FINITE VOLUME SCHEMES FOR NONLINEAR PARABOLIC PROBLEMS: ANOTHER REGULARIZATION METHOD

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ABSTRACT. On one hand, the existence of a solution to degenerate parabolic equations, without a nonlinear convection term, can be proven using the results of Alt and Luckhaus, Minty and Kolmogorov. On the other hand, the proof of uniqueness of an entropy weak solution to a nonlinear scalar hyperbolic equation, first provided by Krushkov, has been extended in two directions: Carrillo has handled the case of degenerate parabolic equations including a nonlinear convection term, whereas Di Perna has proven the uniqueness of weaker solutions, namely Young measure entropy solutions. All of these results are reviewed in the course of a convergence result for two regularizations of a degenerate parabolic problem including a nonlinear convective term. The first regularization is classicaly obtained by adding a minimal diffusion, the second one is given by a finite volume scheme on unstructured meshes. The convergence result is therefore only based on $L^\infty(\Omega \times (0,T))$ and $L^2(0,T;H^1(\Omega))$ estimates, associated with the uniqueness result for a weaker sense for a solution.

1. Introduction

The aim of this paper is to review a chain of various results obtained after 1960, for the approximation of the solution u to the following nonlinear parabolic/hyperbolic problem:

$$u_t + \operatorname{div}\Big(\mathbf{q}\ f(u)\Big) - \Delta\varphi(u) = 0 \quad \text{in } Q, \tag{1.1}$$

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with the initial condition

$$u(\cdot,0)=u_0 \quad \text{on } \Omega, \tag{1.2}$$

and the non homogeneous Dirichlet boundary condition

$$u=\bar{u} \quad \text{on } \partial\Omega\times(0,T), \tag{1.3}$$

denoting by $Q=\Omega\times(0,T)$, under various hypotheses on the domain Ω, the initial data u_0, the boundary conditions $\bar{u}$, the convection velocity $\mathbf{q}$, the nonlinear transport function $f : \mathbb{R}\to\mathbb{R}$ and the degenerate diffusion $\varphi : \mathbb{R}\to\mathbb{R}$. Let us only detail some of these hypotheses:

1. u_0 and $\bar{u}$ are bounded functions with $u_I\le u_0\le u_S$ and $u_I\le\bar{u}\le u_S$ a.e., and $\bar{u}$ is the trace on $\partial\Omega\times(0,T)$ of a regular function defined in Q, also denoted by $\bar{u}$,
2. the velocity field $\mathbf{q}$ is Lipschitz continuous on Q and it satisfies $\mathrm{div}\mathbf{q}=0$ (this hypothesis is not necessary, but it corresponds to a large number of physical situations), and $\mathbf{q}\cdot\mathbf{n}=0$ on $\partial\Omega\times(0,T)$ (this hypothesis prevents from the handling of boundary conditions for nonlinear hyperbolic problems),
3. the function f is Lipschitz continuous and monotonous nondecreasing (this is only assumed to simplify the expression of the Godunov scheme),
4. the function φ is Lipschitz continuous and monotonous nondecreasing, which implies a degenerate diffusion for $(x,t)\in\Omega\times(0,T)$ such that $\varphi'(u(x,t))=0$ (the case $\varphi=0$ is not excluded).

Using such weak hypotheses, it is necessary to introduce the definition of a weak entropy solution u to Problem (1.1)–(1.3):

1. $u\in L^\infty(\Omega\times(0,T))$,
2. thanks to $\mathbf{q}\cdot\mathbf{n}=0$, the Dirichlet boundary condition has only to be taken on $\varphi(u)$, namely: $\zeta(u)-\zeta(\bar{u})\in L^2(0,T;H_0^1(\Omega))$ with $\zeta(s):=\int_0^s\sqrt{\varphi'(a)}\,\mathrm{d}a$ (the function ζ such defined verifies $-\int_\Omega v\Delta\varphi(v)\,\mathrm{d}x=\int_\Omega(\nabla\zeta(v))^2\,\mathrm{d}x$ for all regular function v vanishing at the boundary),

3. to handle the case of strong degeneracy, entropy conditions (necessary to expect a uniqueness property) are introduced:

$$\int_{\Omega\times(0,T)} [\eta(u)\psi_t + \Phi(u)\ \mathbf{q}\cdot\nabla\psi - \nabla\theta(u)\cdot\nabla\psi]\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \eta(u_0(x))\psi(x,0)\,\mathrm{d}x \geq 0, \tag{1.4}$$

$$\forall\ \psi\in\mathcal{C},\quad \forall\eta\in C^1(\mathbb{R},\mathbb{R}),\quad \eta''\geq 0,\quad \Phi' = \eta'(\cdot)f'(\cdot),\quad \theta' = \eta'(\cdot)\varphi'(\cdot),$$

where the space of test functions is given by $\mathcal{C} = \{\psi\in C_c^\infty(\mathbb{R}^d\times\mathbb{R})$, with $\psi\geq 0$ and $\psi = 0$ on $\partial\Omega\times(0,T)\cup\Omega\times\{T\}\}$.

Results of existence and uniqueness were developed for such a solution. Let us first remark that, in the case where $\varphi = 0$, the problem resumes to a scalar nonlinear hyperbolic equation, for which Krushkov's works [7] were fundamental. These works include the introduction of entropies and that of the doubling variable technique for the uniqueness proof of a solution. In the case where $\varphi\neq 0$, Carrillo's works [2] have led to a clever and essential adaptation of Krushkov's method to the presence of a degenerate diffusion term. Let us examine, on a numerical simulation, the effect of a degenerate diffusion on a linear convection problem. We consider the example where $\varphi(u) = \max(u, .5)$, $f(u) = u$, $\Omega = (0,1)\times(0,1)$ and $\mathbf{q}(x_1,x_2) = \mathrm{curl}\,(x_1(1-x_1)x_2(1-x_2))$. Figure 1 shows the approximate solution for u at different times. We see that in such a case, the degenerate parabolic term makes only disappear the initial bump from $u = 0.5$ to $u = 1$ (black color in the figure), whereas the initial bump from $u = 0.5$ to $u = 0$ is convected and only smeared by the numerical diffusion.

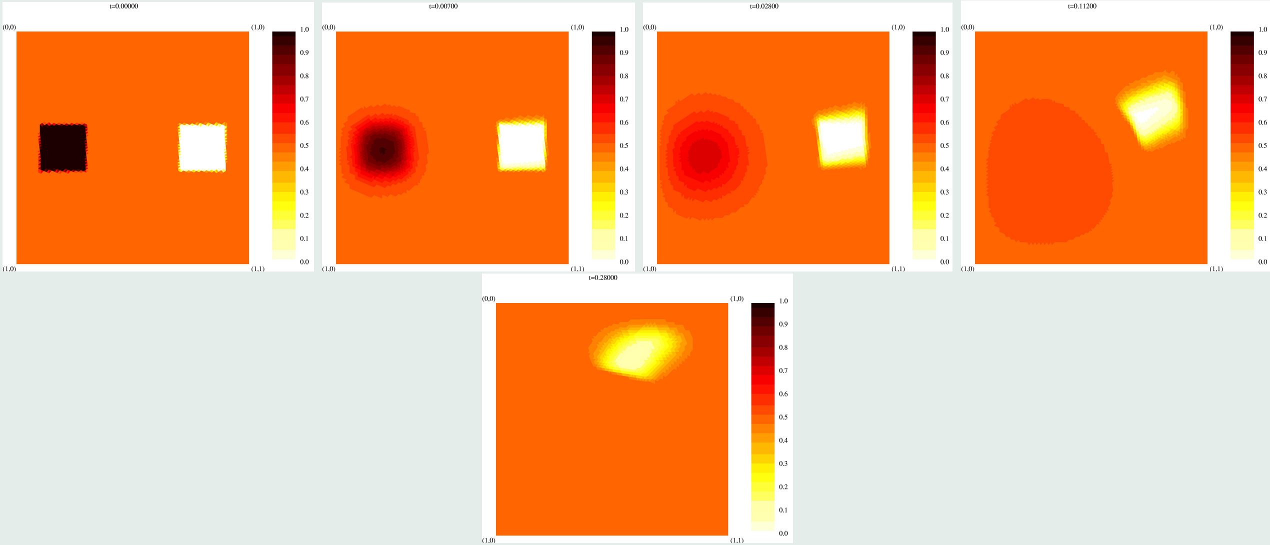

Figure 1.1. Approximate solutions u at times 0.00, 0.01, 0.04, 0.16, 0.40, from left to right. Color white stands for $u = 0$ and black for $u = 1$.

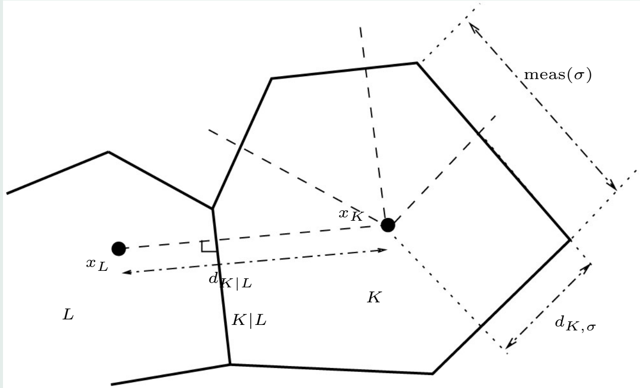

Figure 2.1. Notations and example of two control volumes of an admissible mesh..

2. Two regularization methods

We consider two types of regularized solutions. The first one is the classical strongly parabolic regularization u_ε, for $\varepsilon > 0$, solution of

$$(u_\varepsilon)_t + \mathrm{div}\Big(\mathbf{q}\ f(u_\varepsilon)\Big) - \Delta\Big(\varphi(u_\varepsilon) + \varepsilon u_\varepsilon\Big) = 0 \qquad \text{in } Q, \tag{2.1}$$

with initial and boundary conditions (1.2) and (1.3). The second one is defined using a finite volume scheme. Within the notations of [4], we use an admissible mesh $\mathcal{M}$, the control volumes of which satisfying an orthogonality property between the "centers" of the control volumes and the edges (see Figure 2).

We then introduce a constant (for simplicity) time step $\delta t > 0$, and we define the convected flux $q_{K,L}^{n+1} = \frac{1}{\delta t}\int_{n\delta t}^{(n+1)\delta t}\int_{K|L}\mathbf{q}(x,t)\cdot\mathbf{n}_{K,L}\,\mathrm{d}\gamma(x)\,\mathrm{d}t$ at time step n and at each edge $K|L$, denoting by $\mathbf{n}_{K,L}$ the unit vector, normal to $K|L$ and oriented from K to L. We denote by $\mathcal{N}_K \subset \mathcal{M}$ the set of the neighbours of K, by $\mathcal{E}_{ext} \subset \mathcal{E}$ (resp. $\mathcal{E}_{int}$) the set of the exterior (resp. interior) edges, by $\mathcal{E}_{ext,K} \subset \mathcal{E}_{ext}$ the set of the edges of K belonging to $\mathcal{E}_{ext}$, for all $s \in \mathbb{R}$ we set $s^+ = \max(s,0)$ and $s^- = \max(-s,0)$. Using the notations of Figure 2, we define the finite volume scheme by

$$
\begin{aligned}
(u_K^{n+1} - u_K^n)\,\mathrm{meas}(K) &+ \delta t \sum_{L\in\mathcal{N}_K}\Big((q_{K,L}^{n+1})^+ f(u_K^{n+1}) - (q_{K,L}^{n+1})^- f(u_L^{n+1})\Big)\\
&- \delta t \sum_{L\in\mathcal{N}_K}\frac{\mathrm{meas}(K|L)}{d_{K|L}}\big(\varphi(u_L^{n+1}) - \varphi(u_K^{n+1})\big)\\
&- \delta t \sum_{\sigma\in\mathcal{E}_{ext,K}}\frac{\mathrm{meas}(\sigma)}{d_{K,\sigma}}\big(\varphi(\bar{u}_\sigma^{n+1}) - \varphi(u_K^{n+1})\big) = 0,
\end{aligned}
\tag{2.2}
$$

in association with a standard definition for the approximation of the initial condition u_K^0 for all $K \in \mathcal{M}$, and the boundary condition $\bar{u}_\sigma^{n+1}$ for all exterior edge σ and time step n. Scheme (2.2) appears to be implicit, using the Godunov scheme for the convection term (which is the upstream weighting scheme in the present case where f is non decreasing). It is then possible to show that the implicit scheme (2.2) has at least one solution, which allows to define the function $u_{\mathcal{D}}(x,t)$ by the value u_K^{n+1} for a.e. $x \in K$ and $t \in (n\delta t, (n+1)\delta t)$. The remaining of this paper is devoted to the analysis of the convergence of these regularizations to the weak entropy solution of Problem (1.1)–(1.3).

2.1. $L^\infty(Q)$ estimate

Both regularizations satisfy the same bounds as the initial and boundary conditions:

$$u_I \le u_\varepsilon(x,t) \le u_S, \qquad \text{for a.e. } (x,t) \in Q, \tag{2.3}$$

and, for the discrete approximation,

$$u_I \le u_{\mathcal{D}}(x,t) \le u_S, \qquad \text{for a.e. } (x,t) \in Q. \tag{2.4}$$

These $L^\infty(Q)$ estimates allows for the application of the non linear weak-$\star$ compactness property [3, 4]: for any sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \in L^\infty(Q)$ for all $n \in \mathbb{N}$, which is bounded in $L^\infty(Q)$, one can extract a subsequence, again denoted $(u_n)_{n\in\mathbb{N}}$, and $u \in L^\infty(Q \times (0,1))$, such that for all continuous function $g \in C^0(\mathbb{R})$, $(g(u_n))_{n\in\mathbb{N}}$ converges to $\int_0^1 g(u(\cdot,\alpha))\,\mathrm{d}\alpha$ for the weak-$\star$ topology of $L^\infty(Q)$. This function u is then called a "process limit" of $(u_n)_{n\in\mathbb{N}}$, the word process being used with analogy to the trajectories defined by $u(\cdot,\alpha)$ for a.e. $\alpha \in (0,1)$. This notion of process limit (used in [4]) happens to be a way to define a Young measure $(x,t) \mapsto \mu_{x,t}$ (used in [3]), thanks to the relation $\int g\,\mathrm{d}\mu_{x,t} = \int_0^1 g(u(x,t,\alpha))\,\mathrm{d}\alpha$. The advantage of the notion of process limit is that the measurability properties of the function u become explicit, allowing for easier applications of the theorem of continuity in means during the course of the uniqueness proof.

We thus get the existence of a process limit u_c for u_ε as $\varepsilon \longrightarrow 0$, and u_d for $u_{\mathcal{D}}$ as $\delta(\mathcal{D}) \longrightarrow 0$ (where $\delta(\mathcal{D})$ is the maximum of the space steps and time step).

2.2. $L^2(0,T;H^1(\Omega))$ estimate

We now consider, again using the function defined by $\zeta(s) = \int_0^s \sqrt{\varphi'(a)}\,\mathrm{d}a$, the continuous function $z_\varepsilon = \zeta(u_\varepsilon) - \zeta(\bar{u})$ and the discrete one $z_{\mathcal{D}}$, defined by the discrete values $z_K^{n+1} = \zeta(u_K^{n+1}) - \zeta(\bar{u}_K^{n+1})$ in a same manner as $u_{\mathcal{D}}$. We then get the existence of a real $C_{1c} > 0$, which does not depend on ε and of a real $C_{1d} > 0$, which does not depend on the size of the discretization $\delta(\mathcal{D})$, such that:

$$\|z_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \le C_{1c}, \tag{2.5}$$

and

$$\sum_{n=0}^{N} \delta t \left(\sum_{K|L\in\mathcal{E}_{int}} \frac{\text{meas}(K|L)}{d_{K|L}} (z_K^{n+1} - z_L^{n+1}))^2 + \sum_{\sigma\in\mathcal{E}_{ext}} \frac{\text{meas}(\sigma)}{d_{K,\sigma}} (z_K^{n+1})^2 \right) \leq C_{1d}, \tag{2.6}$$

where $N \in \mathbb{N}$ is such that $N\delta t \leq T < (N+1)\delta t$. Each of these relations implies a space translate estimate, which writes in the first case

$$\int_0^T \int_{\mathbb{R}^d} (z_\varepsilon(x+\xi,t) - z_\varepsilon(x,t))^2 \, \mathrm{d}x \, \mathrm{d}t \leq C_{1c}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \tag{2.7}$$

and in the second one (see [4])

$$\int_0^T \int_{\mathbb{R}^d} (z_\mathcal{D}(x+\xi,t) - z_\mathcal{D}(x,t))^2 \, \mathrm{d}x \, \mathrm{d}t \leq C_{1d}|\xi|(|\xi| + 4\,\delta(\mathcal{D})), \quad \forall \xi \in \mathbb{R}^d. \tag{2.8}$$

Both results are a first step in direction to the application of Kolmogorov's theorem, proving the relative compactness of the families z_ε, for $\varepsilon > 0$ and $z_\mathcal{D}$, for all admissible discretization $\mathcal{D}$. The second step is handled in the next subsection.

2.3. Time translate estimate

The use of time translate estimates for degenerate parabolic equations is first due to Alt and Luckhaus [1], since standard functional arguments cannot be easily adapted to the time derivatives of functions z_ε and $z_\mathcal{D}$. The existence of some $C_{2c} > 0$, which does not depend on ε and of some $C_{2d} > 0$ which does not depend on $\delta(\mathcal{D})$, such that:

$$\int_0^{T-s} \int_{\mathbb{R}^d} (z_\varepsilon(x,t+s) - z_\varepsilon(x,t))^2 \, \mathrm{d}x \, \mathrm{d}t \leq C_{2c}\, s, \quad \forall s \in (0,T) \tag{2.9}$$

and

$$\int_0^{T-s}\int_{\mathbb{R}^d}(z_{\mathcal{D}}(x,t+s)-z_{\mathcal{D}}(x,t))^2\,\mathrm{d}x\,\mathrm{d}t \le C_{2d}\ s, \quad \forall s\in(0,T) \tag{2.10}$$

are proven (in the case of degenerate equations without convective terms, inequality (2.10) has been proven in [6]). Note that in the case of variable time steps, one must replace s in the right hand side of (2.10) by $s+\delta(\mathcal{D})$, which leads to a slight modification in the verification of the hypotheses of Kolmogorov's theorem. It is now possible to express a relative compactness property.

3. Compactness and monotony

Thanks to the space and time translate estimates, we have now got some strong convergence for z_ε and $z_{\mathcal{D}}$. For the continuous regularization, we thus have proven the following results: there exists a sequence $(u_{\varepsilon_n})_{n\in\mathbb{N}}$ with ε_n tends to 0 as $n\to\infty$ such that

1. u_{ε_n} converges to some function $u_c\in L^\infty(Q\times(0,1))$ in the nonlinear weak-$\star$ sense,
2. $z_{\varepsilon_n}=\zeta(u_{\varepsilon_n})-\zeta(\bar u)\to z_c$ in $L^2(Q)$ as $\varepsilon\longrightarrow 0$, and $z_c\in L^2(0,T;H_0^1(\Omega))$.

In the discrete case, we have proven that there exists a sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ with $\delta(\mathcal{D}_n)$ tends to 0 as $n\to\infty$ such that

1. $u_{\mathcal{D}_n}$ converges to some function $u_d\in L^\infty(Q\times(0,1))$ in the nonlinear weak-$\star$ sense,
2. $z_{\mathcal{D}_n}=\zeta(u_{\mathcal{D}_n})-\zeta(\bar u_{\mathcal{D}_n})\to z_d$ in $L^2(Q)$ as $n\to\infty$, and $z_d\in L^2(0,T;H_0^1(\Omega))$.

Then, using the Minty monotony argument [8], classicaly used in this framework, we get that, for a.e. $(x,t,\alpha)\in Q\times(0,1)$, $z_c(x,t)=\zeta(u_c(x,t,\alpha))-\zeta(\bar u(x,t))$ and $z_d(x,t)=\zeta(u_d(x,t,\alpha))-\zeta(\bar u(x,t))$. Intuitively, this result means that the strong convergence of z_ε or $z_{\mathcal{D}}$ prevents u_ε or $u_{\mathcal{D}}$ from oscillating around values such that $\varphi'>0$, which implies that $\zeta(u_c(x,t,\alpha))$ and $\zeta(u_d(x,t,\alpha))$ do not depend on α for a.e. $(x,t)\in Q$. At this stage, there is not yet an evidence that u_c and u_d don't depend on α for a.e. $(x,t)\in Q$. This will be handled in the next section.

4. Uniqueness theorem

Thanks to the passage to the limit in the equations leading to the definition of both regularizations, we show that the functions u_c and u_d are entropy weak process solutions [5] to Problem (1.1)–(1.3), where we say that a function u is an entropy weak process solution to Problem (1.1)–(1.3) if it satisfies

1. $u \in L^\infty(Q \times (0,1))$,
2. $\zeta(u(x,t,\alpha))$ does not depend on α for a.e. $(x,t) \in \Omega \times (0,T)$ and $\zeta(u) - \zeta(\bar{u}) \in L^2(0,T; H_0^1(\Omega))$,
3. a first kind of entropy inequalities is satisfied

$$\int_Q \Big[\int_0^1 (\mu(u(\cdot,\alpha))\ \psi_t + \nu(u(\cdot,\alpha))\ \mathbf{q}\cdot\nabla\psi)\,\mathrm{d}\alpha - \nabla\eta(\varphi(u))\cdot\nabla\psi - \eta''(\varphi(u))(\nabla\varphi(u))^2\psi \Big]\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \mu(u_0)\psi(\cdot,0)\,\mathrm{d}x \geq 0, \tag{4.1}$$

   for all $\psi \in \mathcal{C}$ and for all regular convex function η, setting $\mu' = \eta'(\varphi(\cdot))$, $\nu' = \eta'(\varphi(\cdot))f'(\cdot)$,
4. a second kind of entropy inequalities is satisfied

$$\int_Q \Big[\int_0^1 (|u-\kappa|\psi_t + (f(\max(u,\kappa)) - f(\min(u,\kappa)))\ \mathbf{q}\cdot\nabla\psi)\,\mathrm{d}\alpha - \nabla|\varphi(u)-\varphi(\kappa)|\cdot\nabla\psi \Big]\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega |u_0-\kappa|\psi(\cdot,0)\,\mathrm{d}x \geq 0, \tag{4.2}$$

   for all $\psi \in \mathcal{C}$ and for all $\kappa \in \mathbb{R}$, where one recognizes the Krushkov entropy pair $|\cdot - \kappa|$, $f(\max(\cdot,\kappa)) - f(\min(\cdot,\kappa)) = |f(\cdot) - f(\kappa)|$ in the particular case where f is monotonous nondecreasing (remark that the two entropy criteria cannot be deduced one from each other).

We then have the following result: the entropy weak process solution to Problem (1.1)–(1.3) is unique, and thus does not depend on α, resuming to the entropy weak solution, which is also unique. This result is proven in [5], following the doubling variable technique introduced by Krushkov, adapted to Young measures by Di Perna [3]. The proof uses Carrillo's method, which is an adaptation to the doubling variable technique of the following simple result: for all $\eta \in C^2(\mathbb{R})$ with $\eta'' \geq 0$, and for all u, v such that $u_t - \Delta u = 0$ and $v_t - \Delta v = 0$, then $\eta(u-v)_t - \Delta\eta(u-v) \leq 0$.

5. Conclusion: strong convergence of the regularizations

We have now obtained that both regularizations converge to the entropy weak solution in the nonlinear weak-$\star$ sense. In fact, the uniqueness result implies that the convergence is strong in all $L^p(Q)$, for all $p \in [1, +\infty)$. This result is an immediate consequence of the definition of the nonlinear weak-$\star$ sense and of the fact that $u(x, t, \alpha)$ does not depend on α (see [3] or [4]). This concludes the proof that both regularizations strongly converge to the entropy weak solution of Problem (1.1)–(1.3). This conclusion shows that the finite volume scheme, which permits to define piecewise constant functions and therefore to handle simple real values, indeed behaves as a standard regularization method. A large advantage of such an approximation is that all algebraic operations are possible, without functional space considerations.

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