# THE $p$-LAPLACIAN – MASCOT OF NONLINEAR ANALYSIS

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ABSTRACT. In this paper we give survey of the results concerning the solvability of certain quasilinear boundary value problems which contain the $p$-Laplacian as a main part. Special attention is paid to the resonance problems which lead naturally to the study of spectral properties of nonlinear operators. Besides statements of principal results in the historical context we also mention some open problems which seem to be a great challenge for nonlinear analysts.

## 1. INTRODUCTION

The word "$p$-Laplacian" has become a key word in nonlinear analysis and problems involving this second order quasilinear operator are now extensively studied in the literature. One can ask: "Why there are so many papers dealing with this topic?" We believe that there are several reasons for that. On one hand, there are some serious ones. It appears that certain nonlinear mathematical models lead to differential equations with the $p$-Laplacian. One of them describing the behavior of compressible fluid in a homogeneous isotropic rigid porous medium is presented below. But also some purely mathematical properties of the $p$-Laplacian seem to be a challenge for nonlinear analysts and their study lead to the development of new methods and approaches. On the other hand, there are some less serious reasons for having around so many papers on this topic. The structure of the $p$-Laplacian allows also some "cheap" or "easy doable" generalizations which do not bring anything new but not much interesting extensions from semilinear to quasilinear case. The only purpose of such publications is to collect "points" in order to get better "grade" in the author's promotion, evaluation of her/his research team,

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etc. Unfortunately, the regulations of the "research money flows" in most of the countries cause that the number of such papers is exponentially increasing!

In this paper we want to focus on the results which demonstrate the striking difference between the nonlinear ($p \neq 2$) and linear ($p = 2$) case and which, before they have been proven, required a long time of testing and bugging.

In order to avoid complicated notation and to make our exposition as clear as possible, we do not give precise definitions and do not formulate all assumptions in detail. We handle most of the notions only intuitively and the reader who is interested in precise statements of the results presented in this survey paper is kindly requested to consult the literature we are referring to.

## 2. One mathematical model and functional setting

In this section we present mathematical model of the behavior of compressible fluid in a homogeneous isotropic rigid porous medium. Let $\rho = \rho(x,t)$ denote the density, $\varphi$ be a volumetric moisture content and $\overrightarrow{V} = \overrightarrow{V}(x,t)$ be a seepage velocity. Then the continuity equation reads as follows:

$$\varphi \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \overrightarrow{V}) = 0. \tag{1}$$

In the laminar regime through the porous medium the momentum velocity $\rho\overrightarrow{V}$ and the pressure $P = P(x,t)$ are connected by the Darcy law

$$\rho \overrightarrow{V} = -\lambda \operatorname{grad} P. \tag{2}$$

In turbulent regimes, however, the flow rate is different and several authors proposed a nonlinear relation instead of (2). Namely, the nonlinear Darcy law of the following form is often considered (see e.g. Wu et al. [28]):

$$\rho \overrightarrow{V} = -\lambda \left|\operatorname{grad} P\right|^{\alpha-2} \operatorname{grad} P, \tag{3}$$

where $\alpha > 1$ is a suitable real constant. Taking into account the equation of state for the polytropic gas

$$P = c\,\rho$$

with some constant of proportionality $c > 0$, we get from (1) and (3) the equation

$$\varphi \frac{\partial \rho}{\partial t} = c^{\alpha-1} \lambda \operatorname{div}\left(|\operatorname{grad} \rho|^{\alpha-2} \operatorname{grad} \rho\right).$$

After the change of variables and notations this equation becomes

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2} \nabla u), \tag{4}$$

where $p > 1$ is a real number.

The following notation is widely used in the literature:

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u),$$

and

$$u \longmapsto \Delta_p u \tag{5}$$

is called the $p$-Laplacian or the $p$–Laplace operator. Obviously, $\Delta_2 = \Delta$ is the usual Laplace operator and in the case of one spatial dimension, we have $\Delta_p u = (|u'|^{p-2} u')'$. Note also that for $p \neq 2$ the operator (5) is $(p-1)$-homogeneous but not additive. For this reason some of the authors, in particular those who work in ODEs, call equations involving the $p$-Laplacian "half–linear" equations. The word "half–linear" reflects the fact that "one half" of the properties of linearity (i. e. additivity) is lost, while "one half" of the properties is preserved (i. e. homogeneity).

In the forthcoming text we focus on the stationary case and discuss the solvability of quasilinear boundary value problems of the type

$$
\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}
$$

In this paper we deal exclusively with the Dirichlet boundary conditions and the notion of the solution is always understood in a weak sense. The natural space for such a solution is the Sobolev space $W_0^{1,p}(\Omega)$ and the solution of (6) is defined as a function $u \in W_0^{1,p}(\Omega)$ which satisfies the integral identity

$$
\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f(x,u) v \, \mathrm{d}x
$$

for all test functions $v \in W_0^{1,p}(\Omega)$.

The boundary value problem (6) can be thus formulated as an operator equation

$$
J(u) = F(u),
$$

where $J, F \colon W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ are defined by

$$
(J(u), v) = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, \mathrm{d}x, \quad (F(u), v) = \int_\Omega f(x,u) v \, \mathrm{d}x,
$$

for any $u, v \in W_0^{1,p}(\Omega)$ and $(\cdot,\cdot)$ is the duality pairing between $(W_0^{1,p}(\Omega))^*$ and $W_0^{1,p}(\Omega)$.

Let us emphasize the main troubles which one meets in case $p \neq 2$ and which make the quasilinear case so different from the semilinear one:

- the lack of the Hilbert structure of the space $W_0^{1,p}(\Omega)$ when passing from $p = 2$ to $p \neq 2$;
- the Lyapunov–Schmidt reduction does not “decompose” the operator $J$ to invariant subspaces if $p \neq 2$;
- the spectral properties of $J$ are much more complicated and still not completely understood if $p \neq 2$.

## 3. Eigenvalue problem

The "natural" eigenvalue problem for the $p$-Laplacian reads as follows:

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

The value of parameter $\lambda \in \mathbb{R}$ for which (7) has a nonzero solution $u \neq 0$ is called an eigenvalue of (7) and corresponding $u$ is called an eigenfunction associated with $\lambda$. It is well understood in one dimension (e.g. $\Omega = (0,1)$). In this case the situation is similar for any $p > 1$: the set of all eigenvalues forms an increasing sequence

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \to \infty$$

approaching infinity, every $\lambda_n$ is simple in the sense that there is exactly one eigenfunction $u_n$ associated with $\lambda_n$ normalized by $u_n(0) = 1$. Since every $\lambda_n$ and $u_n$ can be expressed explicitly in terms of a Beta function (see e.g. Elbert [20] as one of the first references in this direction), more precise information about the structure of $u_n$ can be given. Moreover, every $\lambda_n$ can be characterized variationally (see e.g. Cuesta [8]), using the Courant–Weinstein principle in the linear case $p = 2$ and using the Lyusternik–Schmirelmann minimax principle in the nonlinear case $p \neq 2$. Roughly speaking, in one dimension, the properties of the linear eigenvalue problem extend to the nonlinear one.

In higher dimension the situation is different and the above claim is true only for the principal eigenvalue $\lambda_1$ and partially also for the second eigenvalue $\lambda_2$. Namely, in any dimension, we have

$$\lambda_1 = \min_{\substack{u \in W_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_\Omega |\nabla u|^p \,\mathrm{d}x}{\displaystyle\int_\Omega |u|^p \,\mathrm{d}x},$$

it is simple and isolated and the corresponding eigenfunction $\varphi_1$ can be taken positive in $\Omega$. Even Hopf maximum principle holds for general $p > 1$ (there are many references in the literature, let us mention Anane [1] among others). It is also known that there exists the least eigenvalue next to $\lambda_1$, called the second eigenvalue $\lambda_2$ which allows a minimax characterization and every eigenfunction associated with it changes sign in $\Omega$ exactly once (see e.g. Anane and Tsouli [3] or Drábek and Robinson [18]). Using similar minimax formula it is possible to construct a sequence $\lambda_n$ of the so called variational eigenvalues which approach infinity. In case $p = 2$ those are the only eigenvalues of (7) and to prove this fact the linearity plays the key role. The question of the existence of other eigenvalues of (7) than the variational ones remains open if $p \neq 2$. There are some indications showing that if there are nonvariational eigenvalues of (7) they are "less important" than variational ones in a certain sense. For example, we proved in Drábek and Robinson [18] that the number of nodal domains of the eigenfunction associated with (possibly nonvariational) eigenvalue $\leq \lambda_n$ is at most equal to $2n-2$. Under the assumption that unique continuation property holds for the $p$-Laplacian (which is an open problem according to the best author's knowledge) we also showed that this estimate can be improved to $n$. This corresponds to the well known Courant Nodal Domain Theorem for the linear case $p = 2$.

It is worth mentioning explicitly that a lot of well known facts break down when we pass from the linear problem to a nonlinear one. For instance the multiplicity of the eigenvalue becomes to be a very different issue if $p \neq 2$. While in the linear case having two linearly independent eigenfunctions associated with a given eigenvalue, any linear combination is again an eigenfunction associated with the same eigenvalue. This need not be true in the nonlinear case.

In order to summarize the main questions related to the eigenvalue problem (7) with $p \neq 2$ which represent the challenge for nonlinear analysts, we close this section as follows:

- Are there any "nonvariational" eigenvalues of (7) which are not characterized using a suitable minimax formula?
- If there are some nonvariational eigenvalues then how many? Can we have a continuum of nonvariational eigenvalues?

- Is $\lambda_2$ isolated or not? i. e. is there a sequence of (nonvariational) eigenvalues $\lambda_k$ of (7) such that $\lambda_k \searrow \lambda_2$?

## 4. Solvability of nonlinear problems: nonresonance case

In this section we focus on the existence of solutions to the boundary value problem

$$\begin{cases} -\Delta_p u = h(x,u) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

in the case when

$$\lim_{u \to \pm\infty} \frac{h(x,u)}{|u|^{p-2}u} \tag{9}$$

does not "interact" (e.g. it is not equal to) with the eigenvalues of $\Delta_p$. Such condition is easy to formulate when we know precisely the structure of all eigenvalues (e.g. in one dimension) or if the above limits are either below $\lambda_1$ or between $\lambda_1$ and $\lambda_2$ in higher dimensions. Otherwise, such condition is rather abstract and impossible to verify. In principle, the nonresonant assumptions yield the following type of the result:

"For any right hand side $f$ the problem (8) has at least one solution."

The results of this type were studied e.g. in Fučík et al. [22] and Boccardo at al. [7] using the topological argument based on the degree theory. Note also that the nonresonance was formulated more generally by means of lim inf and lim sup rather than (9).

## 5. Solvability of nonlinear problems: resonance case

If $h(x,u) = \lambda|u|^{p-2}u + g(x,u)$ where $g$ is bounded and $\lambda$ is an eigenvalue of $\Delta_p$, then

$$\lim_{u\to\pm\infty} \frac{h(x,u)}{|u|^{p-2}u} = \lambda$$

and so (9) "interact" with eigenvalue $\lambda$. Let us start with the case $\lambda = \lambda_1$ and consider problem

$$\begin{cases} -\Delta_p u = \lambda_1|u|^{p-2}u + g(x,u) + f(x) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where $g$ has finite limits

$$g(x,\pm\infty) := \lim_{s\to\pm\infty} g(x,s).$$

The well–known condition of Landesman and Lazer [23] for semilinear case (p=2),

$$\int_\Omega g(x,+\infty)\varphi_1(x)\,\mathrm{d}x < \int_\Omega f(x)\varphi_1(x)\,\mathrm{d}x < \int_\Omega g(x,-\infty)\varphi_1(x)\,\mathrm{d}x \tag{11}$$

extends also to the quasilinear case ($p \neq 2$) and it was proved that it is sufficient for the existence of at least one solution of (10). This generalization of result of Landesman and Lazer was proved by Boccardo, Drábek and Kučera [6] using the topological argument based on the degree theory and by Anane and Gossez [2] using the variational approach showing the existence of a global minimizer of the energy functional associated with (10).

In the linear case ($p = 2$), Landesman and Lazer [23] showed that also "reversed" condition

$$\int_\Omega g(x,-\infty)\varphi_1(x)\,\mathrm{d}x < \int_\Omega f(x)\varphi_1(x)\,\mathrm{d}x < \int_\Omega g(x,\infty)\varphi_1(x)\,\mathrm{d}x \tag{12}$$

is sufficient for the existence of solution of (10). The authors of [6] were able to prove the same result only in the case $p > N$ (using the embedding $W_0^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$). For general $p > 1$ this was proved using the variational argument based on the saddle point theorem by Arcoya and Orsina [4].

It is an interesting fact that this result can be generalized to arbitrary (possibly nonvariational) eigenvalue. Namely, let us consider problem

$$\left\{ \begin{array}{rcll} -\Delta_p u & = & \lambda |u|^{p-2}u + g(x,u) + f(x) & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{13}$$

Then if one of the inequalities

$$\int_\Omega f(x)\varphi(x)\,\mathrm{d}x > (<) \int_{\{\Omega,\varphi>0\}} g(x,+\infty)\varphi(x)\,\mathrm{d}x + \int_{\{\Omega,\varphi<0\}} g(x,-\infty)\varphi(x)\,\mathrm{d}x$$

holds for any eigenfunction $\varphi$ associated with $\lambda$, the problem (13) has a solution. The proof was given in Drábek and Robinson [17] using the variational argument based on the linking theorem. It is interesting to point out here that even if $\lambda$ can be a nonvariational eigenvalue, the variational eigenvalues of $\Delta_p$ and their characterization played the key role in the proof. This is one of the arguments supporting the fact that if there are some nonvariational eigenvalues they are not “very important”.

Let us close this section by the following observation (cf. Drábek, Girg and Takáč [15]). Consider the problem

$$\left\{ \begin{array}{rl} -\Delta_p u = \lambda_1 |u|^{p-2}u + \varepsilon \arctan(u) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{14}$$

Then referring to the discussion above, if $\varphi_1$ is normalized by $\int_\Omega \varphi_1(x)\,\mathrm{d}x = 1$,

$$-\varepsilon\frac{\pi}{2} < \int_\Omega f(x)\varphi_1(x)\,\mathrm{d}x < \varepsilon\frac{\pi}{2} \tag{15}$$

is sufficient condition for solvability of (14). Letting $\varepsilon \to 0_+$ one could be driven to the conclusion that

$$\int_\Omega f(x)\varphi_1(x)\,\mathrm{d}x = 0 \tag{16}$$

is sufficient condition for solvability of

$$\begin{cases} -\Delta_p u &= \lambda_1 |u|^{p-2}u + f(x) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{cases} \tag{17}$$

However, the argument is not so simple! First of all, condition (15) is empty for $\varepsilon = 0$. Second, even if for any $\varepsilon > 0$ there is a solution $u_\varepsilon$ of (14) for $f$ satisfying (16), we cannot pass to the limit for $\varepsilon \to 0_+$ because of the lack of a priori estimates of $u_\varepsilon$. These two facts make the study of (17) more difficult and more interesting.

## 6. Fredholm alternative

Let us consider the problem

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

For $p = 2$ it is well known that whenever $\lambda$ is not an eigenvalue of $\Delta$, problem (18) has a unique solution for arbitrary right hand side $f$. Let us consider general $p > 1$. Due to the strong monotonicity argument for any $f$ the problem (18) has unique solution whenever $\lambda \le 0$. However, the situation is much different if $\lambda > 0$ and $p \ne 2$! In fact, for $\lambda > 0$, $p \ne 2$ there always exists $f$ such that the problem (18) has at least two distinct solutions. This observation was made first for $p > 2$ and $\lambda \in (0, \lambda_1)$ by Del Pino, Elgueta and Manásevich [11], then it was extended to $\lambda \in (0, \lambda_1)$ and $1 < p < 2$ by Fleckinger at al. [21] and finally to $\lambda > \lambda_1$, $p \ne 2$ by Drábek and Takáč [19]. It followed from here that the Fredholm alternative type result for (18) will not extend directly from the linear case $p = 2$ to a nonlinear one $p \ne 2$.

As we already addressed in the previous section it is legitimate to relate the condition

$$\int_\Omega f(x)\varphi_1(x)\,\mathrm{d}x = 0 \tag{19}$$

and the existence of a solution of problem (18) with $\lambda = \lambda_1$. However, as we also pointed out it is not so easy and straightforward to prove that for $p \neq 2$ this condition implies the existence of a solution. It is a well known fact that for $p = 2$ condition (19) is necessary and sufficient for solvability of

$$\begin{cases} -\Delta u - \lambda_1 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Actually, it was not difficult to show that for $p \neq 2$ there is $f$ which violates (19) and the problem

$$\begin{cases} -\Delta_p u - \lambda_1 |u|^{p-2}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{20}$$

still has a solution (see e.g. Binding, Drábek and Huang [5]). This is equivalent to saying that (19) is not necessary condition for solvability of (20) if $p \neq 2$. It took a while to prove that (19) is sufficient condition for solvability of (20) and to understand the structure of all right hand sides $f$ for which (20) has a solution was not an easy issue. Probably the paper by Del Pino, Drábek and Manásevich [10] was the first one which threw the light into this problem and which showed how the problem behaves. However, the authors of [10] were able to deal only with one dimensional case and with the right hand side $f$ being of class $C^1$. This smoothness assumption was removed by Manásevich and Takáč [24] and the authors touched also the Fredholm alternative at the second eigenvalue. While the methods used in [10] rely on the classical shooting argument, the main tool of [24] is the nonlinear version of the Prüfer transformation. The paper by Drábek, Girg and Manásevich [13] then brought characterization of the set of all right hand sides $f$ for which problem (20) has a solution and presented also some multiplicity results.

All these results do not extend easily to higher dimensional case since the methods used in [10, 13, 24] depend essentially on the fact that the problem is one dimensional. Let us mention the following three pioneering papers which handled the higher dimensional case: Drábek and Holubová [16], Takáč [25, 26] and Drábek [12]. Let us point out that a very accurate analysis had to be done in order to prove results which extend from the one dimensional case to higher dimensions. The approach combines the variational methods, topological arguments and the methods of lower and upper solutions. One of the most delicate issues consists in the fact that linearizing the $p$-Laplacian around the first eigenfunction one has to change the functional setting and pass from the Sobolev space $W_0^{1,p}(\Omega)$ and the Lebesgue space $L^p(\Omega)$ to suitable weighted spaces, where the norms are defined by

$$\left(\int_\Omega |\nabla\varphi_1|^{p-2}|\nabla u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \text{ and } \left(\int_\Omega \varphi_1^{p-2}u^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

This fact makes the problem difficult but extremely interesting. For more details we refer to the survey paper Takáč [27].

Now, it is time to explain, without too many technical details, what are the main results concerning the solvability of problem (20). First of all, condition (19) is sufficient for the existence of solution of (20).

Let us denote by $\mathcal{R}$ the set of all $f \in L^\infty(\Omega)$ for which (20) has a solution. Then as we mentioned above,

$$\{f^\top \in L^\infty(\Omega)\colon \int_\Omega f^\top\varphi_1 \,\mathrm{d}x = 0\} \subset \mathcal{R}.$$

But even more is true for $p \neq 2$. Namely, given $f^\top \neq 0, \int_\Omega f^\top\varphi_1 \,\mathrm{d}x = 0$ there exists $\rho > 0$ such that

$$\{f \in L^\infty(\Omega)\colon \|f - f^\top\|_{L^\infty(\Omega)} \leq \rho\} \subset \mathcal{R}.$$

In other words, the set $\mathcal{R}$ has a nonempty interior with respect to the topology given by the norm $\|\cdot\|_{L^\infty(\Omega)}$.

Obviously, the set $\mathcal{R}$ is homogeneous, i. e. $f \in \mathcal{R}$ implies that $tf \in \mathcal{R}$ for any $t \in \mathbb{R}$. The structure of the set $\mathcal{R}$ can be also characterized as follows.

*For any $f^\top \neq 0, \int_\Omega f^\top \varphi_1 \,\mathrm{d}x = 0$ there exist $T_1 < 0 < T_2$ such that problem* (20) *with $f \in L^\infty(\Omega)$ written as $f = t + f^\top$*

(i) *has no solution if $t \in (-\infty, T_1) \cup (T_2, +\infty)$;*
(ii) *at least one solution if $t = T_i, i = 1, 2,$ or $t = 0$;*
(iii) *at least two solutions if $t \in (T_1, 0) \cup (0, T_2)$.*

Note that the problem (20) has a variational structure and the corresponding energy functional associated with (20) has the form

$$E(u) := \frac{1}{p} \int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda_1}{p} \int_\Omega |u|^p \,\mathrm{d}x - \int_\Omega f\, u \,\mathrm{d}x. \tag{21}$$

The following observations (b) and (c) concerning the energy functional $E$ for $p \neq 2$ are interesting and they were made in above mentioned papers. Let $\int_\Omega f \varphi_1 \,\mathrm{d}x = 0$. Then

(a) *for $p = 2$ the functional $E$ is bounded from below, it has an affine one dimensional set of global minimizers which are the only critical points of $E$* (this is a well known fact);
(b) *for $p > 2$ the functional $E$ is bounded from below, it has a global minimizer and the set of all its critical points is bounded;*
(c) *for $1 < p < 2$ the functional $E$ is unbounded from below, the set of its critical points is nonempty and bounded.*

Also this section can be closed by open questions which challenge nonlinear analysts and force them to look for new methods and approaches:

- Find characterization of all $f$ for which (18) has a solution where $\lambda$ is a higher eigenvalue of $\Delta_p$.
- Is the condition $\int_\Omega f \varphi \,\mathrm{d}x = 0$, for any eigenfunction $\varphi$ associated with the eigenvalue $\lambda$ (different from $\lambda_1$) of $\Delta_p$, sufficient for the solvability of (18)?

## 7. Asymptotic bifurcation approach

In this section we discuss some results based on the bifurcation from infinity which provide the information about the existence and multiplicity of the solution of the problem

$$
\begin{cases} -\Delta_p u - \lambda |u|^{p-2} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{22}
$$

where $\lambda$ is close or possibly equal to $\lambda_1$. The situation is precisely described by the Fredholm alternative in the linear case $p = 2$ and it can be expressed as follows:

(i) if $\int_\Omega f\varphi_1 \,\mathrm{d}x \neq 0$ then (22) has unique solution when $\lambda \neq \lambda_1, \lambda$ close to $\lambda_1$, and the norm of solution blows up if $\lambda \to \lambda_1$; there is no solution for $\lambda = \lambda_1$;

(ii) if $\int_\Omega f\varphi_1 \,\mathrm{d}x = 0$ then (22) has infinitely many solutions for $\lambda = \lambda_1$.

In paper Drábek, Girg, Takáč and Ulm [14] we deal with case $p \neq 2$ and study how the situation expressed by (i) and (ii) changes. In order to explain the results we need some notation. We shall write solution of (22) as $u = t^{-1}(\varphi_1 + v^\top), t \in \mathbb{R} \setminus \{0\}, \int_\Omega v^\top \varphi_1 \,\mathrm{d}x = 0$ and $\lambda = \lambda_1 + \mu$. We show that there are solutions of (22) with $\mu$ and $t$ small (this is done by the argument based on the bifurcation from infinity at $\lambda = \lambda_1$) and that they obey the following asymptotics

$$
\begin{aligned}
\mu = &- |t|^{p-2} t \int_\Omega f\varphi_1 \,\mathrm{d}x + (p-2)|t|^{2(p-1)} Q_0(V^\top, V^\top) \\
&+ (p-1)|t|^{2(p-1)} \left( \int_\Omega f\varphi_1 \,\mathrm{d}x \right) \left( \int_\Omega \varphi_1^{p-1} V^\top \,\mathrm{d}x \right) \\
&+ o(|t|^{2(p-1)})
\end{aligned} \tag{23}
$$

as $t \to 0$. Here $|t|^{-p} t v^\top \to V^\top$ in a certain sense, $V^\top \neq 0$ and $Q_0$ is a bilinear form satisfying $Q_0(V^\top, V^\top) > 0$.

This estimate provides the information about the behavior of bifurcation branch on one hand but also it is an a priori estimate for solutions of (22) on the other hand. For example, it follows immediately from (23) that for $f$ satisfying $\int_\Omega f \varphi_1 \, \mathrm{d}x = 0$ all solutions of (20) are a priori bounded if $p \neq 2$. It also follows that for such $f$ there are no large solutions of (22) for $1 < p < 2$, $\lambda > \lambda_1$ and for $p > 2$, $\lambda < \lambda_1$. One can also "read" from (23) also other properties of the bifurcation branch and the reader can find many instructive pictures in [14] and also in Čepička, Drábek and Girg [9] where we included also several numerical experiments.

In order to formulate some of the results which follow from [14], we write $f$ as $f = f^\top + a\varphi_1$, $\int_\Omega f^\top \varphi_1 \, \mathrm{d}x = 0$, $f^\top \neq 0$, $a \in \mathbb{R}$.

We have the following existence and multiplicity results:

(E1) *For $\lambda = \lambda_1$, $a = 0$, problem (22) has at least one solution; all possible solutions of (22) are a priori bounded in $C^{1,\beta}(\overline{\Omega})$, $0 < \beta < 1$, by a constant which depends on $f^\top$.*

(E2) *There exist $a_0 = a_0\left(f^\top\right) > 0$ and $\delta = \delta\left(f^\top\right) > 0$ such that*

- *if either $\lambda \in (\lambda_1 - \delta, \lambda_1)$ and $a \geq a_0$, or else $\lambda \in (\lambda_1, \lambda_1 + \delta)$ and $a \leq -a_0$, then problem (22) can have only positive solutions;*
- *if either $\lambda \in (\lambda_1 - \delta, \lambda_1)$ and $a \leq -a_0$, or else $\lambda \in (\lambda_1, \lambda_1 + \delta)$ and $a \geq a_0$, then problem (22) can have only negative solutions.*

(M1) *There exists $\eta = \eta\left(f^\top\right) > 0$ such that for $a = 0$ problem (22) has at least three distinct solutions (among them at least one positive and one negative) provided either $1 < p < 2$ and $\lambda \in (\lambda_1 - \eta, \lambda_1)$, or $p > 2$ and $\lambda \in (\lambda_1, \lambda_1 + \eta)$.*

(M2) *There exists $\varepsilon > 0$ with the following properties:*

- *for every $\varepsilon' \in (0, \varepsilon)$, there is $\eta = \eta\left(f^\top, \varepsilon, \varepsilon'\right) > 0$ such that $\varepsilon' < |a| < \varepsilon$ and $\lambda \in (\lambda_1 - \eta, \lambda_1) \cup (\lambda_1, \lambda_1 + \eta)$ imply that problem (22) has at least three distinct solutions, of which at least one is positive and at least one is negative;*

- *$\lambda = \lambda_1$ and $0 < |a| < \varepsilon$ imply that problem (22) has at least two distinct solutions, of which at least one is negative if $(p-2)a < 0$, and at least one is positive if $(p-2)a > 0$.*

Numerical experiments presented in [9] allowed to draw some bifurcation diagrams from higher eigenvalues of one dimensional $p$-Laplacian and to get some idea about the global behavior of the bifurcation branches. However, also here we are left with open questions like the following ones:

- Consider solutions $u = t^{-1}(\varphi_1 + v^\top)$ of (22), large in the norm, which obey the asymptotic estimate (23). Is the solution $u$ unique for a given (small) $t$?
- What is the global behavior of the bifurcation branch?

## 8. Resonance problems revisited

The asymptotic estimate (23) can be actually modified to get new results for the existence and multiplicity of solutions of the problem

$$\begin{cases} -\Delta_p u = \lambda_1 |u|^{p-2}u + g(u) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{24}$$

where $g$ is bounded and continuous function having the limits $g(\pm\infty) = \lim\limits_{s\to\pm\infty} g(s)$. This is done in paper Drábek, Girg and Takáč [15] and below we shall list some of them.

(E3) *Let $g(-\infty) = g(+\infty) = 0$, $f \in L^\infty(\Omega)$, $g(s)s \geq 0$ for all $s \in \mathbb{R}$ if $1 < p < 2$ or $g(s)s \leq 0$ for all $s \in \mathbb{R}$ if $2 < p < \infty$. Then problem (24) has at least one solution provided $\int_\Omega f\,\varphi_1 \mathrm{d}x = 0$.*

(E4) *Let $g(-\infty) = g(+\infty) = 0$, $f \in L^\infty(\Omega)$, $1 < p < 2$ or $2 < p < 3$, and either $\lim\limits_{s\to\pm\infty} (g(s)|s|^{p-2}s) = +\infty$ or $\lim\limits_{s\to\pm\infty} (g(s)|s|^{p-2}s) = -\infty$. Then problem (24) has at least one solution provided $\int_\Omega f\,\varphi_1 \mathrm{d}x = 0$.*

(E5) *Let* $g(-\infty) = g(+\infty) = 0$, $f \in L^\infty(\Omega)$, $\liminf_{s\to\pm\infty}(g(s)|s|^{p-2}s) \ge 0$ *if* $1 < p < 2$ *or* $\limsup_{s\to\pm\infty}(g(s)|s|^{p-2}s) \le 0$ *if* $2 < p < 3$. *Then problem* (24) *has at least one solution provided* $\int_\Omega f\,\varphi_1 \mathrm{d}x = 0$.

(E6) *Let* $g(-\infty) = g(+\infty) = 0$, $f \in L^\infty(\Omega)$, $3 \le p < +\infty$, *and*

(i) $\underline{g}_{+\infty} \stackrel{\text{def}}{=} \liminf_{s\to+\infty} \left(g(s)\,|s|^{p-2}s\right)$ *satisfies either* $\underline{g}_{+\infty} > 0$, *or else* $\underline{g}_{+\infty} = 0$ *and*
- $\int_0^\infty g(s)\,s\,\mathrm{d}s = +\infty$ *if* $p = 3$,
- $g(s) \not\equiv 0$ *for* $s \ge 0$ *if* $3 < p < \infty$;

(ii) $\underline{g}_{-\infty} \stackrel{\text{def}}{=} \liminf_{s\to-\infty} \left(g(s)\,|s|^{p-2}s\right)$ *satisfies either* $\underline{g}_{-\infty} > 0$, *or else* $\underline{g}_{-\infty} = 0$ *and*
- $\int_{-\infty}^0 g(s)\,s\,\mathrm{d}s = +\infty$ *if* $p = 3$,
- $g(s) \not\equiv 0$ *for* $s \le 0$ *if* $3 < p < \infty$.

*Then problem* (24) *has at least one solution provided* $\int_\Omega f\,\varphi_1 \mathrm{d}x = 0$.

Notice that we can combine these assertions in the sense that possibly different type of asymptotic behavior of $g$ is assumed at $+\infty$ and $-\infty$. For example, if we combine (E4) and (E5), we obtain the following assertion:

(E7) *Let* $g(-\infty) = g(+\infty) = 0$, $f \in L^\infty(\Omega)$, $2 < p < 3$, *and* $\lim_{s\to+\infty}(g(s)|s|^{p-2}s) = -\infty$ *together with* $\limsup_{s\to-\infty}(g(s)|s|^{p-2}s) \le 0$. *Then problem* (24) *has at least one solution provided* $\int_\Omega f\varphi_1 \mathrm{d}x = 0$.

Let us point out two interesting facts which illustrate the striking difference between the semilinear and quasilinear case.

At first we consider problem

$$
\begin{cases} -\Delta u = \lambda_1 u - e^{-u^2} + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{25}
$$

Multiplying the equation by $\varphi_1$ and integrating by parts we immediately obtain that $\int_\Omega f\varphi_1 \,\mathrm{d}x > 0$ is necessary condition for solvability of (25). In particular, (25) has no solution if $\int_\Omega f\varphi_1 \,\mathrm{d}x = 0$! On the other hand, according to (E5), for $1 < p < 2,\ \ 2 < p < 3$, the problem

$$\begin{cases} -\Delta_p u = \lambda_1 |u|^{p-2}u - e^{-u^2} + f & \text{in } \Omega,\\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution provided $\int_\Omega f\varphi_1 \,\mathrm{d}x = 0$. The reason consists in the different asymptotics for $p \neq 2$ which combined with the asymptotics of $g(s) = \mathrm{e}^{-s^2}$ yields the existence of a solution.

As a second interesting fact we address the necessity of the Landesman–Lazer condition. Namely, let us consider the problem

$$\begin{cases} -\Delta_p u = \lambda_1 |u|^{p-2}u + \arctan |u|^{q-1}u + f & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{26}$$

with $q > p - 1$. Assume that $\varphi_1$ is normalized by $\int_\Omega \varphi_1 \,\mathrm{d}x = 1$. It follows from results mentioned in Section 5 that (26) has a solution for any $f$ satisfying

$$0 < \int_\Omega f\varphi_1 \,\mathrm{d}x < \pi. \tag{27}$$

Since $0 < \arctan |s|^{q-1}s < \pi$ for any $s \in \mathbb{R}$, condition (27) is also necessary in the case $p = 2$. As it is proved in [15] the different asymptotics in case $p \neq 2$ yields that for $p \in (1,2) \cup (2,3)$ problem (26) has a solution also for

$$\int_\Omega f\varphi_1 \,\mathrm{d}x = 0,$$

i. e. condition (27) is not necessary in this case.

Let us close this section by the following challenge:

- Remove the technical assumption $p < 3$ from above mentioned results.

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