ON PREŠIĆ TYPE GENERALIZATION OF THE BANACH CONTRACTION MAPPING PRINCIPLE

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ABSTRACT. Let (X,d) be a metric space, k a positive integer and T a mapping of X^k into X. In this paper we proved that if T satisfies conditions (2.1) and (2.2) below, then there exists a unique x in X such that $T(x,x,\dots,x)=x$. This result generalizes the corresponding theorems of the second author [4], [5] and the theorem of Dhage [3].

1. Introduction

The well known Banach contraction mapping principle states that if (X,d) is a complete metric space and $T: X \to X$ is a self mapping such that

$$d(Tx,Ty) \le \lambda d(x,y)$$

for all $x,y \in X$, where $0 \le \lambda < 1$, then there exists a unique $x \in X$ such that $T(x)=x$. In recent years many generalizations of this principle have appeared ([1], [2], [6]). A special type generalization was introduced by the second author [4], [5].

Considering the convergence of ceratin sequences Prešić proved the following result.

Theorem 1. *Let (X,d) be a complete metric space, k a positive integer and $T: X^k \to X$ a mapping satisfying the following contractive type condition*

$$d(T(x_1,x_2,x_3,\dots,x_k),T(x_2,x_3,\dots,x_k,x_{k+1})) \le q_1 d(x_1,x_2)+q_2 d(x_2,x_3)+\dots+q_k d(x_k,x_{k+1}), \tag{1.1}$$

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for every $x_1, \ldots, x_{k+1}$ *in* X, *where* $q_1, q_2, \ldots, q_k$ *are non-negative constants such that* $q_1 + q_2 + \ldots + q_k < 1$. *Then there exists a unique point* x *in* X *such that* $T(x, x, \ldots, x) = x$. *Moreover, if* $x_1, x_2, x_3, \ldots, x_k$ *are arbitrary points in* X *and for* $n \in \mathbb{N}$,

$$x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1}),$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ *is convergent and*

$$\lim x_n = T(\lim x_n, \lim x_n, \ldots, \lim x_n).$$

Remark that condition (1.1) in the case $k = 1$ reduces to the well-known Banach contraction mapping principle. So, Theorem 1 is a generalization of the Banach fixed point theorem.

2. Main theorem

Inspired with the results in Theorem 1 we shall prove the following theorem.

Theorem 2. *Let* (X, d) *be a complete metric space,* k *a positive integer and* $T : X^k \to X$ *a mapping satisfying the following contractive type condition*

$$d(T(x_1, x_2, \ldots, x_k), T(x_2, \ldots, x_k, x_{k+1})) \le \lambda \max\{d(x_i, x_{i+1}) : 1 \le i \le k\}, \tag{2.1}$$

where $\lambda \in (0, 1)$ *is constant and* $x_1, \ldots, x_{k+1}$ *are arbitrary elements in* X. *Then there exists a point* x *in* X *such that* $T(x, \ldots, x) = x$. *Moreover, if* $x_1, x_2, x_3, \ldots, x_k$ *are arbitrary points in* X *and for* $n \in \mathbb{N}$,

$$x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1}),$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ *is convergent and*

$$\lim x_n = T(\lim x_n, \lim x_n, \ldots, \lim x_n).$$

If in addition we suppose that on diagonal $\Delta \subset X^k$,

$$d(T(u, \ldots, u), T(v, \ldots, v)) < d(u, v) \tag{2.2}$$

holds for all $u, v \in X$, with $u \neq v$, then x is the unique point in X with $T(x, x, \ldots, x) = x$.

Proof. Let $x_1, \ldots, x_k$ be k arbitrary points in X. Using these points define a sequence $\{x_n\}$ as follows:

$$x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1}) \qquad (n = 1, 2, \ldots).$$

For simplicity set $\alpha_n = d(x_n, x_{n+1})$. We shall prove by induction that for each $n \in \mathbb{N}$:

$$\alpha_n \leq K\theta^n \qquad (\text{where } \theta = \lambda^{1/k},\ K = \max\{\alpha_1/\theta, \alpha_2/\theta^2, \ldots, \alpha_k/\theta^k\}). \tag{2.3}$$

According to the definition of K we see that (2.3) is true for $n = 1, \ldots, k$. Now let the following k inequalities:

$$\alpha_n \leq K\theta^n,\ \alpha_{n+1} \leq K\theta^{n+1},\ \ldots,\ \alpha_{n+k-1} \leq K\theta^{n+k-1}$$

be the induction hypotheses. Then we have:

$$\begin{aligned}
\alpha_{n+k} &= d(x_{n+k}, x_{n+k+1}) \\
&= d(T(x_n, x_{n+1}, \ldots, x_{n+k-1}), T(x_{n+1}, x_{n+2}, \ldots, x_{n+k})) \\
&\leq \lambda \max\{\alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+k-1}\} \qquad (\text{by } (2.1) \text{ and the definition of } \alpha_i) \\
&\leq \lambda \max\{K\theta^n, K\theta^{n+1}, \ldots, K\theta^{n+k-1}\} \qquad (\text{by the induction hypotheses}) \\
&= \lambda K\theta^n \qquad (\text{as } 0 \leq \theta < 1) \\
&= K\theta^{n+k} \qquad (\text{as } \theta = \lambda^{1/k})
\end{aligned}$$

and the inductive proof of (2.3) is complete. Next using (2.3) for any $n, p \in \mathbb{N}$ we have the following argument:

$$\begin{aligned}
d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \\
&\leq K\theta^n + K\theta^{n+1} + \ldots + K\theta^{n+p-1} \\
&\leq K\theta^n(1 + \theta + \theta^2 + \ldots) \\
&= K\theta^n/(1 - \theta)
\end{aligned}$$

by which we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is a complete space, there exists x in X such that

$$x = \lim_{n\to\infty} x_n.$$

Then for any integer n we have

$$\begin{aligned} d(x, T(x, \ldots, x) &\leq d(x, x_{n+k}) + d(x_{n+k}, T(x. \ldots, x)) \\ &= d(x, x_{n+k}) + d(T(x_n, \ldots, x_{n+k-1}), T(x, \ldots, x)) \\ &\leq d(x, x_{n+k}) + d(T(x, \ldots, x, x), T(x, \ldots, x, x_n)) + \\ &\quad d(T(x, \ldots, x, x_n), T(x, \ldots, x_n, x_{n+1})) + \ldots \\ &\quad + d(T(x, x_n, x_{n+1}, x_{n+k-2}), T(x_n, x_{n+1}, \ldots \cdot x_{n+k-1})) \\ &\leq d(x, x_{n+k}) + \lambda d(x, x_n) + \lambda \max\{d(x, x_n), d(x_n, x_{n+1})\} + \ldots \\ &\quad + \lambda \max\{d(x, x_n), d(x_n, x_{n+1}), \ldots, d(x_{n+k-2}, x_{n+k-1})\}. \end{aligned}$$

Taking the limit when n tends to infinity we obtain $d(x, T(x, \ldots, x)) \leq 0$, which implies $T(x, \ldots, x) = x$. Thus we proved that

$$\lim x_n = T(\lim x_n, \lim x_n, \ldots, \lim x_n).$$

Now suppose that (2.2) holds. To prove the uniqueness of the fixed point, let us assume that for some $y \in X$, $y \neq x$, we have $T(y, \ldots, y) = y$. Then by (2.2), $d(x, y) = d(T(x, \ldots, x), T(y, \ldots, y)) < d(x, y)$, which is a contradiction. So, x is the unique point in X such that $T(x, x, \ldots, x) = x$. □

Remark 1. Theorem 2 is a generalization of Theorem 1, as the condition (1.1) implies the conditions (2.1) and (2.2). Indeed, since

$$\begin{aligned} &q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + \ldots + q_k d(x_k, x_{k+1}) \\ \leq\ &(q_1 + q_2 + \ldots + q_k) \max\{d(x_1, x_2), d(x_2, x_3), \ldots, d(x_k, x_{k+1})\} \end{aligned}$$

and $q_1 + q_2 + \ldots + q_k < 1$, we conclude the implication (1.1) $\Rightarrow$ (2.2). Next, for any $u, v \in X$ with $u \neq v$, from (1.1) we have

$$\begin{aligned} & d(T(u,u,\ldots,u), T(v,v,\ldots,v)) \\ \leq\ & d(T(u,\ldots,u), T(u,\ldots,u,v)) + d(T(u,\ldots,u,v), T(u,\ldots,u,v,v)) + \ldots \\ & + d(T(u,v,\ldots,v), T(v,v,\ldots,v)) \\ \leq\ & q_k d(u,v) + q_{k-1} d(u,v) + \ldots + q_1 d(u,v) \\ =\ & (q_k + q_{k-1} + \ldots + q_1) d(u,v) < d(u,v), \end{aligned}$$

and consequently we conclude the implication (1.1) $\Rightarrow$ (2.2).

The following example shows that the condition (2.2) in Theorem 2 can not be omitted.

Example 1. Let $X = [0,1] \cup [2,3]$ and let $T : X^2 \to X$ be a mapping defined by

$$\begin{aligned} T(x,y) &= \frac{x+y}{4}, && \text{if}(x,y) \in [0,1] \times [0,1], \\ T(x,y) &= 1 + \frac{x+y}{4}, && \text{if}(x,y) \in [2,3] \times [2,3], \\ T(x,y) &= \frac{x+y}{4} - \frac{1}{2}, && \text{if } (x,y) \in [0,1] \times [2,3], \quad \text{or} \quad (x,y) \in [2,3] \times [0,1]. \end{aligned}$$

Then for any $x, y \in [0,1]$ we have $T(x,y) = z \in [0,1]$ and for $x, y \in [2,3]$ we have $T(x,y) = z \in [2,3]$. Thus, for $x, y \in [0,1]$, or $x, y \in [2,3]$, we have

$$\begin{aligned} d(T(x,y), T(y,z)) &= \left|\frac{x+y}{4} - \frac{y+z}{4}\right| = \left|\frac{x-y}{4} + \frac{y-z}{4}\right| \\ &\leq \left|\frac{x-y}{4}\right| + \left|\frac{y-z}{4}\right| \leq \frac{1}{2} \max\{d(x,y),\ d(y,z)\}. \end{aligned}$$

For $(x,y)\in[0,1]\times[2,3]$, or $(x,y)\in[2,3]\times[0,1]$ we have $T(x,y)=z\in[0,1]$. Therefore, if $y\in[2,3]$, then

$$d(T(x,y),T(y,z))=|\frac{x+y}{4}-\frac{y+z}{4}|\leq\frac{1}{2}\max\{d(x,y),\ d(y,z)\}.$$

If $y\in[0,1]$, then

$$\begin{aligned}d(T(x,y),T(y,z))&=|\frac{x+y}{4}-\frac{1}{2}-\frac{y+z}{4}|=|\frac{x-y}{4}-\frac{1}{2}+\frac{y-z}{4}|\\&\leq|\frac{x-y}{4}-\frac{1}{2}|+|\frac{y-z}{4}|<|\frac{x-y}{4}|+|\frac{y-z}{4}|\\&\leq\frac{1}{2}\max\{d(x,y),\ d(y,z)\}.\end{aligned}$$

Thus, T satisfies (2.1) with $\lambda=1/2$, but for $x=0$ and $y=2$ we have $T(0,0)=0$ and $T(2,2)=2$.

3. Applications

We shall illustrate an application of Theorem 2 to the convergence problem of real sequences.

Let $\{x_n\}_1^\infty$ be a real sequence, $x_1,\dots,x_k$ be a given its k members and let x_n, for $n\geq k+1$, be defined by a recursive relation:

$$x_n=\rho(x_{n-k},x_{n-k+1},\dots,x_{n-1}).$$

To investigate the convergence of $\{x_n\}_1^\infty$, it suffices to substitute T for ρ in a recursive relation assuming earlier that $T:\mathbb{R}^k\to\mathbb{R}$. If we find that T satisfies (2.1), then the convergence of $\{x_n\}_1^\infty$ will follow from Theorem 2.

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