that* m, n, r *are increasing functions defined on* $[0,\infty[$ *such that* $m(0)=n(0)=r(0)=0$, $\lim\limits_{x\to\infty} m(x)=\lim\limits_{x\to\infty} n(x)=\lim\limits_{x\to\infty} r(x)=\infty$, *and* f, g, h *satisfy*

$$\int_0^\infty (m(x))^{\frac{q_1}{p_1}(1+\frac{1}{pq})}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})}|f(x)|^{\frac{kq_1}{2}}\,\mathrm{d}x<\infty \tag{8}$$

$$\int_0^\infty (m(x))^{\frac{p_1}{q_1}-\frac{1}{qk}}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-p_1}{q_1}}|f(x)|^{\frac{pp_1}{2}}\,\mathrm{d}x<\infty, \tag{9}$$

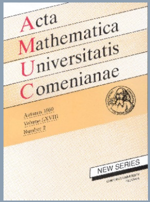
$$\int_0^\infty (n(y))^{\frac{q_1}{p_1}(1+\frac{1}{qk})}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})}|g(y)|^{\frac{pq_1}{2}}\,\mathrm{d}y<\infty, \tag{10}$$

$$\int_0^\infty (n(y))^{\frac{p_1}{q_1}-\frac{1}{pk}}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p_1}{q_1}}|g(y)|^{\frac{qp_1}{2}}\,\mathrm{d}y<\infty, \tag{11}$$

$$\int_0^\infty (r(z))^{\frac{q_1}{p_1}(1+\frac{1}{pk})}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})}|h(z)|^{\frac{qq_1}{2}}\,\mathrm{d}z<\infty, \tag{12}$$

$$\int_0^\infty (r(z))^{\frac{p_1}{q_1}-\frac{1}{pk}}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-p_1}{q_1}}|h(z)|^{\frac{kp_1}{2}}\,\mathrm{d}z<\infty, \tag{13}$$

where $\dfrac{1}{p}+\dfrac{1}{q}+\dfrac{1}{k}=1,\quad \dfrac{1}{p_1}+\dfrac{1}{q_1}=1,$ *then*



$$
\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x)g(y)h(z)|}{m(x)+n(y)+r(z)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \le \left(\frac{\pi}{\sin\frac{\pi}{qk}}\beta\left(\frac{1}{q_1qk},\frac{1}{p_1qk}\right)\right)^{\frac{1}{p}}\\
&\times\left(\int_0^\infty (m(x))^{\frac{p_1}{q_1}-\frac{1}{qk}}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-p_1}{q_1}}|f(x)|^{\frac{pp_1}{2}}\,\mathrm{d}x\right)^{\frac{1}{pp_1}}\\
&\times\left(\int_0^\infty (n(y))^{\frac{q_1}{p_1}(1+\frac{1}{qk})}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})}|g(y)|^{\frac{pq_1}{2}}\,\mathrm{d}y\right)^{\frac{1}{pq_1}}\\
&\times\left(\frac{\pi}{\sin\frac{\pi}{pk}}\beta\left(\frac{1}{q_1pk},\frac{1}{p_1pk}\right)\right)^{\frac{1}{q}}\times\left(\int_0^\infty (n(y))^{\frac{p_1}{q_1}-\frac{1}{pk}}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p_1}{q_1}}|g(y)|^{\frac{qp_1}{2}}\,\mathrm{d}y\right)^{\frac{1}{qp_1}}\\
&\times\left(\int_0^\infty (r(z))^{\frac{q_1}{p_1}(1+\frac{1}{pk})}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})}|h(z)|^{\frac{qq_1}{2}}\,\mathrm{d}z\right)^{\frac{1}{qq_1}}\\
&\times\left(\frac{\pi}{\sin\frac{\pi}{pq}}\beta\left(\frac{1}{q_1pq},\frac{1}{p_1pq}\right)\right)^{\frac{1}{k}}\left(\int_0^\infty (r(z))^{\frac{p_1}{q_1}-\frac{1}{pk}}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-p_1}{q_1}}|h(z)|^{\frac{kp_1}{2}}\,\mathrm{d}z\right)^{\frac{1}{kp_1}}\\
&\times\left(\int_0^\infty (m(x))^{\frac{q_1}{p_1}(1+\frac{1}{pq})}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})}|f(x)|^{\frac{kq_1}{2}}\,\mathrm{d}x\right)^{\frac{1}{kq_1}},
\end{aligned}
\tag{14}
$$

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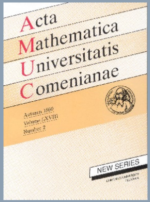

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where the constant factors

$$\frac{\pi}{\sin\frac{\pi}{qk}}\beta\left(\frac{1}{q_1qk},\frac{1}{p_1qk}\right),\qquad \frac{\pi}{\sin\frac{\pi}{pk}}\beta\left(\frac{1}{q_1pk},\frac{1}{p_1pk}\right),\qquad \frac{\pi}{\sin\frac{\pi}{pq}}\beta\left(\frac{1}{q_1pq},\frac{1}{p_1pq}\right)$$

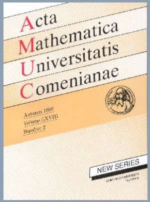
are best possible.

Proof. Since

$$\begin{aligned}
I &= \int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x)\ g(y)h(z)|}{m(x)+n(y)+r(z)}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&= \int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x)|^{\frac12}\ |g(y)|^{\frac12}}{(m(x)+n(y)+r(z))^{\frac1p}}\left(\frac{m(x)+n(y)}{r(z)}\right)^{\frac{1}{pqk}}\left(\frac{n(y)}{m(x)+n(y)}\right)^{\frac{1}{pqk}}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac1p}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-1}{k}}\\
&\qquad\times\frac{|g(y)|^{\frac12}\ |h(z)|^{\frac12}}{(m(x)+n(y)+r(z))^{\frac1q}}\left(\frac{n(y)+r(z)}{m(x)}\right)^{\frac{1}{pqk}}\left(\frac{r(z)}{n(y)+r(z)}\right)^{\frac{1}{pqk}}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac1q}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-1}{p}}\qquad(15)\\
&\qquad\times\frac{|f(x)|^{\frac12}\ |h(z)|^{\frac12}}{(m(x)+n(y)+r(z))^{\frac1k}}\left(\frac{r(z)+m(x)}{n(y)}\right)^{\frac{1}{pqk}}\left(\frac{m(x)}{r(z)+m(x)}\right)^{\frac{1}{pqk}}\\
&\qquad\times\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac1k}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-1}{q}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.
\end{aligned}$$

Applying Hölder's inequality on (15) we get

$$I\le I_1^{\frac1p}I_2^{\frac1q}I_3^{\frac1k}\qquad(16)$$



where

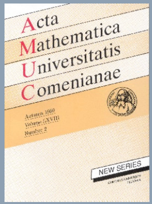
$$I_1=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{|f(x)|^{\frac{p}{2}}\ |g(y)|^{\frac{p}{2}}}{m(x)+n(y)+r(z)}\left(\frac{m(x)+n(y)}{r(z)}\right)^{\frac{1}{qk}}\left(\frac{n(y)}{m(x)+n(y)}\right)^{\frac{1}{qk}} \times\frac{\mathrm{d}r(z)}{\mathrm{d}z}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p}{k}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{17}$$

$$I_2=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{|g(y)|^{\frac{q}{2}}\ |h(z)|^{\frac{q}{2}}}{m(x)+n(y)+r(z)}\left(\frac{n(y)+r(z)}{m(x)}\right)^{\frac{1}{pk}}\left(\frac{r(z)}{n(y)+r(z)}\right)^{\frac{1}{pk}} \times\frac{\mathrm{d}m(x)}{\mathrm{d}x}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-q}{p}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{18}$$

$$I_3=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{|f(x)|^{\frac{k}{2}}\ |h(z)|^{\frac{k}{2}}}{m(x)+n(y)+r(z)}\left(\frac{r(z)+m(x)}{n(y)}\right)^{\frac{1}{pq}}\left(\frac{m(x)}{r(z)+m(x)}\right)^{\frac{1}{pq}} \times\frac{\mathrm{d}n(y)}{\mathrm{d}y}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-k}{q}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{19}$$

Consider the weight coefficient

$$w_1(x,y)=\int_0^{\infty}\frac{1}{m(x)+n(y)+r(z)}\left(\frac{m(x)+n(y)}{r(z)}\right)^{\frac{1}{qk}}\frac{\mathrm{d}r(z)}{\mathrm{d}z}\mathrm{d}z. \tag{20}$$



Let $v = \dfrac{r(z)}{m(x)+n(y)}$ in (20) then we obtain

$$w_1(x,y) = \frac{\pi}{\sin \dfrac{\pi}{qk}}. \tag{21}$$

Similarly,

$$w_2(y,z) = \int_0^\infty \frac{1}{m(x)+n(y)+r(z)} \left(\frac{n(y)+r(z)}{m(x)}\right)^{\frac{1}{pk}} \mathrm{d}x = \frac{\pi}{\sin \dfrac{\pi}{pk}} \tag{22}$$

and

$$w_3(z,x) = \int_0^\infty \frac{1}{m(x)+n(y)+r(z)} \left(\frac{r(z)+m(x)}{n(y)}\right)^{\frac{1}{pq}} \mathrm{d}x = \frac{\pi}{\sin \dfrac{\pi}{pq}}. \tag{23}$$

Combining (21), (22), (23) and (16) we get

$$\begin{aligned} I \le {} & \left(\frac{\pi}{\sin\frac{\pi}{qk}} \int_0^\infty\!\!\int_0^\infty \left(\frac{n(y)}{m(x)+n(y)}\right)^{\frac{1}{qk}} |f(x)|^{\frac{p}{2}}\, |g(y)|^{\frac{p}{2}} \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p}{k}} \mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}} \\ & \times \left(\frac{\pi}{\sin\frac{\pi}{pk}} \int_0^\infty\!\!\int_0^\infty \left(\frac{r(z)}{n(y)+r(z)}\right)^{\frac{1}{pk}} |g(y)|^{\frac{q}{2}}\, |h(z)|^{\frac{q}{2}} \left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-q}{p}} \mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{q}} \\ & \times \left(\frac{\pi}{\sin\frac{\pi}{pq}} \int_0^\infty\!\!\int_0^\infty \left(\frac{m(x)}{r(z)+m(x)}\right)^{\frac{1}{pq}} |f(x)|^{\frac{k}{2}}\, |h(z)|^{\frac{k}{2}} \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-k}{q}} \mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{k}}. \end{aligned} \tag{24}$$

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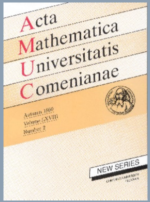

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Applying Hölder's inequality with $p_1 > 1, \frac{1}{p_1} + \frac{1}{q_1} = 1$ on the first integral on the right side in (24),we have

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty \left(\frac{n(y)}{m(x)+n(y)}\right)^{\frac{1}{qk}} |f(x)|^{\frac{p}{2}}\,|g(y)|^{\frac{p}{2}} \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p}{k}} \mathrm{d}x\,\mathrm{d}y\\
&= \int_0^\infty\int_0^\infty \frac{|f(x)|^{\frac{p}{2}} \left(\frac{n(y)}{m(x)}\right)^{\frac{1}{p_1q_1qk}-\frac{1}{p_1}} m(x)^{\frac{1}{q_1}-\frac{1}{p_1}}}{(m(x)+n(y))^{\frac{1}{p_1qk}}} \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{1}{p_1}} \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-1}{q_1}}\\
&\qquad \times \frac{|g(y)|^{\frac{p}{2}} (n(y))^{\frac{1}{qk}+\frac{1}{p_1}-\frac{1}{q_1}}}{(m(x)+n(y))^{\frac{1}{q_1qk}}} \left(\frac{m(x)}{n(y)}\right)^{\frac{1}{p_1q_1qk}-\frac{1}{q_1}} \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p}{k}-\frac{1}{p_1}} \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{1}{q_1}} \mathrm{d}y\mathrm{d}x\\
&\le \left(\int_0^\infty\int_0^\infty \frac{|f(x)|^{\frac{pp_1}{2}}}{(m(x)+n(y))^{\frac{1}{qk}}} \left(\frac{n(y)}{m(x)}\right)^{\frac{1}{q_1qk}-1} (m(x))^{\frac{p_1}{q_1}-1} \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-p_1}{q_1}} \frac{\mathrm{d}n(y)}{\mathrm{d}y}\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{p_1}}\\
&\quad \times \left(\int_0^\infty\int_0^\infty \frac{|g(y)|^{\frac{pq_1}{2}} (n(y))^{\frac{q_1}{qk}+\frac{q_1}{p_1}-1}}{(m(x)+n(y))^{\frac{1}{qk}}} \left(\frac{m(x)}{n(y)}\right)^{\frac{1}{p_1qk}-1} \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})} \frac{\mathrm{d}m(x)}{\mathrm{d}x}\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{q_1}}.
\end{aligned}
\tag{25}
$$

Let

$$
w_4(x) = \int_0^\infty \frac{1}{\left(1+\frac{n(y)}{m(x)}\right)^{\frac{1}{qk}}} \left(\frac{n(y)}{m(x)}\right)^{\frac{1}{q_1qk}-1} \frac{\mathrm{d}n(y)}{\mathrm{d}y}\mathrm{d}y
\tag{26}
$$

Putting $v_1 = \dfrac{n(y)}{m(x)}$ in (26) we have

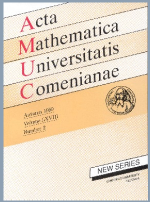
$$w_4(x) = m(x)\ \beta\left(\frac{1}{q_1 qk}, \frac{1}{qk} - \frac{1}{q_1 qk}\right) = m(x)\ \beta\left(\frac{1}{q_1 qk}, \frac{1}{p_1 qk}\right). \tag{27}$$

Similarly

$$\begin{aligned} w_5(y) &= \int_0^\infty \frac{1}{\left(1 + \dfrac{m(x)}{n(y)}\right)^{\frac{1}{qk}}} \left(\frac{m(x)}{n(y)}\right)^{\frac{1}{p_1 qk}-1} \frac{\mathrm{d}m(x)}{\mathrm{d}x}\mathrm{d}x \\ &= n(y)\ \beta\left(\frac{1}{q_1 qk}, \frac{1}{p_1 qk}\right). \end{aligned} \tag{28}$$

From (27), (28) and (25) we get

$$\begin{aligned} &\int_0^\infty\int_0^\infty \left(\frac{n(y)}{m(x)+y}\right)^{\frac{1}{qk}} |f(x)|^{\frac{p}{2}}\ |g(y)|^{\frac{p}{2}} \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p}{k}} \mathrm{d}x\,\mathrm{d}y \\ &\le \beta\left(\frac{1}{q_1 qk}, \frac{1}{p_1 qk}\right) \left(\int_0^\infty (m(x)^{\frac{p_1}{q_1} - \frac{1}{qk}} \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-p_1}{q_1}} |f(x)|^{\frac{pp_1}{2}}\ \mathrm{d}x\right)^{\frac{1}{p_1}} \\ &\quad\times \left(\int_0^\infty (n(y))^{\frac{q_1}{p_1}(1+\frac{1}{qk})} \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})} |g(y)|^{\frac{pq_1}{2}}\ \mathrm{d}y\right)^{\frac{1}{q_1}}. \end{aligned} \tag{29}$$



For the second and third integrals on the right side of (24), following the same steps used for obtaining (29), we obtain

$$\int_0^\infty\int_0^\infty \left(\frac{r(z)}{n(y)+r(z)}\right)^{\frac{1}{pk}} |g(y)|^{\frac{q}{2}} \; |h(z)|^{\frac{q}{2}} \left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-q}{p}} \mathrm{d}y\mathrm{d}z$$

$$\leq \beta\left(\frac{1}{q_1qk},\frac{1}{p_1qk}\right)\left(\int_0^\infty (n(y))^{\frac{p_1}{q_1}-\frac{1}{pk}}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p_1}{q_1}} |g(y)|^{\frac{qp_1}{2}}\,\mathrm{d}y\right)^{\frac{1}{p_1}} \tag{30}$$

$$\times\left(\int_0^\infty (r(z))^{\frac{q_1}{p_1}(1+\frac{1}{pk})}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})} |h(z)|^{\frac{qq_1}{2}}\,\mathrm{d}z\right)^{\frac{1}{q_1}},$$

and

$$\int_0^\infty\int_0^\infty \left(\frac{m(x)}{r(z)+m(x)}\right)^{\frac{1}{pq}} |h(z)|^{\frac{k}{2}} \; |f(x)|^{\frac{k}{2}} \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-k}{q}} \mathrm{d}z\mathrm{d}x$$

$$\leq \beta\left(\frac{1}{q_1pq},\frac{1}{p_1pq}\right)\left(\int_0^\infty (r(z))^{\frac{p_1}{q_1}-\frac{1}{pk}}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-p_1}{q_1}} |h(z)|^{\frac{kp_1}{2}}\,\mathrm{d}z\right)^{\frac{1}{p_1}} \tag{31}$$

$$\times\left(\int_0^\infty (m(x))^{\frac{q_1}{p_1}(1+\frac{1}{pq})}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})} |f(x)|^{\frac{kq_1}{2}}\,\mathrm{d}x\right)^{\frac{1}{q_1}}.$$

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Let

$$\gamma_1 = \beta\left(\frac{1}{q_1 qk}, \frac{1}{p_1 qk}\right), \qquad \gamma_2 = \beta\left(\frac{1}{q_1 qk}, \frac{1}{p_1 qk}\right), \qquad \gamma_3 = \beta\left(\frac{1}{q_1 pq}, \frac{1}{p_1 pq}\right).$$

To prove the constant factors γ_1, γ_2 and γ_3 are best possible. Assume that the constant factor γ_1 is not the best possible, then there exists a positive constant λ_1 with $\lambda_1 < \gamma_1$, such that (29) is still valid if we replace γ_1 by λ_1. Without loss of generality, we assume that $m(1) = n(1) = 1$. For $0 < \varepsilon < 1$, setting f_ε and g_ε as $f_\varepsilon(x) = g_\varepsilon(x) = 0$, for $x \in (0,1)$, and for $x \in [1, \infty)$

$$|f_\varepsilon(x)| = (m(x))^{\frac{2}{pp_1}\left(-\frac{p_1}{q_1}+\frac{1}{qk}-\varepsilon-1\right)}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{2}{pp_1}\left(\frac{p_1}{q_1}+1\right)}$$

$$|g_\varepsilon(x)| = (n(x))^{\frac{2}{pq_1}\left(-\frac{q_1}{p_1}\left(1+\frac{1}{qk}\right)-\varepsilon-1\right)}\left(\frac{\mathrm{d}n(x)}{\mathrm{d}x}\right)^{\frac{2}{pq_1}\left(q_1\left(\frac{p}{k}+\frac{1}{p_1}\right)+1\right)},$$

then we obtain

$$\begin{aligned}
&\lambda_1\left(\int_0^\infty (m(x))^{\frac{p_1}{q_1}-\frac{1}{qk}}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-p_1}{q_1}}|f(x)|^{\frac{pp_1}{2}}\,\mathrm{d}x\right)^{\frac{1}{p_1}} \\
&\qquad\times\left(\int_0^\infty (n(y))^{\frac{q_1}{p_1}\left(1+\frac{1}{qk}\right)}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{-q_1\left(\frac{p}{k}+\frac{1}{p_1}\right)}|g(y)|^{\frac{pq_1}{2}}\,\mathrm{d}y\right)^{\frac{1}{q_1}} \\
&= \lambda_1\left(\int_1^\infty (m(x))^{-\varepsilon-1}\frac{\mathrm{d}m(x)}{\mathrm{d}x}\mathrm{d}x\right)^{\frac{1}{p_1}}\left(\int_1^\infty (n(y))^{-\varepsilon-1}\frac{\mathrm{d}n(y)}{\mathrm{d}y}\mathrm{d}y\right)^{\frac{1}{q_1}} = \frac{\lambda_1}{\varepsilon}.
\end{aligned}$$

But we have

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty \left(\frac{n(y)}{m(x)+y}\right)^{\frac{1}{qk}} |f(x)|^{\frac{p}{2}}\ \ |g(y)|^{\frac{p}{2}} \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p}{k}} \mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_1^\infty\int_1^\infty \left(\frac{n(y)}{m(x)+y}\right)^{\frac{1}{qk}} (m(x))^{\frac{1}{p_1 qk}-\frac{\varepsilon}{p_1}-1} (n(y))^{-\frac{1}{p_1 qk}-\frac{\varepsilon}{q_1}-1} \\
&\qquad\qquad \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)\mathrm{d}x\,\mathrm{d}y = J_1 \\
&\quad \geq \frac{1}{\varepsilon}(\gamma_1 + o(1)) - O(1).
\end{aligned}
$$

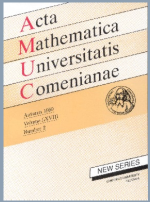
Hence we find

$$
\frac{1}{\varepsilon}(\gamma_1 + o(1)) - O(1) < \frac{\lambda_1}{\varepsilon} \tag{32}
$$

or

$$
\gamma_1 + o(1) - \varepsilon O(1) < \lambda_1. \tag{33}
$$

For $\varepsilon \to 0^+$, it follows that $\gamma_1 \leq \lambda_1$. This contradicts the fact that $\lambda_1 < \gamma_1$. Hence the constant factor γ_1 in (29) is the best possible. Similarly γ_2 and γ_3 are the best possible. Substituting from (29), (30), (31) in (24), the result of the theorem follows. □



Remark 1. Putting $p = q = k = \frac{1}{3}$ in (14), we get a new inequality in the form

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x)\,g(y)h(z)|}{m(x)+n(y)+r(z)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\leq \left(\frac{\pi}{\sin\frac{\pi}{9}}\beta\left(\frac{1}{9q_1},\frac{1}{9p_1}\right)\right)\left(\int_0^\infty (m(x))^{\frac{p_1}{q_1}-\frac{1}{9}}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-p_1}{q_1}}|f(x)|^{\frac{3p_1}{2}}\,\mathrm{d}x\right)^{\frac{1}{3p_1}} \\
&\quad\times\left(\int_0^\infty (n(y))^{\frac{10q_1}{9p_1}}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{-q_1(1+\frac{1}{p_1})}|g(y)|^{\frac{3q_1}{2}}\,\mathrm{d}y\right)^{\frac{1}{3q_1}} \\
&\quad\times\left(\int_0^\infty (n(y))^{\frac{p_1}{q_1}-\frac{1}{9}}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p_1}{q_1}}|g(y)|^{\frac{3p_1}{2}}\,\mathrm{d}y\right)^{\frac{1}{3p_1}} \\
&\quad\times\left(\int_0^\infty (r(z))^{\frac{10q_1}{9p_1}}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{-q_1(1+\frac{1}{p_1})}|h(z)|^{\frac{3q_1}{2}}\,\mathrm{d}z\right)^{\frac{1}{3q_1}} \\
&\quad\times\left(\int_0^\infty (r(z))^{\frac{p_1}{q_1}-\frac{1}{9}}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-p_1}{q_1}}|h(z)|^{\frac{kp_1}{2}}\,\mathrm{d}z\right)^{\frac{1}{3p_1}} \\
&\quad\times\left(\int_0^\infty (m(x))^{\frac{10q_1}{9p_1}}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{-q_1(1+\frac{1}{p_1})}|f(x)|^{\frac{3q_1}{2}}\,\mathrm{d}x\right)^{\frac{1}{3q_1}}.
\end{aligned} \tag{34}
$$

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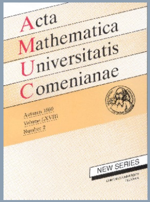
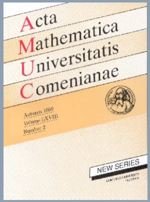


Remark 2. Putting $p_1 = q_1 = 2$ in (14) we obtain

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x)\,g(y)h(z)|}{m(x)+n(y)+r(z)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\leq \left(\frac{\pi}{\sin\frac{\pi}{qk}}\beta\left(\frac{1}{2qk},\frac{1}{2qk}\right)\right)^{\frac{1}{p}} \left(\int_0^\infty (m(x))^{1-\frac{1}{qk}}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{-1}|f(x)|^q\,\mathrm{d}x\right)^{\frac{1}{2p}} \\
&\qquad\times\left(\int_0^\infty (n(y))^{(1+\frac{1}{qk})}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{-2(\frac{p}{k}+\frac{1}{2})}|g(y)|^p\,\mathrm{d}y\right)^{\frac{1}{2p}} \\
&\quad\times\left(\frac{\pi}{\sin\frac{\pi}{pk}}\beta\left(\frac{1}{2pk},\frac{1}{2pk}\right)\right)^{\frac{1}{q}} \left(\int_0^\infty (n(y))^{1-\frac{1}{pk}}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p_1}{q_1}}|g(y)|^p\,\mathrm{d}y\right)^{\frac{1}{2q}} \\
&\qquad\times\left(\int_0^\infty (r(z))^{(1+\frac{1}{pk})}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{-2(\frac{p}{k}+\frac{1}{2})}|h(z)|^q\,\mathrm{d}z\right)^{\frac{1}{2q}} \\
&\quad\times\left(\frac{\pi}{\sin\frac{\pi}{pq}}\beta\left(\frac{1}{2pq},\frac{1}{2pq}\right)\right)^{\frac{1}{k}} \left(\int_0^\infty (r(z))^{1-\frac{1}{pk}}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{-1}|h(z)|^k\,\mathrm{d}z\right)^{\frac{1}{2k}} \\
&\qquad\times\left(\int_0^\infty (m(x))^{(1+\frac{1}{pq})}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{-2\left(\frac{p}{k}+\frac{1}{2}\right)}|f(x)|^k\,\mathrm{d}x\right)^{\frac{1}{2k}},
\end{aligned}
\tag{35}
$$



which is a new inequality.

Remark 3. Let $m(x) = x$, $n(y) = y$ and $r(z) = z$ in (34) and (35) we get respectively

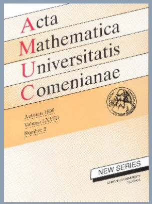
$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{|f(x)\, g(y) h(z)|}{x+y+z} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

$$\leq \frac{\pi}{\sin \frac{\pi}{9}} \beta\left(\frac{1}{9q_1}, \frac{1}{9p_1}\right) \left(\left(\int_0^\infty x^{\frac{p_1}{q_1}-\frac{1}{9}} |f(x)|^{\frac{3p_1}{2}} \, \mathrm{d}x\right) \left(\int_0^\infty y^{\frac{p_1}{q_1}-\frac{1}{9}} |g(y)|^{\frac{3p_1}{2}} \, \mathrm{d}y\right)\right.$$

$$\left. \times \left(\int_0^\infty z^{\frac{p_1}{q_1}-\frac{1}{9}} |h(z)|^{\frac{3p_1}{2}} \, \mathrm{d}z\right)\right)^{\frac{1}{3p_1}}$$

$$\times \left(\left(\int_0^\infty x^{\frac{10q_1}{9p_1}} |f(x)|^{\frac{3q_1}{2}} \, \mathrm{d}x\right) \left(\int_0^\infty y^{\frac{10q_1}{9p_1}} |g(y)|^{\frac{3q_1}{2}} \, \mathrm{d}y\right)\right.$$

$$\left. \times \left(\int_0^\infty z^{\frac{10q_1}{9p_1}} |h(z)|^{\frac{3q_1}{2}} \, \mathrm{d}z\right)\right)^{\frac{1}{3q_1}}.$$



and

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x)\,g(y)h(z)|}{x+y+z}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$\le \left(\frac{\pi}{\sin\frac{\pi}{qk}}\beta\left(\frac{1}{2qk},\frac{1}{2qk}\right)\right)^{\frac{1}{p}}\left(\left(\int_0^\infty x^{1-\frac{1}{qk}}\,|f(x)|^p\,\mathrm{d}x\right)\left(\int_0^\infty y^{(1+\frac{1}{qk})}\,|g(y)|^p\,\mathrm{d}y\right)\right)^{\frac{1}{2p}}$$

$$\times\left(\frac{\pi}{\sin\frac{\pi}{pk}}\beta\left(\frac{1}{2pk},\frac{1}{2pk}\right)\right)^{\frac{1}{q}}\left(\left(\int_0^\infty y^{1-\frac{1}{pk}}\,|g(y)|^q\,\mathrm{d}y\right)\left(\int_0^\infty z^{1+\frac{1}{pk}}\,|h(z)|^q\,\mathrm{d}z\right)\right)^{\frac{1}{2q}}$$

$$\times\left(\frac{\pi}{\sin\frac{\pi}{pq}}\beta\left(\frac{1}{2pq},\frac{1}{2pq}\right)\right)^{\frac{1}{k}}\left(\left(\int_0^\infty z^{1-\frac{1}{pk}}\,|h(z)|^k\,\mathrm{d}z\right)\left(\int_0^\infty x^{1+\frac{1}{pq}}\,|f(x)|^k\,\mathrm{d}x\right)\right)^{\frac{1}{2k}}.$$

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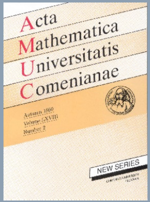

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