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TWO-GENERATED IDEALS OF LINEAR TYPE

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Abstract. We show that in a local S_1 ring every two-generated ideal of linear type can be generated by a two-element sequence of linear type and give an example which illustrates that the S_1 condition is essential. We also show that every Noetherian local ring in which every two-element sequence is of linear type is an integrally closed integral domain and every two-generated ideal of it can be generated by a two-element d-sequence. Finally, we investigate two-element c-sequences and characterize them under some mild assumptions.

1. Introduction

Let R be a commutative ring, $\langle \mathbf{a} \rangle = \langle a_1, \dots, a_n \rangle$ a sequence of elements of R, $I = (a_1, \dots, a_n)$ the ideal generated by the a_i's and $I_i = (a_1, \dots, a_i)$, $i = 0, 1, \dots, n$, the ideal generated by the first i elements of the sequence. Let $S(I)$ be the symmetric algebra of the ideal I, $R[It] = \bigoplus_{i \geq 0} I^i t^i$ its Rees algebra and $\alpha : S(I) \to R[It]$ the canonical map, which maps $a_i \in S^1(I)$ to $a_i t$. The ideal I is said to be *an ideal of linear type* if α is an isomorphism.

Let us mention a simple property of ideals of linear type that we are going to use later.

Lemma 1.1 ([**3**, Theorem 4(i)])**.** *If $I = (a_1, \dots, a_n)$ is an ideal of linear type, then*

$$I_{n-1} I^{k-1} : a_n^k = I_{n-1} : a_n$$

for every $k \geq 1$.

Here the notation $J : x$, where J is an ideal and x an element of a commutative ring R, means $\{r \in R | \ rx \in J\}$. We also use the notation $(J : x)$ and $[J : x]$ for the same thing.

We say that $\langle \mathbf{a} \rangle$ is a *d-sequence* ([**5**]) if

(1) $$[I_{i-1} : a_i] : a_j = I_{i-1} : a_j$$

for every $i, j \in \{1, 2, \dots, n\}$ with $j \geq i$. Equivalently

(2) $$[I_{i-1} : a_i] \cap I = I_{i-1}$$

for every $i \in \{1, 2, \dots, n\}$.

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The notion of a d-sequence is a useful tool in many questions in commutative algebra. Huneke [**6**] and G. Valla [**13**], showed that ideals generated by d-sequences are of linear type, thus generalizing a result of A. Micali [**8**], who proved the same statement for regular sequences.

We say that $\langle \mathbf{a} \rangle$ is a *sequence of linear type* ([**3**]) if I_i is an ideal of linear type for every $i = 1, 2, \dots, n$.

Conditions for a two-generated ideal to be of linear type are first investigated in detail by Ratliff [**11**]. Similar type of results can also be found in Shimoda's paper [**12**]. A nice overview of results about ideals and sequences of linear type, including those of two elements, is given by Cipu and Fiorentini [**1**]. We should also mention two papers by Planas-Vilanova, namely [**9**] and [**10**], where among other things two-generated ideals of linear type are considered.

2. Two-generated ideals of linear type in local rings

We start with an example of a two-generated ideal in a local ring, which is of linear type, but cannot be minimally generated by a sequence of linear type.

Example 2.1. Let $R = k[[X, Y, U, V]]/(XU - YV, XV, YU, U^2, V^2, UV) = k[[x, y, u, v]]$, where k is a field and $I = (x, y)$. Then I is an ideal of linear type which cannot be minimally generated by a sequence of linear type.

Proof. Let $A = k[u, v]$ with $u^2 = v^2 = uv = 0$. Then the ring $S = A[X, Y]/(vX, uY, uX - vY)$ is a symmetric algebra of an A-module (namely, $A^2/(A(v, 0) + A(0, u) + A(u, -v))$) and so its augmentation ideal is of linear type ([**4**]). Hence the "polynomial version" of the ideal I is of linear type and so I in the above ring R is of linear type.

Now note that $ux^2 = 0$ and $vy^2 = 0$. (Indeed, $ux^2 = ux \cdot x = yv \cdot x = 0$. Similarly $vy^2 = 0$.)

Every element of I has the form $fx+gy$, where f, g are power series in x,y,u and v. Suppose $fx+gy$ is a minimal generator of I and $(0 : fx+gy) = (0 : (fx+gy)^2)$. Since

$$(fx + gy)^2 \cdot u = (f^2x^2 + 2fgxy + g^2y^2) \cdot u = f^2x^2u = 0,$$
$$(fx + gy)^2 \cdot v = (f^2x^2 + 2fgxy + g^2y^2) \cdot v = g^2y^2v = 0,$$

we would have

$$(fx + gy) \cdot u = fxu = 0,$$
$$(fx + gy) \cdot v = gyv = 0.$$

The first of these equalities would imply $f \in m_R$ (all terms that have either x, or y, or u, or v, when multiplied by xu would give 0 and the constant term c would give $cxu \neq 0$, so there is no constant term) and the second one would imply $g \in m_R$. Hence $fx + gy \in m_R I$ and could not be a minimal generator, that is a contradiction. □

The next theorem shows that by adding a very mild condition, namely that the ring is S_1 (which means that the associated primes of the ring are minimal), we can guarantee that every two-generated ideal of linear type in a local ring can necessarily be minimally generated by a sequence of linear type.

Theorem 2.2. *Let R be a local S_1 ring. Then every two generated ideal of R of linear type can be generated by a sequence of linear type of two elements.*

Proof. For every element $a \in R$ we have $\mathrm{Ass}(R/(0:a)) \subset \mathrm{Ass}(R) = \mathrm{Min}(R)$ (the last equality since R is S_1). Hence all the associated primes of $(0:a)$ are of height 0. Now (a) is of linear type if and only if $(0:a^2) = (0:a)$, i.e., if and only if a is in no associated prime of $(0:a)$, i.e., if and only if a is in no minimal prime of R containing $(0:a)$.

Let $Q_1, Q_2, \cdots, Q_s$ be the minimal primes of R that do not contain I. By the prime avoidance lemma we can choose an element a so that

$$a \in I \setminus [(\cup_{i=1}^{s} Q_i) \cup m_R I].$$

We claim that a is in no minimal prime of R containing $(0:a)$. Suppose the contrary. Let $P \in \mathrm{Min}(R)$ with $P \supset (0:a)$ and $a \in P$. Since a is in no minimal prime of R which does not contain I, we have $P \supset I$. But then, since I is of linear type, by [**4**, Proposition 2.4] IR_P can be generated by $\mathrm{ht}(P) = 0$ elements, so $IR_P = 0$. Hence $PR_P \supset (0:a)_P = (0:a/1) = (0:0) = R_P$, this is a contradiction. Thus (a) is of linear type. Now since $a \notin m_R I$, a is a minimal generator of I, hence we can add one more generator $b \in I$ so that $\{a, b\}$ is a minimal system of generators of I and $\langle a, b\rangle$ is a sequence of linear type. □

Now we characterize local rings in which every two-element sequence is of linear type. (Note that every one-element sequence $\langle a\rangle$ in a ring R is of linear type if and only if R is *reduced*.)

Theorem 2.3. *Let R be a Noetherian local ring in which every sequence $\langle a, b\rangle$ is of linear type. Then R is an integrally closed integral domain and every two generated ideal of R can be generated by a d-sequence of two elements.*

Proof. We first show that R is an integral domain. Suppose the contrary. Let a, b be nonzero elements of R such that $ab = 0$ (so $a, b \in m_R$) and let $I = (a, b)$. By the Costa-Kühl criterion ([**3**, Theorem 1] or [**7**, Theorem 1.2]), the sequence

$$0 \longrightarrow N \longrightarrow I \times I \xrightarrow{f} I^2 \longrightarrow 0$$

is exact, where $f(x, y) = ax + by$ and N is the submodule of $I \times I$ generated by the trivial syzygy $(-b, a)$. Since $(b, 0) \in N$, we have $(b, 0) = r(-b, a)$ for some $r \in R$. So $rb = -b$ and $ra = 0$. Since $a \neq 0$, $r \in m_R$. Hence $1 + r$ is a unit and since $(1 + r)b = 0$, we have $b = 0$, this is a contradiction. Thus R is an integral domain.

Now by [**8**, page 38, Proposition 1], $S(I)$ is an integral domain for every two generated ideal of R. Hence, by [**2**, Theorem 3], R is integrally closed. Finally, by [**5**, Proposition 1.5], every two-generated ideal of R can be generated by a d-sequence of two elements. □

3. C-SEQUENCES OF TWO ELEMENTS

It was proved in [**3**] that d-sequences satisfy the following property:

$$[I_{i-1}I^k : a_i] \cap I^k = I_{i-1}I^{k-1}$$

for every $i \in \{1, \ldots, n\}$ and every $k \geq 1$. It was also proved ([**3**, Theorem 3]) that, if a sequence satisfies this property, it generates an ideal of linear type. We call the sequences that satisfy this property c-sequences.

Definition 3.1. We say that $\langle \mathbf{a} \rangle$ is a *c-sequence* if

$$[I_{i-1}I^k : a_i] \cap I^k = I_{i-1}I^{k-1} \tag{3}$$

for every $i \in \{1, \ldots, n\}$ and every $k \geq 1$.

We say that a sequence is an *unconditioned c-sequence* if it is a c-sequence in any order.

For one-element sequences the notions of c- and d-sequences coincide, but already among two-element sequences it is possible to find an example of a c-sequence that is not a d-sequence.

Example 3.2. Let $R = k[X, Y, Z, U]/(XU - Y^2Z) = k[x, y, z, u]$, where k is a field. Consider the sequence $\langle x, y \rangle$ and the ideal $I = (x, y)$. This sequence is not a d-sequence (since $z \in (x) : y^2$ and $z \notin (x) : y$), although I is an ideal of linear type (as it was shown in [**13**, Example 3.16]).

Let us show that $\langle x, y \rangle$ is a c-sequence. We should show two relations:

$$[0 : x] \cap I^k = 0, \qquad k \geq 1$$
$$\big[xI^k : y\big] \cap I^k = xI^{k-1}, \quad k \geq 1,$$

the first of which is trivial since R is an integral domain. For the second one, note that $[xI^k : y] \cap I^k$=$[xI^k : y] \cap ((xI^{k-1} + (y)^k) = [xI^k : y] \cap (y)^k + xI^{k-1}$. So it is enough to prove that $[xI^k : y] \cap (y)^k \subset xI^{k-1}$. Let $\alpha = ay^k$, $a \in R$, be an element of $[xI^k : y] \cap (y)^k$. Then $ay^{k+1} \in xI^k$, i.e., $a \in xI^k : y^{k+1} = (x) : y$ by Lemma 1.1. Hence $ay \in (x)$ and so $\alpha = ay^k = ay \cdot y^{k-1} \in xI^{k-1}$.

For one-element sequences the notions of a sequence of linear type and a c-sequence coincide. For two-element sequences, every c-sequence $\langle a, b \rangle$ is a sequence of linear type. Indeed, the $k = 1$ condition for a c-sequence implies $(0 : a) \cap (a) = 0$, which is equivalent with $(0 : a) = (0 : a^2)$. Hence (a) is an ideal of linear type. Also (a, b) is of linear type since every c-sequence generates an ideal of linear type by the above mentioned [**3**, Theorem 3]. Now we show that the notion of a sequence of linear type is strictly weaker than the notion of a c-sequence.

Example 3.3. Let $R = k[X, Y, U, V]/(UX, VX, UY, U^2, V^2, UV) = k[x, y, u, v]$ where k is a field. Then $\langle x, y \rangle$ is a sequence of linear type which is not a c-sequence.

Indeed, let us first show that $I = (x, y)$ is an ideal of linear type. We can write

$$R = A[X, Y]/(uX, vX, uY),$$

where $A = k[u, v]$ with $u^2 = v^2 = uv = 0$. Hence R is a symmetric algebra of an A-module (namely $A^2/(A(u,0)+A(v,0)+A(0,u))$ and so (by [**4**, page 87]) its augmentation ideal $I = (x, y)$ is an ideal of linear type.

Also it is easy to verify that $(0 : x) = (0 : x^2) = (u, v)$. Thus $\langle x, y\rangle$ is a sequence of linear type.

But $(0 : x) \cap (x, y)$ contains a nonzero element vy and thus the first condition for $\langle x, y\rangle$ to be a c-sequence is not satisfied.

Now we characterize two-element c-sequences under some mild assumptions.

Theorem 3.4. *Let* $I = (a, b)$ *be an ideal of* R. *Suppose* $(0 : a) \cap I = 0$. *Then the following conditions are equivalent:*

(i) I *is of linear type,*

(ii) $\langle a, b\rangle$ *is a sequence of linear type,*

(iii) $\langle a, b\rangle$ *is a c-sequence,*

and they imply the following conditions:

(iv) $aI^k \cap bI^k = abI^{k-1}$, $k \geq 1$,

(v) $aI^{k-1} \cap (b)^k \subset a(b)^{k-1}$, $k \geq 1$.

If we also suppose that $(0 : b) \cap I = 0$, *then all five conditions are equivalent to each other.*

Proof. (i) $\Rightarrow$(iii): Assume I that is of linear type. Since the first condition for $\langle a, b\rangle$ to be a c-sequence, namely $(0 : a) \cap I = 0$, is assumed, we only need to show the second condition, namely

$$[aI^k : b] \cap I^k \subset aI^{k-1}, \qquad k \geq 1,$$

or equivalently

$$[aI^k : b] \cap [aI^{k-1} + (b)^k] \subset aI^{k-1}, \qquad k \geq 1.$$

Since $aI^{k-1} \subset aI^k : b$, this is equivalent with

$$[aI^k : b] \cap (b)^k + aI^{k-1} \subset aI^{k-1}, \qquad k \geq 1,$$

i.e., with

$$[aI^k : b] \cap (b)^k \subset aI^{k-1}, \qquad k \geq 1.$$

Let $x = rb^k$, $r \in R$, be such that $xb = rb^{k+1} \in aI^k$. Then $r \in aI^k : b^{k+1}$ and so by Lemma 1.1, $r \in (a) : b$, i.e., $rb \in (a)$. Hence $x = rb^k = rbb^{k-1} \in aI^{k-1}$.

(iii)$\Rightarrow$ (ii) and (ii)$\Rightarrow$(i): clear.

(ii)$\Rightarrow$(v): Let $x = rb^k \in aI^{k-1} \cap (b)^k$.Then $r \in aI^{k-1} : b^k = (a) : b$ by Lemma 1.1. Hence $rb \in (a)$ and so $x = rb^k \in a(b)^{k-1}$.

(v)$\Rightarrow$(iv): We have

$$\begin{aligned} aI^k \cap bI^k &= aI^k \cap b(aI^{k-1} + (b)^k) \\ &= aI^k \cap (abI^{k-1} + (b)^{k+1}) \\ &= aI^k \cap (b)^{k+1} + abI^{k-1} \\ &\subset a(b)^k + abI^{k-1} \\ &= abI^{k-1}, \end{aligned}$$

where the the inclusion follows from the assumption (v).

(iv)⇒(iii): Assume now that $(0:b)\cap I=0$ and suppose that (iv) holds. Let $k\geq 1$ and let $x\in[aI^k:b]\cap I^k$. Then $bx\in aI^k$ and also $bx\in bI^k$. So $bx\in aI^k\cap bI^k\subset abI^{k-1}$ by the assumption. Hence $bx=aby$, $y\in I^{k-1}$. Now $b(x-ay)=0$ and, since $x-ay\in I$ and $(0:b)\cap I=0$, we have $x=ay\in aI^{k-1}$. Hence $\langle a,b\rangle$ is a c-sequence. □

Remark 3.5. All five of the above conditions are equivalent, for example, when R is an integral domain.

Corollary 3.6 ([**2**, Theorem 2])**.** *Let R be an integral domain, $a,b,\in R$, $I=(a,b)$. Then $S_R(I)$ is an integral domain if and only if $aI^k\cap bI^k=abI^{k-1}$ for all $k\geq 1$.*

Proof. Follows from Theorem 3.4 and [**8**, Proposition 1]. □

Corollary 3.7. *In an integral domain every two-generated ideal of linear type, is generated by an unconditioned c-sequence of two elements.*

Remark 3.8. This is an analogue of [**5**, Proposition 1.5] which says that in an integrally closed integral domain *every* two-generated ideal is generated by a d-sequence of two elements.

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