FUNCTIONALS ON SEQUENCE SPACES CONNECTED WITH THE EXPONENTIAL STABILITY OF EVOLUTIONARY PROCESSES

P. PREDA, A. POGAN AND C. PREDA

ABSTRACT. We consider certain functionals on the space of all real, positive sequences and in terms of these we can characterize the exponential stability of evolutionary processes.

1. Introduction

The problem of input-output stability was first studied by O. Perron in 1930 [11] for the case of linear finite-dimensional continuous-time systems $x'(t) = A(t)x(t) + f(t)$. In his paper, a central concern is the relationship, for linear equations, between the condition that the non-homogenous equation has some bounded solution for every bounded "second member" on the one hand and a certain form of stability of the solution of the homogenous equation on the other.

For the case of discrete-time systems analogous results was first obtained by Li Ta in 1934 [16]. This idea was later extensively developed for the discrete-time systems in the infinite-dimensional case by Ch. V. Coffman and J.J. Schäffer in 1967 [2] and D. Henry in 1981 [4] and more recently we refer the readers to the papers due to A. Ben-Artzi [1], I. Gohberg [1], M. Pinto [12], J. P. La Salle [5].

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Also, applications of this "discrete-time" case to stability theory of linear infinite-dimensional continuous-time systems have been presented by Henry [4], Przyluski and Rolewicz in [15]. Also using a discrete time argument, an extension of the well-known Datko's result [3] was obtained in [13].

More recently an useful approach was given for the continuous case by Van Minh, Rabiger and Schnaubelt in [6]. Using a different technique, related results were obtained in [14] and for a nonuniform case in [7] , and also a version the theorem due to Van Minh, Rabiger and Schnaubelt, for discrete case was pointed out in [8]. Beside this line of results, in the recent years another approach was initiated by J. M. A. M van Neerven for the case of strongly continuous semigroups.

We note that the admissibility spaces used in many of the above papers are not just l^∞ but also we can find over there some spaces as l^p with $p \in [1, \infty)$. An important case of evolutionary process are the processes which can be expressed by the formulae $U(t,s) = T(t-s)$ where $\mathbf{T}$ is a C_0-semigroup (also these processes are called stationary evolutionary processes).

We recall that a family of bounded linear operators $\mathbf{T} = \{T(t)\}_{t \geq 0}$ is a C_0-semigroup on X if:

(s_1) $T(0) = I$ (where I is the identity operator on X);

(s_2) $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$;

(s_3) $\lim\limits_{t \to 0_+} T(t)x = x$, for all $x \in X$.

One can easily remark that a sequence belongs to l^p if and only if there exists a positive functional which is finite on the respective sequence. This idea was developed successfully on the case of C_0-semigroups by Jan van Neerven in [10] where there is proved the following result:

If $\mathbf{T} = \{T(t)\}_{t \geq 0}$ is a C_0-semigroup on X, then $J : C^+([0,\infty)) \to [0,\infty]$ is a lower semi-continuous, nondecreasing functional which satisfies the property that

$$J(f) = \infty, \quad \text{for all} \quad f \in C^+([0,\infty)) \quad \text{with} \quad \liminf_{t \to \infty} f(t) > 0,$$

and if

$$J(||T * f||) < \infty \quad \text{for all} \quad f \in C_c((0,\infty), X),$$

then $\mathbf{T}$ is exponentially stable (here $C^+([0,\infty)$ denotes the space of all continuous, positive functions on $[0,\infty)$ and $C_c((0,\infty), X)$ is the space of all continuous functions with compact support on $(0,\infty)$), also $(T * f)(t) = \int_0^t T(t-s)f(s)ds$.)

Also, this kind of results was touched in [9] for the case of strongly continuous semigroups, by Megan M., Sasu A. L., Sasu B. and Pogan A.

The present paper is more related to this last type of results from [5, 9] than to the results from [6, 7, 8, 14]. In this spirit the first aim of this paper is to extend the Neerven's type analysis to the general case of evolutionary processes, using another type of functionals. Thus there are obtained some new characterizations of exponential stability of evolutionary processes, using a discrete-time argument, in terms of admissibility of certain functionals.

2. Preliminaries

First of all let us remind some definitions and standard notations. Throughout this paper X will be a Banach space, $B(X)$ the Banach algebra of all bounded linear operator from X into itself. We recall that a function $U : \{(t,s) \in \mathbf{R}^2 : t \geq s \geq 0\} \to B(X)$ is called an evolutionary process if

(ep$_1$) $U(t,t) = I$ (the identity operator on X), for all $t \geq 0$;

(ep$_2$) $U(t,s) = U(t,r)U(r,s)$, for all $t \geq r \geq s \geq 0$;

(ep$_3$) There exist $M \geq 1, \omega > 0$ such that

$$\|U(t,s)x\| \leq Me^{\omega(t-s)}\|x\|, \quad \text{for all } t \geq s \geq 0, \quad x \in X.$$

We note that in many works concerning the asymptotic behavior of the evolution families there are different continuity conditions dictated by some local interest, but we do not need here any continuity hypothesis. Now we give

Definition 2.1. The evolutionary process $\mathcal{U}$ is said to be *uniformly exponentially stable (u.e.s.)* if there exist $N, \nu > 0$ such that

$$\|U(t,t_0)x\| \leq Ne^{-\nu(t-t_0)}\|x\|, \quad \text{for all} \quad t \geq t_0 \geq 0, \quad x \in X \setminus \{0\}.$$

Proposition 2.1. *The evolutionary process $\mathcal{U}$ is u.e.s. if and only if there exists a positive numbers sequence $(a_n)_{n\in\mathbf{N}}$ such that*

$$\inf_{n\in N} a_n = 0, \quad \|U(n,m)\| \leq a_{n-m},$$

for all $n, m \in \mathbf{N}$ with $n \geq m$.

Proof. The necessity. It is obvious from Definition 2.1.

The sufficiency. Let $n_0 = \inf\{n \in \mathbf{N}^* : a_n < e^{-1}\}$, $t_0 \geq 0$, $t \geq t_0 + 2n_0$, $n = [t/n_0]$, $m = [t_0/n_0]$. We have that $m + 2 \leq \frac{t_0+2n_0}{n_0} \leq \frac{t}{n_0}$, so it is obvious that $n \geq m+2$.

On the other hand we have

$$\begin{aligned}
\|U(t,t_0)\| &= \|U(t,nn_0)U(nn_0,(m+1)n_0)U((m+1)n_0,t_0)x\| \\
&\leq \|U(t,nn_0)\|\|U((m+1)n_0,t_0)\| \prod_{k=m+2}^{n} \|U(kn_0,(k-1)n_0)\| \\
&\leq Me^{(t-nn_0)\omega}Me^{((m+1)n_0-t_0)\omega} \prod_{k=m+2}^{n} a_{n_0} \\
&\leq M^2e^{n_0\omega+((m+1)n_0-t_0)\omega}e^{-(n-m-1)} \\
&\leq M^2e^{2n_0\omega+(mn_0-t_0)\omega}e^{-(\frac{t}{n_0}-1)}e^{\frac{t_0}{n_0}+1} \\
&\leq M^2e^{2n_0\omega+2}e^{-\frac{1}{n_0}(t-t_0)}.
\end{aligned}$$

If $t_0 \geq 0$ and $t \in [t_0, t_0 + 2n_0)$ then

$$\|U(t,t_0)\| \leq Me^{\omega(t-t_0)} \leq M^2 e^{2n_0\omega} \leq M^2 e^{2n_0\omega+2} e^{-\frac{1}{n_0}(t-t_0)}.$$

It follows that

$$\|U(t,t_0)\| \leq Ne^{-\nu(t-t_0)},$$

for all $t \geq t_0 \geq 0$, where $N = M^2 e^{2n_0\omega+2}$, $\nu = \frac{1}{n_0}$. □

In what follows we will denote by $\mathcal{S}(\mathbf{R})$ the set of all real numbers sequences and by $\mathcal{S}^+(\mathbf{R})$ the set of all $s \in \mathcal{S}(\mathbf{R})$ with $s(n) \geq 0$, for all $n \in \mathbf{N}$. For $s \in \mathcal{S}(\mathbf{R})$ we will denote by s$'$ the unique real sequence which satisfies the equality:

$$s(n) = \sum_{k=0}^{n} s'(k), \qquad \text{for all} \quad n \in \mathbf{N}.$$

Inductively we will define $s^{(k+1)} = (s^{(k)})'$, for $k = 0, 1, \ldots$

Also we will denote by $\mathcal{S}^+_{k,0}(\mathbf{R})$ the space of all $s \in \mathcal{S}(\mathbf{R})$ which satisfy the properties

- $s^{(k)}(n) \geq 0$, for all $n \in \mathbf{N}$,
- $\operatorname{card}\{n \in \mathbf{N} : s^{(k)}(n) > 0\} < \aleph_0$.

Let $\mathcal{F}$ be the set of all functions $F : \mathcal{S}^+(\mathbf{R}) \to [0,\infty]$ such that the following statements hold:

(f$_1$) If $s_1, s_2 \in \mathcal{S}^+(\mathbf{R})$ with $s_1 \leq s_2$ then $F(s_1) \leq F(s_2)$;

(f$_2$) $F(s) < \infty$ for all $s \in \mathcal{S}^+_{0,0}(\mathbf{R})$;

(f$_3$) there exists $c > 0$ such that $F(\alpha\chi_{\{n\}}) \geq c\alpha$, for all $\alpha > 0$ and all $n \in \mathbf{N}$;

(f$_4$) there exist $a > 0, j \in \mathbf{N}, \psi \in \mathcal{S}^+_{j,0}(\mathbf{R})$ such that

$$\inf_{n\in\mathbf{N}} \psi(n) \geq a \quad \text{and} \quad \lim_{n\to\infty} F(\alpha\psi\chi_{\{0,\ldots,n\}}) = \infty, \quad \text{for all} \;\; \alpha > 0.$$

Example 2.1. The function $F : \mathcal{S}^+(\mathbf{R}) \to [0,\infty]$, $F(s) = \sum_{n=0}^{\infty} s(n)$ belongs to $\mathcal{F}$.

Proposition 2.2. *If $F \in \mathcal{F}$ then we have that:*

$$\lim_{n\to\infty} \inf_{\alpha\in(0,1]} \frac{F(\alpha\psi\chi_{\{0,\ldots n\}})}{\alpha^2} = \infty.$$

Proof. From (f_1) it results immediately that the map $r : \mathbf{N} \to \mathbf{R}_+$

$$r(n) = \inf_{\alpha\in(0,1]} \frac{F(\alpha\psi\chi_{\{0,\ldots,n\}})}{\alpha^2}$$

is nondecreasing. Let $l = \lim_{n\to\infty} r(n)$. We shall prove that $l = \infty$. Assume for a contradiction that $l < \infty$. Then it is easy to see that for every $n \in \mathbf{N}$ there exists $\alpha_n \in (0,1]$ with

$$\frac{ac}{\alpha_n} = \frac{ac\alpha_n}{\alpha_n^2} \leq \frac{F(a\alpha_n\chi_{\{0\}})}{\alpha_n^2} \leq \frac{F(\alpha_n\psi\chi_{\{0,\ldots,n\}})}{\alpha_n^2} \leq r(n) + \frac{1}{n+1}$$

and hence

$$\alpha_n \geq \frac{ac}{r(n) + \frac{1}{n+1}} \quad \text{for all} \quad n \in \mathbf{N}.$$

Using the fact that $l < \infty$ we obtain that $\lim_{n\to\infty} \inf \alpha_n > 0$ which implies that there exist $n_0 \in \mathbf{N}$ and $\alpha > 0$ such that $\alpha_n \geq \alpha$, for all $n \in \mathbf{N}$ with $n \geq n_0$. Then

$$F(\alpha\psi\chi_{\{0,\ldots,n\}}) \leq \frac{F(\alpha_n\psi\chi_{\{0,\ldots,n\}})}{\alpha_n^2} \leq r(n) + \frac{1}{n+1},$$

for all $n \in \mathbf{N}$ with $n \geq n_0$ and so

$$\lim_{n\to\infty} F(\alpha\psi\chi_{\{0,\ldots,n\}}) \leq l < \infty,$$

which is the required contradiction. $\square$

A map $N : \mathcal{S}(\mathbf{R}) \to [0, \infty]$ is called a generalized norm on $\mathcal{S}(\mathbf{R})$ if

(n_1) $N(s) = 0$ if and only if $s = 0$;

(n_2) $N(s_1 + s_2) \leq N(s_1) + N(s_2)$, for all $s_1, s_2 \in \mathcal{S}(\mathbf{R})$;

(n_3) $N(\alpha s) = |\alpha| N(s)$, for all $\alpha \in \mathbf{R}$ and all $s \in \mathcal{S}(\mathbf{R})$ with $N(s) < \infty$;

(n_4) If $s_1, s_2 \in \mathcal{S}(\mathbf{R})$ with $|s_1| \leq |s_2|$ then $N(s_1) \leq N(s_2)$.

Remark 2.1. If N is a generalized norm on $\mathcal{S}(\mathbf{R})$ then $E = \{s \in \mathcal{S}(\mathbf{R}) : N(s) < \infty\}$ is a normed function space with the norm $\|s\|_E = N(s)$.

We will denote by $\xi(\mathbf{N})$ the set of all normed sequence spaces E with the properties.

(e_1) $\chi_{\{0,\ldots,m\}} \in E$, for all $m \in \mathbf{N}$;

(e_2) $\inf\limits_{n \in \mathbf{N}} \|\chi_{\{n\}}\|_E > 0$;

(e_3) there exist $a > 0$, $j \in \mathbf{N}$, $\psi \in \mathcal{S}^+_{j,0}(\mathbf{R})$ such that

$$\inf_{n \in \mathbf{N}} \psi(n) \geq a \quad \text{and} \quad \lim_{n \to \infty} \|\psi \chi_{\{0,\ldots,n\}}\|_E = \infty.$$

Example 2.2. We note that $l^p \in \xi(\mathbf{N})$ for all $p \in [1, \infty]$.
Indeed (e_1) and (e_2) are trivial to verify in this case. In order to verify (e_3) take $a = 1$, $j = 2$, $\psi(n) = n + 1$.

Example 2.3. Another example of normed sequences space which belongs to $\xi(\mathbf{N})$ is

$$E = \{s \in \mathcal{S}(\mathbf{R}) : \sup_{n \in \mathbf{N}} (n+1)|s(n)| < \infty\}$$

with the norm

$$\|s\|_E = \sup_{n \in \mathbf{N}} (n+1)|s(n)|.$$

Remark 2.2. If $E \in \xi(\mathbf{N})$ then the function $F_E : \mathcal{S}^+(\mathbf{R}) \to [0,\infty]$ given by $F_E(s) = N(s)$, belongs to $\mathcal{F}$, where N is defined above.
Next, we define $V_m : X^N \to X^N, (V_m f)(n) = \sum_{k=0}^{n} U(n+m, m+k) f(k)$, where $X^{\mathbf{N}}$ denotes the space of all functions from $\mathbf{N}$ to X.

Definition 2.2. (i) $F \in \mathcal{F}$ is said to be admissible to $\mathcal{U}$ if there exists $K > 0$ such that

$$F(\|V_m f\|) \leq K F(\|f\|),$$

for all $m \in \mathbf{N}$ and all $f \in X^{\mathbf{N}}$ with $F(\|f\|) < \infty$.

(ii) $E \in \xi(\mathbf{N})$ is said to be admissible to $\mathcal{U}$ if F_E is admissible to $\mathcal{U}$, where F_E is defined in the Remark 2.2.

Remark 2.3. If $E \in \xi(\mathbf{N})$ then

$$E(X) = \{f \in X^{\mathbf{N}} : \|f\| \in E\},$$

is a normed space with the norm

$$\|f\|_{E(X)} = \|\|f\|\|_E.$$

Remark 2.4. $E \in \xi(\mathbf{N})$ is admissible to $\mathcal{U}$ if and only if there exists $K > 0$ such that $V_m f \in E(X)$ for all $m \in \mathbf{N}$ and all $f \in E(X)$ and

$$\|V_m f\|_{E(X)} \leq K \|f\|_{E(X)},$$

for all $m \in \mathbf{N}$ and all $f \in E(X)$.

Finally we will denote by Φ the set of all non-decreasing bijective functions φ from $\mathbf{R}_+$ into itself which satisfy the condition

$$\sup_{\beta>0} \frac{\varphi^{-1}(\alpha\beta)}{\varphi^{-1}(\beta)} < \infty, \quad \text{for all} \quad \alpha \geq 0.$$

Definition 2.3. $\varphi \in \Phi$ is admissible to $\mathcal{U}$ if there exists $K > 0$ such that

$$\sum_{n=0}^{\infty} \varphi(\|(V_m f)(n)\|) \leq K \sum_{n=0}^{\infty} \varphi(\|f(n)\|),$$

for all $m \in \mathbf{N}$ and all $f \in X^{\mathbf{N}}$ with $\sum_{n=0}^{\infty} \varphi(\|f(n)\|) < \infty$.

3. The main result

For $\mathcal{U}$ an evolutionary process we denote by $s_{x,m} : \mathbf{N} \to \mathbf{R}_+$ the sequence defined by

$$s_{x,m}(n) = \frac{\|U(m+n,m)x\|}{\|x\|},$$

where $m \in \mathbf{N}$ and $x \in X \backslash \{0\}$. Without any loss of generality we may assume that $s_{x,m}(n) \neq 0$, for all $m, n \in \mathbf{N}$ and all $x \in X \backslash \{0\}$.

Theorem 3.1. *The evolutionary process $\mathcal{U}$ is u.e.s. if and only if there exists $F \in \mathcal{F}$ admissible to $\mathcal{U}$.*

Proof. *The necessity.* As we already mentioned in Example 2.1, the map $F : \mathcal{S}^+(\mathbf{R}) \to [0, \infty]$, $F(s) = \sum_{n=0}^{\infty} s(n)$, belongs to $\mathcal{F}$. Also

$$
\begin{aligned}
F(\|(V_m f)\|) &= \sum_{n=0}^{\infty} \|(V_m f)(n)\| \le \sum_{n=0}^{\infty} \sum_{k=0}^{n} \|U(m+n, m+k) f(k)\| \\
&\le \sum_{n=0}^{\infty} \sum_{k=0}^{n} N e^{-\nu(n-k)} \|f(k)\| = \sum_{k=0}^{\infty} (\sum_{n=k}^{\infty} N e^{-\nu(n-k)} \|f(k)\|) \\
&= \sum_{k=0}^{\infty} \frac{N}{1 - e^{-\nu}} \|f(n)\| = K F(\|f\|),
\end{aligned}
$$

for all $m \in \mathbf{N}$, and all $f \in X^{\mathbf{N}}$ with $F(\|f\|) < \infty$.

The sufficiency. First we consider the map $f_{x,m} : \mathbf{N} \to X$, defined by

$$
f_{x,m}(n) = \frac{1}{\|x\|} \psi^{(j)}(n) U(n+m, m) x.
$$

A simple computation shows that

$$
\|f_{x,m}\| \le M e^{\omega i_0} \psi^{(j)},
$$

where $i_0 = \max\{n \in \mathbf{N} : \psi^{(j)}(n) > 0\}$ and that $V_m^j f_{x,m} = \psi s_{x,m}$, and hence by the admissibility condition we obtain that

$$
F(\psi s_{x,m}) \le L = K^j F(M e^{\omega i_0} \psi^{(j)}) < \infty,
$$

for all $m \in \mathbf{N}$ and all $x \in X \setminus \{0\}$. Also the fact that

$$
a c s_{x,m}(n) \le F(a s_{x,m}(n) \chi_{\{n\}}) \le F(\psi s_{x,m}) \le L
$$

implies that

$$s_{x,m}(n) \leq \frac{L}{ac}, \quad \text{for all} \quad m,n \in \mathbf{N}, \quad x \in X \backslash \{0\}.$$

On the other hand we have that

$$\frac{\|U(n+m,m)x\|}{\|U(k+m,m)x\|} = s_{U(k+m,m)x,k+m}(n-k),$$

for all $m,n \in \mathbf{N}, k \in \mathbf{N}$ with $k \leq n$ and all $x \in X \backslash \{0\}$ and so

$$\|U(n+m,m)x\| \leq \frac{L}{ac}\|U(k+m,m)x\|,$$

for all $m,n \in \mathbf{N}, k \in \mathbf{N}$ with $k \leq n$ and all $x \in X \backslash \{0\}$. It results that

$$\begin{aligned} s_{x,m}(n)\chi_{\{0,\ldots,n\}} &= \sum_{k=0}^{n} s_{x,m}(n)\chi_{\{k\}} \leq \frac{L}{ac}\sum_{k=0}^{n} s_{x,m}(k)\chi_{\{k\}}l \\ &\leq \frac{L}{ac}\sum_{k=0}^{\infty} s_{x,m}(k)\chi_{\{k\}} = \frac{L}{ac}s_{x,m}, \end{aligned}$$

for all $m,n \in \mathbf{N}, x \in X \backslash \{0\}$ and by this inequality we obtain that

$$\left(\frac{ac}{L}s_{x,m}(n)\right)^2 r(n) \leq F(\frac{ac}{L}s_{x,m}(n)\psi\chi_{\{0,\ldots,n\}}) \leq F(\psi s_{x,m}) \leq L$$

for all $m,n \in \mathbf{N}, x \in X \backslash \{0\}$ and hence

$$\|U(n+m,m)\| \leq a_n, \quad \text{for all} \quad m,n \in \mathbf{N},$$

where $a_n = \frac{L}{ac}(1+L^{1/2})\frac{1}{1+\sqrt{r(n)}}$. By Proposition 2.1 and Proposition 2.2 it follows that $\mathcal{U}$ is u.e.s. $\square$

Corollary 3.2. *The evolutionary process $\mathcal{U}$ is u.e.s. if and only if there exists $E \in \xi(\mathbf{N})$ admissible to $\mathcal{U}$.*

Proof. The proof follows easily from Theorem 3.1 and Definition 2.2. □

Corollary 3.3. *The evolutionary process $\mathcal{U}$ is u.e.s. if and only if there exists $p \in [1, \infty]$ such that l^p is admissible to $\mathcal{U}$.*

Proof. It results from Corollary 3.2 and Example 2.2. □

Corollary 3.4. *The evolutionary process $\mathcal{U}$ is u.e.s. if and only there exists $\varphi \in \Phi$ admissible to $\mathcal{U}$.*

Proof. It follows easily from Theorem 3.1 and Definition 2.3 using the map $F_\varphi : \mathcal{S}^+(\mathbf{R}) \to [0, \infty]$ defined by

$$F_\varphi(s) = \varphi^{-1}\Big(\sum_{n=0}^{\infty} \varphi(s(n))\Big)$$

which belongs to $\mathcal{F}(a = 1, j = 1, \psi(n) = 1)$. □

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P. Preda, Department of Mathematics, West University of Timişoara, Bd. V. Pârvan, No 4, Timişoara 1900, Romania,
e-mail: `preda@math.uvt.ro`

A. Pogan, Department of Mathematics, University of Missouri, 202, Mathematical Sciences Bldg, Columbia, MO 65211, U.S.A,
e-mail: `pogan@math.missouri.edu`

C. Preda, Department of Electrical Engineering, University of California, Los Angeles (UCLA), CA 90095, U.S.A.,
e-mail: `preda@ee.ucla.edu`