Acta Math. Univ. Comenianae
Vol. LXXV, 2(2006), pp. 241–252

A CLASSIFICATION OF TRIANGULAR MAPS OF THE SQUARE

V. KORNECKÁ

Abstract. It is well-known that, for a continuous map φ of the interval, the condition P1 φ has zero topological entropy, is equivalent, e.g., to any of the following: P2 any ω-limit set contains a unique minimal set; P3 the period of any cycle of φ is a power of two; P4 any ω-limit set either is a cycle or contains no cycle; P5 if $\omega_\varphi(\xi) = \omega_{\varphi^2}(\xi)$, then $\omega_\varphi(\xi)$ is a fixed point; P6 φ has no homoclinic trajectory; P7 there is no countably infinite ω-limit set; P8 trajectories of any two points are correlated; P9 there is no closed invariant subset A such that $\varphi^m|A$ is topologically almost conjugate to the shift, for some $m \geq 1$. In the paper we exhibit the relations between these properties in the class $(x, y) \mapsto (f(x), g_x(y))$ of triangular maps of the square. This contributes to the solution of a longstanding open problem of Sharkovsky.

1. Introduction

For a continuous map φ of the interval there is a long list of properties equivalent to zero topological entropy. About 40 of them are applicable to triangular (or, skew-product) maps but only few of them are equivalent in this more general setting. In the eighties, A. N. Sharkovsky proposed the problem of classification of the triangular maps of the square with respect to such properties. More than 30 conditions were already considered, cf., e.g., [**1**], [**5**], [**6**], [**7**], [**8**] and [**9**]. In this paper we consider another four properties that were not studied before in this context:

- if $\omega_\varphi(z) = \omega_{\varphi^2}(z)$, then $\omega_\varphi(z)$ is a fixed point
- φ has no countably infinite ω-limit set
- the trajectories of any two points are correlated
- for a closed invariant set A and $m \geq 1$, $\varphi^m|A$ is not topologically almost conjugate to the shift

We exhibit the relations between them and five other properties that already have been studied. Using this and the results obtained by other authors it is now

Received December 16, 2005.
2000 *Mathematics Subject Classification.* Primary 37B20, 37B40, 54H20.
Key words and phrases. Triangular map, topological entropy, *omega*-limit set.
The research was supported, in part, by projects 201/03/1153 from the Czech Science Foundation and MSM4781305904 from the Czech Ministry of Education.

possible to find the position of any of these conditions among more than 30 other ones.

The paper is organized as follows. In Section 2 we introduce basic terminology. Section 3 contains our main Theorem 3.1 summarizing the old and new relations between the considered properties. In Section 4 we prove new implications and give several examples disproving other implications. Proof of the main theorem is in Section 5 where all relations are summarized in a table.

2. Basic definitions and notation

Denote by $\mathcal{C}(X)$ the class of continuous maps $X \to X$ of a compact metric space (X, ρ); in the sequel X will be either the unit interval $I = [0,1]$ or the unit square I^2. For $\varphi \in \mathcal{C}(X)$, let φ^n denote the *n-th iterate* of φ. The set of cluster points of the *trajectory* $\{\varphi^n(\xi)\}_{n=0}^{\infty}$ of a $\xi \in X$ is the *ω-limit set* $\omega_\varphi(\xi)$ of ξ. An $M \subset X$ is a *minimal set*, if $M = \omega_\varphi(\xi)$, for any $\xi \in M$.

Denote by $\mathrm{Fix}(\varphi)$ the set of *fixed points*, and by $\mathrm{Per}(\varphi)$ the *set of periodic points* of φ. A point $\xi \in X$ is in the set $\mathrm{CR}(\varphi)$ of *chain recurrent points* of φ if for any $\varepsilon > 0$, there exists an *ε-chain* from ξ into itself, i.e. there is a sequence of points $\{\xi_i\}_{i=0}^{n}$, with $\xi_0 = \xi_n = \xi$ and $\rho(\xi_{i+1}, \varphi(\xi_i)) < \varepsilon$ for any $i \in \{0, \dots, n-1\}$.

A set $A \subset X$ is *(n, ε)-separated* if, for any distinct points $\xi_1, \xi_2 \in A$, there exists i such that $0 \le i < n$ and $\rho(\varphi^i(\xi_1), \varphi^i(\xi_2)) > \varepsilon$. For $Y \subset X$, denote by $s_n(\varepsilon, Y, \varphi)$ the maximum possible number of points in an (n, ε)-separated subset of Y. The *topological entropy of φ with respect to Y* and the *topological entropy of the map φ* are defined by

$$h(\varphi|Y) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon, Y, \varphi), \text{ and } h(\varphi) = h(\varphi|X).$$

Let $\xi \in \mathrm{Fix}(\varphi)$, and let ξ_n, $n = 1, 2, \dots$, be distinct points in X such that $\varphi(\xi_{n+1}) = \xi_n$, for any n, $\varphi(\xi_1) = \xi$, and $\lim_{n\to\infty} \xi_n = \xi$. Then $\{\xi_n\}_{n=1}^{\infty}$ is a *homoclinic trajectory related to the point* ξ. A homoclinic trajectory related to a periodic orbit is defined similarly, cf., e.g., [**2**]. Trajectories of the points $\xi, \zeta \in X$ are *correlated*, if either $\omega_\varphi(\xi)$ or $\omega_\varphi(\zeta)$ is a fixed point or

$$\omega_{\varphi\times\varphi}(\xi, \zeta) \neq \omega_\varphi(\xi) \times \omega_\varphi(\zeta),$$

where the map $\varphi \times \varphi : X \times X \to X \times X$ is given by $(\xi, \zeta) \mapsto (\varphi(\xi), \varphi(\zeta))$.

Denote by (Σ, σ) the shift of the space of sequences of two symbols. Thus, $\Sigma = \{0,1\}^{\mathbb{N}}$, and $\sigma : \xi_1\xi_2 \dots \mapsto \xi_2\xi_3 \dots$. Then Σ is a compact metric space with metric $\rho(\xi, \zeta) = \max\{\frac{1}{i}; \xi_i \neq \zeta_i\}$. Denote by 0^n and 1^n the finite sequence of n zeros or ones, respectively, and define 0^∞ and 1^∞ similarly. A map $\varphi \in \mathcal{C}(X)$ is *topologically almost conjugate* to the shift, if there exist a continuous surjective map $\psi : X \to \Sigma$, such that $\psi \circ \varphi = \sigma \circ \psi$ and any point $\xi \in \Sigma$ has at most two preimages in X.

In the sequel we denote by $\mathbb{N}$ the set of positive integers, and by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Other terminology is given in the remainder of the paper, or can be found in standard books like [**2**], [**12**].

The following properties of φ are considered in this paper:

P1 $h(\varphi) = 0$;
P2 any ω-limit set contains a unique minimal set;
P3 the period of any cycle of φ is a power of two;
P4 any ω-limit set either is a cycle or contains no cycle;
P5 if $\omega_\varphi(\xi) = \omega_{\varphi^2}(\xi)$, then $\omega_\varphi(\xi)$ is a fixed point;
P6 φ has no homoclinic trajectory;
P7 there is no infinite countable ω-limit set;
P8 trajectories of any two points are correlated;
P9 for any closed invariant set A and any $m \in \mathbb{N}$, the map $\varphi^m|A$ cannot be topologically almost conjugate to the shift.

The following proposition summarizes known results, see, e.g., [**2**], [**12**].

Proposition 2.1. *For a $\varphi \in \mathcal{C}(I)$, the properties P1 –P9 are mutually equivalent.*

3. The Main Theorem

We denote by $\mathcal{T}$ the class of *triangular maps* of the square. Thus, $\mathcal{T} \subset \mathcal{C}(I^2)$ is the family of maps F such that $F : (x, y) \mapsto (f(x), g_x(y))$. The map $f : I \to I$ is the *base* of F, and $g_x : I_x \to I$ maps the *fibre* $I_x = \{x\} \times I$ to I. We denote by π_1, π_2 the canonical projections $(x, y) \mapsto x$ resp. $(x, y) \mapsto y$ of I^2 onto I. The following is our main result.

Theorem 3.1. *The relations between the properties $P1 - P9$ of a map $F \in \mathcal{T}$ are displayed by the following graph. The particular implications are represented by double arrows. There are no other implications except for these following by transitivity.*

$$\begin{tikzcd}
1 \arrow[dr, Rightarrow] & & & 2 \arrow[dll, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
& 3 \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & 4 \arrow[dl, Rightarrow] & 5 \arrow[d, Rightarrow] \\
6 \arrow[dr, Rightarrow] & & 7 \arrow[dl, Rightarrow] & & 8 \\
& 9 & & &
\end{tikzcd}$$

Theorem 3.1 contains results that has been already proved by other authors. They are summarized in the following

Proposition 3.2. (See, e.g., [**1**], [**5**], [**6**], [**7**], [**8**] and [**10**].) *The relations between properties $P1$ - $P4$, and $P6$ of an $F \in \mathcal{T}$ are displayed by the following graph. The particular implications are represented by double arrows. There are no other implications except for these following by transitivity.*

$$\begin{tikzcd}
1 \arrow[dr, Rightarrow] & & 2 \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
& 3 \arrow[d, Rightarrow] & & 4 \\
& 6 & &
\end{tikzcd}$$

4. Properties of triangular maps

Lemma 4.1. *Let $F \in \mathcal{T}$ with base map f, and let $\bar{x} \in \mathrm{Fix}(f)$. If $\omega_F(x,y) \subset I_{\bar{x}}$, for some $(x,y) \in I^2$, then $\omega_F(x,y) \subset \mathrm{CR}(F|I_{\bar{x}})$.*

Proof. Let $\bar{x} \in I$ be a fixed point of the base f. For simplicity, consider $F|I_{\bar{x}}$ as a map $I \to I$. Take a point $(x,y) \in I^2$ such that $\omega_F(x,y) \subset I_{\bar{x}}$. Denote by $\{F^n(x,y)\}_{n=0}^{\infty} = \{(x_n,y_n)\}_{n=0}^{\infty}$ the trajectory of (x,y). Take $(\bar{x},\bar{y}) \in \omega_F(x,y)$ arbitrary. We show that $\bar{y} \in \mathrm{CR}(F|I_{\bar{x}})$. Since $\lim_{n\to\infty} x_n = \bar{x}$ and $(\bar{x},\bar{y}) \in \omega_F(x,y)$, for any $\varepsilon > 0$ there is an $i \in \mathbb{N}$ such that, for any $j \geq i$,

$$||g_{x_j} - g_{\bar{x}}|| < \varepsilon, \quad \text{and} \quad |\bar{y} - y_i| < \varepsilon,$$

and there is a $k > i$ such that $|\bar{y} - y_k| < \varepsilon$. Then

$$|\bar{y} - g_{x_{k-1}}(\ldots(g_{x_{i+1}}(g_{x_i}(\bar{y})))\ldots)| < \varepsilon,$$

and

$$\{\bar{y}, g_{x_i}(\bar{y}), g_{x_{i+1}}(g_{x_i}(\bar{y})), \ldots, g_{x_{k-1}}(\ldots(g_{x_i}(\bar{y}))\ldots)\}$$

is an ε-chain from $\bar{y}$ to $\bar{y}$. Hence $\bar{y} \in \mathrm{CR}(F|I_{\bar{x}})$. Consequently, $\omega_F(x,y)$ is a subset of the set of chain recurrent points on $I_{\bar{x}}$. $\square$

Let $\varphi \in \mathcal{C}(I)$. An interval $J \subset I$ is *periodic* if there is an $n \in \mathbb{N}$ such that $\varphi^n(J) = J$ and $\mathrm{int}(J) \cap \mathrm{int}\,\varphi^i(J) = \emptyset$ for any $i \in \{1,\ldots,n-1\}$. A set $S \subset I$ is a *solenoid* for φ if there is a sequence $\{I_k\}_{k=1}^{\infty}$ of compact periodic intervals such that

$$S = \bigcap_{k=1}^{\infty} \bigcup_{n=0}^{2^k-1} \varphi^n(I_k),\ I_k \supset I_{k+1},\ I_k \text{ has period } 2^k,\ k \geq 1.$$

Lemma 4.2. (See, e.g., [**12**].) *Let $\varphi \in \mathcal{C}(I)$ with $h(\varphi) = 0$. Then $\mathrm{CR}(\varphi) = \mathrm{Per}(\varphi) \cup \bigcup_{t\in T} S_t$ where every S_t is a minimal solenoid (i.e., a solenoid which is a minimal set).*

Lemma 4.3. (Cf. [**4**].) *Let $\varphi \in \mathcal{C}(I)$ with $h(\varphi) = 0$, and let $\omega_\varphi(x)$ be infinite. Let $U = [u,v]$ be the convex hull of $\omega_\varphi(x)$, and $V = [a,b]$ the minimal compact invariant interval containing U. Then*

(i) *$V \setminus U$ contains no fixed point of φ;*
(ii) *there is an interval J relatively open in I such that $U \subset J$, $J \setminus U$ contains no fixed point of φ, and $\varphi(\overline{J}) \subset J$.*

Lemma 4.4. *$P2 \Rightarrow P5$: Assume any ω-limit set of an $F \in \mathcal{T}$ contains a unique minimal set. If $\omega_F(z) = \omega_{F^2}(z)$, for a $z \in I^2$, then $\omega_F(z)$ is a fixed point.*

Proof. Assume there is a $z = (x,y) \in I^2$ such that $\omega_F(z) = \omega_{F^2}(z)$ and $\omega_F(z)$ contains more than one point. If the base map f has positive topological entropy then, by Proposition 2.1, f and consequently, F has an ω-limit set containing two minimal sets. So assume $h(f) = 0$. Then, by Proposition 2.1, $\omega_f(x) = \omega_{f^2}(x) = \{\bar{x}\}$, for some $\bar{x} \in \mathrm{Fix}(f)$. Hence, $\omega_F(z) \subset I_{\bar{x}}$ and

$$\omega_F(z) \subset \mathrm{CR}(F|I_{\bar{x}}), \tag{1}$$

by Lemma 4.1. Moreover, we may assume that $h(F|I_{\bar{x}}) = 0$ since otherwise, by Proposition 2.1, F would have an ω-limit set containing more then one minimal subset. Then $\omega_F(z)$ contains no solenoid S. Indeed, if

$$S \subset \omega_F(z), \tag{2}$$

where S is a solenoid then there are disjoint compact intervals $L_0, L_1 \subset I_{\bar{x}}$ forming a periodic orbit of period 2 such that $S \subset L_0 \cup L_1$. Apply Lemma 4.3 to $F^2|I_{\bar{x}}$ to obtain relatively open disjoint intervals $J_i \supset L_i$, $i \in \{0, 1\}$, i.e.

$$S \subset L_0 \cup L_1 \subset J_0 \cup J_1. \tag{3}$$

Denote

$$\delta_i = \rho(F^2(\overline{J}_i),\ I \setminus J_i) > 0,\ i \in \{0, 1\} \tag{4}$$

and put

$$\delta = \min\{\delta_0, \delta_1\}.$$

Since F is continuous, there is some $n_0 \in \mathbb{N}_0$ such that

$$||g_{f^n(x)} - g_{\bar{x}}|| < \delta,\ \text{for any } n \geq n_0. \tag{5}$$

By (2), (3), (4) and (5), there is some $m \geq n_0$, such that $\pi_2(F^m(z)) \in J_0$ and $\omega_{F^2}(F^m(z)) \subset \{\bar{x}\} \times J_0$. Since J_0 is periodic of period 2 then $\omega_F(z) \neq \omega_{F^2}(z)$, which is a contradiction. Thus, by (1) and Lemma 4.3,

$$\omega_F(z) \subset \mathrm{Per}(F|I_{\bar{x}}). \tag{6}$$

To finish the proof it suffices to show that $\omega_F(z)$ contains more than one cycle. So assume that $\omega_F(z)$ is a cycle. Since $h(F|I_{\bar{x}}) = 0$ then, by Proposition 2.1, its period is 2^n for some $n \in \mathbb{N}_0$. By the hypothesis $\omega_F(z)$ is not a fixed point hence, $\omega_F(z) \neq \omega_{F^2}(z)$ which is a contradiction. □

Lemma 4.5. *P5 $\Rightarrow$ P8: Let $F \in \mathcal{T}$. If $\omega_F(z) = \omega_{F^2}(z)$ implies that $\omega_F(z)$ is a fixed point, then the trajectories of any two points are correlated.*

Proof. Assume that, for any $z \in I^2$, $\omega_F(z) = \omega_{F^2}(z)$ implies that $\omega_F(z)$ is a fixed point. Take $z_1, z_2 \in I^2$ such that neither $\omega_F(z_1)$ nor $\omega_F(z_2)$ is a fixed point. Then $\omega_F(z_1) \neq \omega_{F^2}(z_1)$ and $\omega_F(z_2) \neq \omega_{F^2}(z_2)$. Let

$$\bar{z}_1 \in \omega_F(z_1) \setminus \omega_{F^2}(z_1),\ \bar{z}_2 \in \omega_F(z_2) \setminus \omega_{F^2}(F(z_2)).$$

Then $(\bar{z}_1, \bar{z}_2) \in \omega_F(z_1) \times \omega_F(z_2)$.

It suffices to show that $(\bar{z}_1, \bar{z}_2) \notin \omega_{F\times F}(z_1, z_2)$. Assume the contrary. Then $(\bar{z}_1, \bar{z}_2)$ is a cluster point of the sequence

$$\{(F \times F)^n(z_1, z_2)\}_{n=0}^{\infty} = \{(F^n(z_1), F^n(z_2))\}_{n=0}^{\infty},$$

and consequently, a cluster point of either

$$\{(F^{2n}(z_1), F^{2n}(z_2))\}_{n=0}^{\infty} \text{ or } \{(F^{2n+1}(z_1), F^{2n+1}(z_2))\}_{n=0}^{\infty}.$$

Since $\bar{z}_1 \notin \omega_{F^2}(z_1)$, $(\bar{z}_1, \bar{z}_2)$ cannot be a cluster point of $\{(F^{2n}(z_1), F^{2n}(z_2))\}_{n=0}^{\infty}$. Similarly, since $\bar{z}_2 \notin \omega_{F^2}(F(z_2))$, $(\bar{z}_1, \bar{z}_2)$ cannot be a cluster point of the sequence $\{(F^{2n+1}(z_1), F^{2n+1}(z_2))\}_{n=0}^{\infty}$. □

Proposition 4.6. (Cf. [**10**].) *Let $F \in \mathcal{T}$, with base f. Then*

$$h(f) + \sup_{x\in I} h(F|I_x) \geq h(F) \geq \max\{h(f), \sup_{x\in I} h(F|I_x)\}.$$

Lemma 4.7. *$P8 \not\Rightarrow P5, P8 \not\Rightarrow P4, P1 \not\Rightarrow P5$: There is an $F \in \mathcal{T}$ such that*

(i) *$h(F) = 0$,*
(ii) *there is an ω-limit set which is not a cycle but contains a cycle,*
(iii) *there is a point $z \in I^2$, such that $\omega_F(z) = \omega_{F^2}(z)$ and $\omega_F(z)$ is not a fixed point,*
(iv) *trajectories of any two points are correlated.*

Proof. Our construction of $F(x, y) = (f(x), g_x(y))$ is inspired by [**8**]. Let

$$f(x) = \begin{cases} 0 & \text{for } x = 0, \\ 1/2^{i+2} & \text{for } x \in \left(1/2^{i+1}, 1/2^{i+1} + 1/2^{i+2}\right], \\ x - 1/2^{i+1} & \text{for } x \in \left(1/2^{i+1} + 1/2^{i+2}, 1/2^{i}\right], \end{cases}$$

for any $i \in \mathbb{N}_0$. For $\delta \in (0, 1)$, let $\tau_\delta, \tau^*_\delta \in \mathcal{C}(I)$ be given by

$$\tau_\delta(y) = \begin{cases} y + \delta & \text{for } y \in [0, 1 - \delta], \\ 1 & \text{for } y \in (1 - \delta, 1], \end{cases} \tag{7}$$

$$\tau^*_\delta(y) = \begin{cases} 0 & \text{for } y \in [0, \delta), \\ y - \delta & \text{for } y \in [\delta, 1]. \end{cases}$$

Let g_0 be the identity map $I \to I$. Let $n_0 = 0$, and $n_i = n_{i-1} + 2^{i+1}$, for $i \in \mathbb{N}$. For any $n \in \mathbb{N}_0$ and $y \in I$, put

$$g_{f^n(1)}(y) = \begin{cases} \tau_{1/2^i}(y) & \text{for } n_i \leq n < \frac{1}{2}(n_i + n_{i+1}), \\ \tau^*_{1/2^i}(y) & \text{for } \frac{1}{2}(n_i + n_{i+1}) \leq n < n_{i+1}, \end{cases} \tag{8}$$

and extend g_x linearly for any $x \in I$.

Since f and $F|I_x$ are monotone, $h(f) = h(F|I_x) = 0$ and, by Proposition 4.6, $h(F) = 0$. This proves (i) Since $\omega_F(x, y) = \omega_{F^2}(x, y) = \{0\} \times I$, but $\omega_F(x, y) \notin \mathrm{Fix}(F)$ whenever $x \in I \setminus \{0\}$ and $y \in I$, the map F satisfies (ii) and (iii).

It remains to prove (iv). Assume $\omega_F(x_1, y_1), \omega_F(x_2, y_2)$ are not fixed points. In particular, we have $x_1, x_2 \neq 0$. It is easy to see that any point in $(0, 1]$ is eventually mapped by f onto an image of 1. Hence, there are $j_1 \leq k_1$ and $j_2 \leq k_2 \in \mathbb{N}_0$ such that

$$f^{j_1}(x_1) = f^{k_1}(1) \quad \text{and} \quad f^{j_2}(x_2) = f^{k_2}(1). \tag{9}$$

There is an $i_0 \in \mathbb{N}_0$, such that

$$k_1, k_2 \leq \frac{1}{2}(n_{i_0-1} + n_{i_0}). \tag{10}$$

Then, by (7), (8), (9) and (10),

$$\begin{aligned} &F^{n_{i_0}-k_1+j_1}(x_1, y_1) = (f^{n_{i_0}}(1), 0), \qquad \text{and} \\ &F^{n_{i_0}-k_2+j_2}(x_2, y_2) = (f^{n_{i_0}}(1), 0). \end{aligned} \tag{11}$$

Without loss of generality, we may assume that $k_1 - j_1 \geq k_2 - j_2$. Put

$$k = k_1 - j_1 - (k_2 - j_2). \tag{12}$$

Then, by (11) and (12),

$$F^{n_{i_0}-k_2+j_2}(x_1,y_1) = F^k(f^{n_{i_0}}(1),0) = F^{k+n_{i_0}-k_2+j_2}(x_2,y_2). \tag{13}$$

Thus, (x_1,y_1) is eventually mapped by F onto an image of (x_2,y_2). Then, by (7), (8) and (13),

$$|\pi_2(F^m(x_1,y_1)) - \pi_2(F^m(x_2,y_2))| \le k\frac{1}{2^i},$$

whenever $m \ge n_{i_0} - k_2 + j_2$ and $m \ge n_i$, $i \in \mathbb{N}$. For $m \to \infty$,

$$|\pi_2(F^m(x_1,y_1)) - \pi_2(F^m(x_2,y_2))| \to 0,$$

so

$$\omega_{F\times F}((x_1,y_1),(x_2,y_2)) = \{((0,\bar{y}),(0,\bar{y}))|\bar{y} \in I\}. \tag{14}$$

It is easy to see that

$$\omega_F(x_1,y_1) \times \omega_F(x_2,y_2) = (\{0\}\times I)\times(\{0\}\times I). \tag{15}$$

Thus, by (14) and (15),

$$\omega_{F\times F}((x_1,y_1),(x_2,y_2)) \ne \omega_F(x_1,y_1)\times\omega_F(x_2,y_2),$$

and trajectories of $(x_1,y_1),(x_2,y_2)$ are correlated. □

Lemma 4.8. *$P1 \not\Rightarrow P8$: There is $F \in \mathcal{T}$ with $h(F) = 0$ and two points in I^2 whose trajectories are not correlated.*

Proof. We use modified triangular map from Lemma 4.7. Put $f(x) = \frac{1}{2}x$. For $\delta \in (0,1)$, let $\tau_\delta, \tau_\delta^* : I \mapsto I$ be as in Lemma 4.7. Let $n_0 = 0$, and $n_i = n_{i-1} + 4^i$ for $i \in \mathbb{N}$. Let

$$g_{f^n(1)}(y) = \begin{cases} \tau_{1/4^{i+1}}(y), & \text{for } n_i \le n < \frac{1}{2}(n_i+n_{i+1}), \\ \tau^*_{1/4^{i+1}}(y), & \text{for } \frac{1}{2}(n_i+n_{i+1}) \le n < n_{i+1}, \end{cases}$$

and

$$g_{f^n(\frac{3}{4})}(y) = \begin{cases} \tau_{1/2^{i+1}}(y), & \text{for } n_i + j2^{i+1} \le n < n_i + 2^i + j2^{i+1}, \\ \tau^*_{1/2^{i+1}}(y), & \text{for } n_i + 2^i + j2^{i+1} \le n < n_i + 2^{i+1} + j2^{i+1}, \end{cases}$$

for any $n \in \mathbb{N}_0$ and any j, $0 \le j < 2^{i+1}$. Put $g_0(y) = y$ and extend g_x linearly for any $x \in I$.

Since f and $F|I_x$ are monotone, $h(f) = h(F|I_x) = 0$ for any $x \in I$ and consequently, by Proposition 4.6, $h(F) = 0$. We show that there are points $z_1, z_2 \in I^2$ whose trajectories are not correlated. Denote

$$I_i = (f^{\frac{1}{2}(n_i+n_{i+1})}(1), f^{n_i}(1)],\ I_i^* = (f^{n_{i+1}}(1), f^{\frac{1}{2}(n_i+n_{i+1})}(1)],\ i \in \mathbb{N}_0.$$

For any j, $0 \le j < 2^{i+1}$, let

$$J_{i,j} = (f^{n_i+2^i+j2^{i+1}}(1), f^{n_i+j2^{i+1}}(1)],\ J^*_{i,j} = (f^{n_i+2^{i+1}+j2^{i+1}}(1), f^{n_i+2^i+j2^{i+1}}(1)].$$

Thus, $J_{i,j}, J^*_{i,j}$ are subintervals of $I_i \cup I_i^*$. Then, for any $i \in \mathbb{N}_0$,

$$g_{f^m(1)} = \begin{cases} \tau_{1/4^{i+1}} & \text{if } f^m(1) \in I_i, \\ \tau^*_{1/4^{i+1}} & \text{if } f^m(1) \in I_i^*. \end{cases}$$

For any $i \in \mathbb{N}_0$, and any j, $0 \le j < 2^{i+1}$,

$$g_{f^m(\frac{3}{4})} = \begin{cases} \tau_{1/2^{i+1}} & \text{if } f^m(\frac{3}{4}) \in J_{i,j}, \\ \tau^*_{1/2^{i+1}} & \text{if } f^m(\frac{3}{4}) \in J^*_{i,j}. \end{cases}$$

Any interval $I_i \cup I_i^*$ contains 2^{i+1} intervals $J_{i,j}$ and 2^{i+1} intervals $J^*_{i,j}$, which regularly change. Any interval $J_{i,j}$ and any $J^*_{i,j}$ contains 2^i images of 1 and 2^i images of $\frac{3}{4}$ both under f. Take $z_1 = (1, 0)$, and $z_2 = (\frac{3}{4}, 0)$. By this construction, the trajectory of the second coordinate of z_1 is equal to

$$\left\{0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, \frac{1}{16}, \cdots, \frac{7}{16}, \frac{1}{2}, \frac{7}{16}, \cdots, \frac{1}{16}, 0, \frac{1}{64}, \cdots, \frac{31}{64}, \frac{1}{2}, \frac{31}{64} \cdots, \frac{1}{64}, 0 \cdots \right\}$$

while the trajectory of the second coordinate of z_2 is

$$\left\{0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{8} \cdots \right\}.$$

We get

$$\omega_{F\times F}(z_1, z_2) = \left(\{0\} \times \left[0, \frac{1}{2}\right]\right) \times \left(\{0\} \times \left[0, \frac{1}{2}\right]\right). \tag{16}$$

It is easy to see that

$$\omega_F(z_1) \times \omega_F(z_2) = \left(\{0\} \times \left[0, \frac{1}{2}\right]\right) \times \left(\{0\} \times \left[0, \frac{1}{2}\right]\right). \tag{17}$$

hence by (16) and (17)

$$\omega_F(z_1) \times \omega_F(z_2) = \omega_{F\times F}(z_1, z_2).$$

□

Lemma 4.9. *P4 $\not\Rightarrow$ P8: There is $F \in \mathcal{T}$ such that any ω-limit set either is a cycle or contains no cycle and there are two points whose trajectories are not correlated.*

Proof. There is an $F \in \mathcal{T}$ possessing all periods and no infinite ω-limit set containing periodic points [**7**]. If $z_1, z_2 \in \mathrm{Per}(F)$ with periods 2 and 3, then $\omega_F(z_1) \times \omega_F(z_2) = \omega_{F\times F}(z_1, z_2)$. □

Proposition 4.10. (See, e.g., [**11**].) *Let $\varphi \in \mathcal{C}(X)$, $\xi \in X$. If $\omega_\varphi(\xi)$ is countable, then $\omega_\varphi(\xi)$ contains a cycle.*

Lemma 4.11. *P4 $\Rightarrow$ P7: Let $F \in \mathcal{T}$. If any ω-limit set of F is a cycle,or contains no cycle, then there is no countably infinite ω-limit set.*

Proof. Assume any $\omega_F(z)$ either is a cycle or contains no cycle. Let $z \in I^2$ such that $\omega_F(z)$ is countable. Then, by Proposition 4.10, $\omega_F(z)$ contains a cycle. By the hypothesis, $\omega_F(z)$ is a cycle, hence, a finite set. □

Lemma 4.12. *P3 $\Rightarrow$ P7: If $F \in \mathcal{T}$, and the period of any cycle of F is a power of two then F has no countably infinite ω-limit set.*

Proof. Assume the period of any cycle of F is a power of two, and let $\omega_F(x,y)$ be countable. Since $\pi_1(\omega_F(x,y)) = \omega_f(x)$, and since the period of any cycle of f is a power of two, Proposition 2.1 implies that f has no countably infinite ω-limit set. Thus, $\pi_1(\omega_F(x,y))$ is a cycle of period $m \in \mathbb{N}$. Then $\pi_1(\omega_{F^m}(x,y)) = \{\bar{x}\}$, for some $\bar{x} \in \mathrm{Fix}(f^m)$, $\omega_{F^m}(x,y) \subset I_{\bar{x}}$ and, by Lemma 4.1,

$$\omega_{F^m}(x,y) \subset \mathrm{CR}(F^m|I_{\bar{x}}).$$

By Proposition 2.1, $h(F^m|I_{\bar{x}}) = 0$, since otherwise F would have a cycle of period distinct from 2^n, for any $n \in \mathbb{N}_0$. By Lemma 4.2,

$$\omega_{F^m}(x,y) \subset \mathrm{Per}(F^m|I_{\bar{x}}),$$

since any point in a solenoid S has an uncountable minimal ω-limit set (cf., e.g., [**3**]) but $\omega_{F^m}(x,y)$ is countable. Then $\omega_{F^m}(x,y)$ contains an isolated periodic point, but this is possible only when $\omega_{F^m}(x,y)$ is a cycle. Thus, $\omega_{F^m}(x,y)$ and hence, $\omega_F(x,y)$ is a finite set. □

Lemma 4.13. *P6 $\not\Rightarrow$ P7: There is an $F \in \mathcal{T}$ with an infinite countable ω-limit set and no homoclinic trajectory.*

Proof. Let T be the tent map i.e., $T(x) = 2x$ if $x \in [0, \frac{1}{2}]$, and $T(x) = 2 - 2x$ otherwise, and let

$$F(x,y) = \left(T(x), \frac{1}{3}y + \tau(x)\right), \text{ where } \tau(x) \in \left[0, \frac{1}{2}\right] \text{ and } \tau(0) = 0.$$

It is possible to specify $\tau(x)$ such that F has no homoclinic trajectory, see [**8**]. Consider this $\tau(x)$.

It is well known that there is a point $\bar{x} \in I$, such that $\omega_T(\bar{x}) = \{\frac{1}{2^{j-1}}; j \in \mathbb{N}\} \cup \{0\}$ is a homoclinic trajectory of T. We prove that, for any $y \in I$, $\omega_F(\bar{x}, y)$ is countable.

First we show that

$$\omega_F(\bar{x}, y) \cap I_1 \text{ is a singleton.} \tag{18}$$

For any $i, j \in \mathbb{N}$, let J_j^i be the compact left-hand neighborhood of the point $\frac{1}{2^{j-1}}$ such that $|J_1^i| = \frac{1}{2^i}$ and $T(J_{j+1}^i) = J_j^i$. It follows that, for any $i \in \mathbb{N}$, the trajectory of $\bar{x}$ is eventually in the union of intervals $J_j^i = [\frac{1}{2^{j-1}} - \frac{1}{2^{i+j-1}}, \frac{1}{2^{j-1}}]$, $j \in \mathbb{N}$. Denote by R_1^i the smallest rectangle which contains $F(J_2^i \times I)$, i.e.,

$$R_1^i = J_1^i \times \left[\min\{\tau(x); x \in J_2^i\}, \frac{1}{3} + \max\{\tau(x); x \in J_2^i\}\right].$$

Obviously, $\omega_F(\bar{x}, y) \cap I_1 \subset R_1^i \subset J_1^i \times I$. Since J_3^i is the preimage of J_2^i, $F^2(J_3^i \times I) \subset F(J_2^i \times I) \subset R_1^i$. Let R_2^i be the smallest rectangle which contains $F^2(J_3^i \times I)$, i.e.,

$$R_2^i = J_1^i \times \left[\min_{x \in J_3^i}\left\{\frac{1}{3}\tau(x) + \tau(T(x))\right\}, \frac{1}{9} + \max_{x \in J_3^i}\left\{\frac{1}{3}\tau(x) + \tau(T(x))\right\}\right].$$

Then $\omega_F(\bar{x}, y) \cap I_1 \subset R_2^i \subset R_1^i$. By induction we obtain rectangles $R_1^i \supset R_2^i \supset \cdots \supset R_j^i \supset \cdots$, where

$$R_j^i = J_1^i \times \Big[\min_{x \in J_{j+1}^i} \Big\{ \sum_{k=1}^{j} \frac{1}{3^{j-k}} \tau(T^{k-1}(x)) \Big\},\ \frac{1}{3^j} + \max_{x \in J_{j+1}^i} \Big\{ \sum_{k=1}^{j} \frac{1}{3^{j-k}} \tau(T^{k-1}(x)) \Big\} \Big],$$

and $\omega_F(\bar{x}, y) \cap I_1 \subset R_j^i$, for any $j \in \mathbb{N}$. Since this is true for any $i \in \mathbb{N}$, and since $R_j^{i+1} \subset R_j^i$,

$$\omega_F(\bar{x}, y) \cap I_1 \subset \bigcap_{i=1}^{\infty} R_j^i =: \tilde{R}_j, \text{ for any } j \in \mathbb{N}, \tag{19}$$

where

$$\tilde{R}_j = \{1\} \times \Big[\sum_{k=1}^{j} \frac{1}{3^{j-k}} \tau(T^{k-1}(\frac{1}{2^j})), \frac{1}{3^j} + \sum_{k=1}^{j} \frac{1}{3^{j-k}} \tau(T^{k-1}(\frac{1}{2^j})) \Big]. \tag{20}$$

To see (20) note that $\frac{1}{2^j}$ is the right-hand endpoint of J_{j+1}^i, for any $i \in \mathbb{N}$. Since $\tilde{R}_j$ is a vertical interval with length $\frac{1}{3^j}$, and since $\tilde{R}_j \supset \tilde{R}_{j+1}$, (18) follows by (19), whereas

$$\omega_F(\bar{x}, y) \cap I_1 = \{1\} \times \left\{ \sum_{k=1}^{\infty} \frac{1}{3^{k-1}} \tau \left(\frac{1}{2^k} \right) \right\},$$

which is a singleton.

Since $\frac{1}{2^j}$ is the j-th preimage of 1, $\omega_F(\bar{x}, y) \cap I_{2^{-j}}$ is a singleton, for any $j \in \mathbb{N}$. To finish the argument it suffices to show that $\omega_F(\bar{x}, y) \cap I_0$ is countable. Denote by $\bar{R}_j^i$ the smallest rectangle containing $F(R_j^i)$. Then, for any $i, j \in \mathbb{N}$, the trajectory of $\bar{R}_j^i$ unified with the set of its cluster points contains $\omega_F(\bar{x}, y) \cap I_0$. It follows that

$$\begin{aligned} \omega_F(\bar{x}, y) \cap I_0 &= \bigcap_{j=1}^{\infty} \bigcap_{i=1}^{\infty} \bigcup_{n=0}^{\infty} F^n(\bar{R}_j^i) \cup \omega_F \Big(\bigcap_{j=1}^{\infty} \bigcap_{i=1}^{\infty} \bigcup_{n=0}^{\infty} F^n(\bar{R}_j^i) \Big) \\ &= \Big\{ \Big(0, \frac{1}{3^m} \sum_{k=1}^{\infty} \frac{1}{3^{k-1}} \tau \left(\frac{1}{2^{k-1}} \right) \Big); m \in \mathbb{N}_0 \Big\} \cup \{(0,0)\}, \end{aligned}$$

which is countable set. Hence $\omega_F(\bar{x}, y)$ is a countable set as a countable union of countable sets. $\square$

Lemma 4.14. *P6 $\Rightarrow$ P9: Let $F \in \mathcal{T}$. If F has no homoclinic trajectory then, for any closed invariant set A and any $m \in \mathbb{N}$, the map $F^m|A$ is not topologically almost conjugate to the shift.*

Proof. Assume $F^m|A$ is topologically almost conjugate to the shift σ in Σ. So, there is a continuous surjective map $\psi : A \to \Sigma$, such that $\psi \circ F^m|A = \sigma \circ \psi$ and any point in Σ is the image of at most two points from A. Take a point $\mathbf{z} = 0^\infty \in \Sigma$. Consider the sequence $\{\mathbf{z}_i\}_{i=1}^{\infty} \subset \Sigma$ of preimages of $\mathbf{z}$, such that

$$\mathbf{z}_1 = \mathbf{z},\ \mathbf{z}_2 = 10^\infty \text{ and } \mathbf{z}_n = 0^{n-2}10^\infty, \text{ for } n > 2.$$

Clearly $\lim_{i\to\infty} \mathbf{z}_i = \mathbf{z}$ and the sequence $\{\mathbf{z}_i\}_{i=1}^{\infty}$ forms a homoclinic trajectory of σ related to the fixed point $\mathbf{z}$. Since ψ is continuous and surjective, there exist points $z, z_i \in A \subset I^2$, such that $\psi(z) = \mathbf{z}, \psi(z_i) = \mathbf{z}_i$, for any $i \in \mathbb{N}$. Then z is a periodic point of F with the period m, $z_i = F^m(z_{i+1})$ and $\lim_{i\to\infty} z_i = z$. Hence the sequence $\{z_i\}_{i=1}^{\infty}$ form a homoclinic trajectory of F^m related to z. Consequently there exist a homoclinic trajectory of F related to the cycle generated by z. $\square$

Lemma 4.15. *$P7 \Rightarrow P9$: Let $F \in \mathcal{T}$. If there is no infinite countable ω-limit set then $F^m|A$ is topologically almost conjugate to the shift, for no closed invariant set A and $m \in \mathbb{N}$.*

Proof. Assume $F^m|A$ is topologically almost conjugate to σ. Let ψ be the corresponding continuous surjective map $A \to \Sigma$, and $\mathbf{z} = 010011\dots0^n1^n\dots \in \Sigma$. It is easy to see, that

$$\omega_\sigma(\mathbf{z}) = \{0^\infty, 10^\infty, 110^\infty, 1110^\infty, \dots, 1^\infty, 01^\infty, 001^\infty, 0001^\infty, \dots\}.$$

Since $\omega_\sigma(\mathbf{z})$ is countably infinite, $\psi(z) = \mathbf{z}$ for some $z \in I^2$, and since any point in Σ has at most two preimages in A, $\omega_F(z)$ is countably infinite, too. $\square$

Lemma 4.16. *$P5 \not\Rightarrow P9$: There is an $F \in \mathcal{T}$ such that*

i. *if $\omega_F(z) = \omega_{F^2}(z)$, then $\omega_F(z)$ is a fixed point;*
ii. *there is a closed invariant set $A \subset I^2$ and $m > 0$, such that $F^m|A$ is topologically almost conjugate to the shift.*

Proof. Let $f(x) = 1 - x$ for $x \in I$, and

$$g_x(y) = \begin{cases} T(y) & \text{for } x \in [0, \frac{1}{4}], \\ y & \text{for } x \in [\frac{1}{2}, 1], \end{cases}$$

where T is the tent map. Extend g_x linearly onto I, and take $F \in \mathcal{T}$ such that $(x, y) \mapsto (f(x), g_x(y))$.

If $\omega_F(z) = \omega_{F^2}(z)$ then, for $x = \pi_1(z)$, $\omega_f(x) = \omega_{f^2}(x)$ and, by definition of f, $x = \frac{1}{2}$. Since $g_{\frac{1}{2}}$ is the identity, $\omega_F(\frac{1}{2}, y) = \omega_{F^2}(\frac{1}{2}, y)$ and $\omega_F(\frac{1}{2}, y)$ is a fixed point, for any $y \in I$. Thus F satisfies (i).

Take 2-cycle $\{0, 1\}$ of the base. Since g_1 is the identity and g_0 is the tent map which has cycles of all periods $F^2|I_0$ has a cycle of period other then a power of two. Then, by Proposition 2.1, there is a closed invariant set A and $m > 0$ such that $(F^2|I_0)^m|A$ is topologically almost conjugate to the shift in Σ. Thus $F^{2m}|A$ is topologically almost conjugate to the shift and (ii) follows. $\square$

5. Proof of the main Theorem

Proof of Theorem 3.1 follows by Proposition 3.2 and Lemmas 4.4 – 4.5, 4.7 – 4.9 and 4.11 – 4.16. The next table summarizes all results.

	1	2	3	4	5	6	7	8	9
1	$\bullet$	$\not\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
2	$\not\Rightarrow$	$\bullet$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$
3	$\not\Rightarrow$	$\not\Rightarrow$	$\bullet$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
4	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\bullet$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
5	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\bullet$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$
6	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\bullet$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
7	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\bullet$	$\not\Rightarrow$	$\Rightarrow$
8	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\bullet$	$\not\Rightarrow$
9	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\bullet$

References

1. Balibrea F. and Smítal J., *A triangular map with homoclinic orbits and no infinite ω-limit set containing periodic points*, Topology Appl. **153** (2006), 2092–2103.

2. Block L. S. and Coppel W. A., *Dynamics in One Dimension*, Lecture Notes in Mathematics 1513, Springer, Berlin, Heidelberg, New York, 1992.

3. Bruckner A. M. and Smítal J., *A characterization of omega-limit sets of maps of the interval with zero topological entropy*, Ergodic Theory & Dynam. Syst. **13** (1993), 7–19.

4. Fedorenko V. V., Sharkovsky A. N., and Smítal J., *Characterizations of weakly chaotic maps of the interval*, Proc. Amer. Math. Soc. **110** (1990), 141–148.

5. Forti G.-L., Paganoni L., and Smítal J., *Strange triangular maps of the square*, Bull. Austral. Math. Soc. **51** (1995), 395–415.

6. ______, *Dynamics of homeomorphisms on minimal sets generated by triangular mappings*, Bull. Austral. Math. Soc. **59** (1999), 1–20.

7. ______, *Triangular maps with all periods and no infinite ω-limit set containing periodic points*, Topology Appl., **153** (2005), 818–832.

8. Kočan Z., *The problem of classification of triangular maps with zero topological entropy*, Ann. Math. Silesianae **13** (1999), 181–192.

9. ______, *Triangular maps nondecreasing on the fibres*, Real Analysis Exchange **30** (2004/2005), 519–537.

10. Kolyada S. F., *On dynamics of triangular maps of the square*, Ergodic Theory & Dynam. Syst. **12** (1992), 749–768.

11. Sharkovsky A. N., *Continuous mapping on the set of limit points of an iteration sequence*, Ukrain. Mat. Zh. **18** (1966), 127–130. (in Russian)

12. Sharkovsky A. N., Kolyada S. F., Sivak A. G., and Federenko V. V., *Dynamics of one-dimensional maps*, Mathematics and its Applications. Kluwer Academic Publishers 1997.

V. Kornecká, Mathematical Institute, Silesian University, CZ-746 01 Opava, Czech Republic,
e-mail: `Veronika.Kornecka@math.slu.cz`