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ON THE DUAL SPACE $C_0^*(S,X)$

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Abstract. Let S be a locally compact Hausdorff space and let us consider the space $C_0(S,X)$ of continuous functions vanishing at infinity, from S into the Banach space X. A theorem of I. Singer, settled for S compact, states that the topological dual $C_0^*(S,X)$ is isometrically isomorphic to the Banach space $r\sigma bv(S,X^*)$ of all regular vector measures of bounded variation on S with values in the strong dual X^*. Using the Riesz-Kakutani theorem and some routine topological arguments, we propose a constructive detailed proof which is, as far as we know, different from that supplied elsewhere.

Preliminaries

Let S be a locally compact Hausdorff space equipped with its Borel σ-field $\mathcal{B}_S$, and let X be a Banach space. We denote by $C_0(S,X)$ the Banach space (uniform norm) of all continuous functions $f : S \to X$, vanishing at infinity. If $X = \mathbb{R}$, we put $C_0(S,X) = C_0(S)$. According to the Riesz-Kakutani theorem [**7**, Theorem 6.19], the dual $C_0^*(S)$ is isometric to the Banach space of all scalar regular measures on S with the variation norm. All the measures we will deal with here are supposed to be defined on the σ-field $\mathcal{B}_S$. We denote by X^* the strong dual of X.

If $\lambda : \mathcal{B}_S \to Y$ is an additive set function from $\mathcal{B}_S$ into the Banach space Y, then the variation of λ is usually defined by the extented positive set function $|\lambda|(\bullet)$ given by:

$$|\lambda|(E) = \sup \sum_i \|\lambda(E_i)\|, \qquad E \in \mathcal{B}_S \tag{1}$$

where the supremum is taken over all finite partitions $\{E_i\}$ of E in $\mathcal{B}_S$.

We say that λ is of bounded variation if $|\lambda|(E) < \infty$, for all $E \in \mathcal{B}_S$. It is easy to check that $|\lambda|$ is additive. Moreover, if λ is of bounded variation, then λ is σ-additive if and only if $|\lambda|$ is σ-additive. We say that λ is regular if $|\lambda|$ is regular in the customary sense [**1**]. We denote by $r\sigma bv(\mathcal{S},Y)$ the set of all regular Y-valued vector measures on S. For $\lambda \in r\sigma bv(\mathcal{S},Y)$, put $|\lambda|(S) = \|\lambda\|$, then the following proposition is well known [**1**]:

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Proposition 1.

(a) $\|\lambda\|$ *is a norm making* $r\sigma bv(\mathcal{S},Y)$ *with the usual operations a Banach space.*

(b) *In the specific case* $Y = X^*$, *we have*

$$|\lambda|(E) = \sup\left|\sum_i \lambda(E_i)x_i\right|, \qquad E \in \mathcal{B}_S \tag{2}$$

where the supremum is taken over all finite partitions $\{E_i\}$ *of* E *in* $\mathcal{B}_S$, *and all finite systems* $\{x_i\}$ *of vectors in* X *with* $\|x_i\| \leq 1$ *for each* i.

The RHS of formula (2) is the so called semivariation of λ [**2**]. So Proposition 1(b) says that, for vector measures with values in a dual, the variation is equal to the semivariation.

The theorem of Singer

Theorem 1. *There is an isometric isomorphism between the topological dual* $C_0^*(S,X)$ *of* $C_0(S,X)$ *and the Banach space* $r\sigma bv(\mathcal{S},X^*)$, *where the functional* $U \in C_0^*(S,X)$ *and the corresponding measure* $\lambda \in r\sigma bv(\mathcal{S},X^*)$ *are related by the integral formula*

$$\begin{aligned} Uf &= \int_S f\,\mathrm{d}\lambda, \qquad f \in C_0(S,X) \\ \|U\| &= \|\lambda\|. \end{aligned} \tag{3}$$

where the integral is the termed immediate integral of Dinculeanu [**3**].

Let us recall that this theorem is the basic tool in the proof of the representation theorem of N. Dinculeanu [**2**, Section 19].

Actually the original proof of this theorem [**8**] contains some gaps about the strong σ-additivity and regularity of the measure λ attached to the functional U. These gaps have been filled by J. Gil de Lamadrid in [**5**, pages 775–776]. Another proof using the Hahn-Banach theorem and measures on product spaces, can be found in [**6**]. To settle the proof of the theorem we need some preparatory lemmas. Let us start with a $U \in C_0^*(S,X)$, we will construct a $\lambda \in r\sigma bv(\mathcal{S},X^*)$ such that formula (3) holds.

Lemma 1. *For each* $(f,x) \in C_0(S) \times X$ *we define* $B(f,x)$ *by*

$$B(f,x) = U(f\cdot x), \qquad f \in C_0(S), \quad x \in X. \tag{4}$$

Then B *is a bounded bilinear form on* $C_0(S) \times X$ *with* $\|B\| \leq \|U\|$.

Proof. It is clear that B is bilinear. The norm inequality is immediate from the following estimation: $|B(f,x)| = |U(f\cdot x)| \leq \|U\| \cdot \|f\|_\infty \cdot \|x\|$. □

Lemma 2. *For each fixed* $x \in X$, *let* $W_x(\bullet) = B(\bullet, x)$. *Then there exists a unique scalar regular measure* μ_x *on* $\mathcal{B}_S$ *such that*

$$W_x(f) = \int_S f\mathrm{d}\mu_x, \qquad f \in C_0(S), \text{ and } \|W_x\| = \|\mu_x\|. \tag{5}$$

Proof. From the construction of B in Lemma 1 we have $|W_x(f)| \leq \|U\| \cdot \|f\|_\infty \cdot \|x\|$. So W_x is linear and bounded, that is $W_x \in C_0^*(S)$, and we have $|W_x(f)| \leq \|U\| \cdot \|f\|_\infty \cdot \|x\|$, therefore $\|W_x\| \leq \|U\| \cdot \|x\|$. Moreover, the correspondence $x \longmapsto W_x$ is a bounded linear operator from X into the dual space $C_0^*(S)$ with the norm at most $\|U\|$. By the Riesz-Kakutani theorem, $C_0^*(S)$ is canonically isometric to the respective space of regular measures with the variation norm. Consequently, for each $x \in X$ there is a unique scalar regular measure μ_x on $\mathcal{B}_S$ such that

$$W_x(f) = \int_S f \mathrm{d}\mu_x, \qquad f \in C_0(S) \text{ and } \|W_x\| = \|\mu_x\|$$

□

Lemma 3. *Define the set function λ on $\mathcal{B}_S$ by the following recipe: for $A \in \mathcal{B}_S$, $\lambda(A)$ is the functional on X given by*

$$\lambda(A)x = \mu_x(A), \qquad x \in X \tag{6}$$

where μ_x comes from Lemma 2.

Then $\lambda(A) \in X^*$ for each $A \in \mathcal{B}_S$, moreover, λ is additive.

Proof. Let $x, y \in X$, $A \in \mathcal{B}_S$, then $\lambda(A)(x+y) = \mu_{x+y}(A)$, where μ_{x+y} corresponds to W_{x+y} according to (5), thus $W_{x+y}(f) = \int_S f \ \mathrm{d}\mu_{x+y}$, for all $f \in C_0(S)$. Since

$$W_{x+y}(f) = B(f, x+y) = B(f,x) + B(f,y),$$

we deduce from (5) that

$$W_{x+y}(f) = \int_S f \mathrm{d}\mu_{x+y} = \int_S f \mathrm{d}\mu_x + \int_S f \mathrm{d}\mu_y = \int_S f \mathrm{d}(\mu_x + \mu_y),$$

where the last equality is easy to check by standard method. Thus

$$\int_S f \mathrm{d}\mu_{x+y} = \int_S f \mathrm{d}(\mu_x + \mu_y), \qquad \text{for each } f \in C_0(S).$$

From the fact that $\mu_x + \mu_y$ is regular, the uniqueness part of the Riesz-Kakutani theorem yields $\mu_{x+y} = \mu_x + \mu_y$. Likewise $\mu_{\alpha x} = \alpha \mu_x$, for $\alpha \in \mathbb{R}$. This proves that $\lambda(A)$ is a linear functional on X. On the other hand we have

$$|\lambda(A)x| = |\mu_x(A)| \leq |\mu_x|(A) \leq \|\mu_x\| = \|W_x\| \leq \|U\| \cdot \|x\|$$

(see the proof of Lemma 2). So we deduce that $\lambda(A) \in X^*$ and $\|\lambda(A)\| \leq \|U\|$ for each $A \in \mathcal{B}_S$.

Finally, it is clear that λ is additive. □

The remaining lemmas are intended to prove that the additive set function λ is actually a vector measure. The following lemma is crucial:

Lemma 4. *The set function λ has finite variation. Moreover, we have $\|\lambda\| \leq \|U\|$.*

Proof. We use formula (2) for the variation of λ. Let $A_1, A_2, \ldots, A_n$ be a finite partition of the locally compact space S by sets in $\mathcal{B}_S$ and let $x_1, x_2, \ldots, x_n$ be vectors in X with $\|x_i\| \leq 1$ for all i. We need an estimation of the sum $\sum_1^n \lambda(A_i)x_i$.

Let $\varepsilon > 0$, then by the regularity of the measures μ_{x_i}, there exist compact sets $K_1, K_2, \ldots, K_n$ and open sets $G_1, G_2, \ldots, G_n$ such that

$$K_i \subset A_i \subset G_i \qquad \text{and} \qquad |\mu_{x_i}|(G_i \setminus K_i) < \frac{\varepsilon}{2n}, \quad i = 1, 2, \ldots n.$$

Note that the K_i are pairwise disjoint since A_i are so. Since S is Hausdorff, disjoint compact sets have disjoint neighbourhoods. So, using a simple induction on n, we can construct pairwise disjoint open sets $U_1, U_2, \ldots, U_n$ such that $K_i \subset U_i$ for each i. Letting $V_i = U_i \cap G_i$, we get pairwise disjoint open sets V_i such that $K_i \subset V_i \subset G_i$, for all i.

Now, let $g_i : S \to \mathbb{R}$ be a continuous function such that $0 \le g_i(t) \le 1$ for all $t \in S$, $g_i(t) = 1$ for all $t \in K_i$, support $g_i \subset V_i$ (such functions exist by Urysohn's lemma since S is locally compact). We have

$$\int_S g_i \mathrm{d}\mu_{x_i} = \int_{V_i} g_i \mathrm{d}\mu_{x_i}$$

(since $g_i \equiv 0$ outside V_i), so we deduce that

$$\int_S g_i \mathrm{d}\mu_{x_i} = \int_{V_i \setminus K_i} g_i \mathrm{d}\mu_{x_i} + \int_{K_i} g_i \mathrm{d}\mu_{x_i}.$$

But $\int_{K_i} g_i \mathrm{d}\mu_{x_i} = \mu_{x_i}(K_i)$ (because $g_i \equiv 1$ on K_i). Consequently, we have

$$\int_S g_i \mathrm{d}\mu_{x_i} - \mu_{x_i}(K_i) = \int_{V_i \setminus K_i} g_i \mathrm{d}\mu_{x_i}.$$

This gives the following estimation

$$\begin{aligned}\left|\int_S g_i \mathrm{d}\mu_{x_i} - \mu_{x_i}(K_i)\right| &= \left|\int_{V_i \setminus K_i} g_i \mathrm{d}\mu_{x_i}\right| \le \int_{V_i \setminus K_i} g_i d \cdot |\mu_{x_i}| \\ &\le |\mu_{x_i}|(V_i \setminus K_i) \qquad (\text{since } 0 \le g_i \le 1) \\ &\le |\mu_{x_i}|(G_i \setminus K_i) \qquad (\text{since } V_i \subset G_i) \\ &< \frac{\varepsilon}{2n}\end{aligned}$$

Therefore

$$\left|\int_S g_i \mathrm{d}\mu_{x_i} - \mu_{x_i}(K_i)\right| < \frac{\varepsilon}{2n}, \qquad \text{for each } i. \tag{7}$$

Now, let $f : S \to X$ be the function defined by

$$f(t) = \sum_1^n g_i(t) \cdot x_i, \qquad t \in S$$

then f is continuous and we have $f(t) = 0$ for each t in $S \setminus \cup_1^n V_i$, and $f(t) = g_i(t) \cdot x_i$ for each t in V_i, because V_i are pairwise disjoint and support $g_i \subset V_i$. Then we deduce that $\|f\| \le 1$ and by (5)

$$Uf = \sum_1^n U(g_i \cdot x_i) = \sum_1^n \int_S g_i \mathrm{d}\mu_{x_i}, \qquad \text{since} \qquad U(g_i \cdot x_i) = W_{x_i}(g_i).$$

So

$$\left|Uf-\sum_1^n \mu_{x_i}(K_i)\right| = \left|\sum_1^n \int_S g_i \mathrm{d}\mu_{x_i} - \sum_1^n \mu_{x_i}(K_i)\right|$$
$$\leq \sum_1^n \left|\int_S g_i \mathrm{d}\mu_{x_i} - \mu_{x_i}(K_i)\right| < \sum_1^n \frac{\varepsilon}{2n} = \frac{\varepsilon}{2}$$

Therefore

$$\left|Uf-\sum_1^n \mu_{x_i}(K_i)\right| < \frac{\varepsilon}{2} \tag{8}$$

Now, we turn to the estimation of $|\sum_1^n \lambda(A_i)x_i|$.

$$\left|\sum_1^n \lambda(A_i)x_i\right| - |Uf| \leq \left|\sum_1^n \lambda(A_i)x_i - Uf\right|$$
$$\leq \left|\sum_1^n \lambda(A_i)x_i - \sum_1^n \mu_{x_i}(K_i)\right| + \left|Uf - \sum_1^n \mu_{x_i}(K_i)\right|$$

and

$$\left|\sum_1^n \lambda(A_i)x_i - \sum_1^n \mu_{x_i}(K_i)\right| = \left|\sum_1^n \mu_{x_i}(A_i) - \sum_1^n \mu_{x_i}(K_i)\right|$$
$$\leq \sum_1^n |\mu_{x_i}|(A_i \setminus K_i)$$
$$\leq \sum_1^n |\mu_{x_i}|(G_i \setminus K_i) < \sum_1^n \frac{\varepsilon}{2n} = \frac{\varepsilon}{2}$$

Combining this with (8), we get

$$\left|\sum_1^n \lambda(A_i)x_i\right| - |Uf| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So

$$\left|\sum_1^n \lambda(A_i)x_i\right| < |Uf| + \varepsilon \leq \|U\| \cdot \|f\|_\infty + \varepsilon \leq \|U\| + \varepsilon \qquad (\text{since } \|f\| \leq 1),$$

letting $\varepsilon \searrow 0$ we obtain $|\sum_1^n \lambda(A_i)x_i| \leq \|U\|$.

So, by taking the supremum for all finite partitions $\{A_i\}$ of S in $\mathcal{B}_S$ and all systems $\{x_i\}$ in X with $\|x_i\| \leq 1$, this leads to $|\lambda|(S) \leq \|U\| < \infty$, by formula (2). Then λ has a finite variation. □

Lemma 5. *For each $A \in \mathcal{B}_S$ we have*

$$|\lambda|(A) = \sup\{|\lambda|(K) : K \subset A,\ K \text{ compact}\} \tag{9}$$
$$|\lambda|(A) = \inf\{|\lambda|(G) : A \subset G, G \text{ open}\} \tag{10}$$

In other words the variation measure $|\lambda|$ of λ is regular, and so λ is regular.

Proof. Let $A \in \mathcal{B}_S$, since $|\lambda| < \infty$, (9) is equivalent to the following approximation: For each $\varepsilon > 0$, there is a compact K such that

$$K \subset A, \qquad |\lambda|(A) - \varepsilon < |\lambda|(K) \tag{11}$$

Let $\varepsilon > 0$, again since $|\lambda| < \infty$ there exists a finite partition $E_1, E_2, \ldots, E_n$ of A in $\mathcal{B}_S$ and $x_1, x_2, \ldots, x_n$ in X with $\|x_i\| \le 1$ for all i such that

$$|\lambda|(A) - \frac{\varepsilon}{2} < \left|\sum_1^n \lambda(E_i)x_i\right|, \qquad \text{by formula (2).}$$

By formula (6) the measures $\lambda(\bullet)x_i = \mu_{x_i}(\bullet)$ are regular; consequently, there exist compact sets $K_1, K_2, \ldots, K_n$, with $K_i \subset E_i$ and $|\lambda(E_i \setminus K_i)x_i| < \dfrac{\varepsilon}{2n}$ for all i. Then we have

$$\begin{aligned} |\lambda|(A) - \frac{\varepsilon}{2} &< \left|\sum_1^n \lambda(E_i)x_i\right| \le \left|\sum_1^n \lambda(K_i)x_i\right| + \left|\sum_1^n \lambda(E_i \setminus K_i)x_i\right| \\ &\le \sum_1^n |\lambda(K_i)x_i| + \sum_1^n |\lambda(E_i \setminus K_i)x_i| < |\lambda|(K) + \frac{\varepsilon}{2}, \end{aligned}$$

where K is defined to be the compact set $\overset{n}{\underset{1}{\cup}} K_i$.

Therefore, (11) is valid and proves (9). We can get (10) by applying (9) to the complement A^c of the set A. □

Lemma 6. *The variation measure $|\lambda|$ is σ-additive.*

Proof. Since λ is additive then so is $|\lambda|$. By the regularity property just proved, the result is a consequence of Alexandroff theorem (see [**4**, p. 138]. □

Lemma 7. *The set function λ is a regular vector measure, that is λ is a member of $r\sigma bv(\mathcal{S}, X^*)$.*

Proof. We know that λ is additive, so to prove the σ-additivity it is enough to prove the continuity at $\emptyset$, that is for every sequence A_n in $\mathcal{B}_S$ decreasing to $\emptyset$, we have $\lambda(A_n) \to 0$. But it is a consequence of the σ-additivity of $|\lambda|$ and the fact that $\|\lambda(A)\| \le |\lambda|(A)$, for each $A \in \mathcal{B}_S$. On the other hand λ is regular since $|\lambda|$ is regular by Lemma 5. □

Lemma 8. *Let υ, $\mu \in r\sigma bv(\mathcal{S}, X^*)$ be such that $\int_S f \mathrm{d}\upsilon = \int_S f \mathrm{d}\mu$ for all $f \in C_0(S, X)$, then $\upsilon \equiv \mu$.*

Proof. Take $f \in C_0(S, X)$ of the form $f(\bullet) = g(\bullet) \cdot x$ where $g \in C_0(S)$ and x fixed in X. Then by standard tools we have $\int_S f \mathrm{d}\upsilon = \int_S g \mathrm{d}\upsilon(\bullet)x$ and $\int_S f \mathrm{d}\mu = \int_S g \mathrm{d}\mu(\bullet)x$. This yields $\int_S g \mathrm{d}\upsilon(\bullet)x = \int_S g \mathrm{d}\mu(\bullet)x$. Since both scalar measures $\upsilon(\bullet)x$ and $\mu(\bullet)x$ are regular and since g is arbitrary, we deduce from Riesz-Kakutani theorem that $\upsilon(\bullet)x = \mu(\bullet)x$ for each $x \in X$. Thus $\upsilon \equiv \mu$. □

Now, we are in a position to give the proof of Theorem 1.

Proof of Theorem 1. First we prove relation (3), i.e, for all $f \in C_0(S,X)$, $Uf = \int_S f\mathrm{d}\lambda$ where λ is the vector measure constructed in Lemma 3.

Let $f \in C_0(S,X)$ be of the form $f(\bullet) = g(\bullet)\cdot x$ for $g \in C_0(S)$ and x fixed in X. Then

$$\begin{aligned} Uf &= W_x(g) \\ &= \int_S g\mathrm{d}\mu_x && \text{by Lemma 2, formula (5)} \\ &= \int_S g\mathrm{d}\lambda(\bullet)x && \text{by Lemma 3, formula (6).} \end{aligned}$$

But we have $\int_S g\mathrm{d}\lambda(\bullet)x = \int_S g{\cdot}x{\cdot}\mathrm{d}\lambda$. Therefore, formula (3) is satisfied for $f = g{\cdot}x$. By linearity we can see that formula (3) is satisfied for all $f \in C_0(S)\otimes X$, the vector space of all $f \in C_0(S,X)$ of the form $f(\bullet) = \sum_1^n g_i(\bullet)\cdot x_i$ with $g_i \in C_0(S)$ for each i. It is well known that $C_0(S)\otimes X$ is dense in $C_0(S,X)$ (see [**2**, Proposition 1 of Section 19]. Consequently, if $f \in C_0(S,X)$, there is a sequence f_n in $C_0(S)\otimes X$ converging to f uniformly on S. By the integration process with respect to an operator valued measure we get

$$\left|\int_S f_n\mathrm{d}\lambda - \int_S f\mathrm{d}\lambda\right| \le \|f_n - f\|_\infty \cdot \tilde{\lambda}(S),$$

where $\tilde{\lambda}$ is the semivariation of λ defined by the RHS of formula (2) and which is, in the present context, equal to the variation $|\lambda|$ (see the Preliminaries). As λ is of finite variation and $\|f_n - f\|_\infty \to 0$, we have $\int_S f_n\mathrm{d}\lambda \to \int_S f\ d\lambda$. But $Uf_n = \int_S f_n\mathrm{d}\lambda$ because $f_n \in C_0(S)\otimes X$ for each n. Since U is bounded and $f_n \to f$ uniformly we get $Uf_n = \int_S f_n\mathrm{d}\lambda \to Uf$.

Hence,

$$Uf = \int_S f\mathrm{d}\lambda, \qquad \text{for all } f \in C_0(S,X).$$

By Lemma 8, λ is the unique measure in $r\sigma bv(S,X^*)$ satisfying relation (3). This proves that the correspondence $U \xrightarrow{\varphi} \lambda$ from $C_0^*(S,X)$ into $r\sigma bv(S,X^*)$ is well-defined. Moreover, we have

$$|Uf| = |\int_S f\mathrm{d}\lambda| \le \|f\|_\infty \cdot \widetilde{\lambda}(S) = \|f\|_\infty \cdot \|\lambda\|,$$

so $\|U\| \le \|\lambda\|$ and by Lemma 4 we get $\|U\| = \|\lambda\|$. This implies that φ is an isometry and then it is one-one. It is not difficult to show that φ is linear (make use of Lemma 8). To complete the proof, we must show that φ is onto. To this end, let us start with $\mu \in r\sigma bv(S,X^*)$, to which we associate the functional on $C_0(S,X)$ given by $Uf = \int_S f\mathrm{d}\mu$, $f \in C_0(S,X)$. It is clear that U is linear and bounded, so $U \in C_0^*(S,X)$. We show that $\varphi(U) = \mu$. Put $\varphi(U) = \lambda$, that is λ is the vector measure constructed along Lemmas 3–7. Then by formula (3), $Uf = \int_S f\mathrm{d}\lambda$ for all $f \in C_0(S,X)$, which yields $\int_S f\ \mathrm{d}\mu = \int_S f\mathrm{d}\lambda$ for all $f \in C_0(S,X)$. From Lemma 8, we deduce that $\mu = \lambda$, and this complete the proof of Theorem 1. $\square$

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