COEFFICIENT ESTIMATES IN SUBCLASSES
OF THE CARATHÉODORY CLASS RELATED
TO CONICAL DOMAINS

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ABSTRACT. We study some properties of subclasses of of the Carathéodory class of functions, related to conic sections, and denoted by $\mathcal{P}(p_k)$. Coefficients bounds, estimates of some functionals are given.

1. INTRODUCTION

We denote by $\mathcal{P}$ the class of Carathéodory functions analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$, e.g.

$$(1.1) \qquad \mathcal{P} = \{p : p \text{ analytic in } \mathcal{U}, \ p(0) = 1, \ \operatorname{Re} p(z) > 0\}.$$

Some special subclasses of $\mathcal{P}$ play an important role in geometric function theory because of their relations with subclasses of univalent functions. Many such classes have been introduced and studied; some became the well-known, for instance, the class of analytic functions p in the unit disk $\mathcal{U}$ such that $p(0) = 1$ and $p \prec (1+Az)/(1+Bz)$, that is the class of functions for which $p(\mathcal{U})$ is a subset of a disk, or a half-plane. The other choice is the class of all p such that $p \prec [(1+z)/(1-z)]^\gamma$. In this case $p(\mathcal{U})$ is a subset of a sector, contained in a right half-plane with a vertex at the origin and symmetric about the real axis. Here the symbol "$\prec$" denotes the subordinations (cf. e.g. [**7**]).

Let $k \in [0, \infty)$. For arbitrarily chosen k let Ω_k denote the following domain

$$(1.2) \qquad \Omega_k = \left\{ u + iv : u^2 > k^2(u-1)^2 + k^2 v^2 \right\}.$$

Note that Ω_k is convex and symmetric in the real axis and $1 \in \Omega_k$ for all k. Ω_0 is nothing but the right half-plane and when $0 < k < 1$, Ω_k is an unbounded domain enclosed by the right branch of the hyperbola

$$(1.3) \qquad \left(\frac{(1-k^2)u + k^2}{k} \right)^2 - \left(\frac{(1-k^2)v}{\sqrt{1-k^2}} \right)^2 = 1$$

with foci at 1 and $-(1 + k^2)/(1 - k^2)$. When $k = 1$, the domain Ω_1 is still unbounded domain enclosed by the parabola

$$2u = v^2 + 1$$

with the focus at 1. When $k > 1$, the domain Ω_k becomes bounded domain being the interior of the ellipse

$$\left(\frac{(k^2 - 1)u - k^2}{k}\right)^2 - \left(\frac{(k^2 - 1)v}{\sqrt{k^2 - 1}}\right)^2 = 1$$

with foci at 1 and $(k^2 + 1)/(k^2 - 1)$. It should be noted that, for no choice of parameter k, Ω_k reduce to a disk. $\{\Omega_k, k \in [0, \infty)\}$ forms the family of domains bounded by conic sections, convergent in the sense of the kernel convergence.

Let p_k denote the conformal mapping of $\mathcal{U}$ onto Ω_k determined by conditions $p_k(0) = 1$, $p_k'(0) > 0$. The concrete form of p_k was given in [**7**], [**8**], [**11**] and in [**5**].

Theorem 1.1. *Let $k \in [0, \infty)$. The conformal mapping of $\mathcal{U}$ onto Ω_k is of the form*

$$(1.4) \qquad p_k(z) = \begin{cases} \frac{1+z}{1-z} & \text{for} \quad k = 0, \\ 1 + \frac{2}{1-k^2} \sinh^2\left(A(k)\text{arctanh}\sqrt{z}\right) & \text{for} \quad k \in (0, 1), \\ 1 + \frac{8}{\pi^2} \left(\text{arctanh}\sqrt{z}\right)^2 & \text{for} \quad k = 1, \\ 1 + \frac{2}{k^2-1} \sin^2\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t}, t)\right) & \text{for} \quad k > 1, \end{cases}$$

where $A(k) = (2/\pi) \arccos k$, $\mathcal{F}(w, t)$ is the Jacobi elliptic integral of the first kind

$$(1.5) \qquad \mathcal{F}(w, t) = \int_0^w \frac{dx}{\sqrt{(1 - x^2)(1 - t^2 x^2)}},$$

and $k = \cosh \mu(t) = \cosh\left(\frac{\pi \mathcal{K}'(t)}{2\mathcal{K}(t)}\right)$, $t \in (0, 1)$.

By $\mathcal{P}(p_k)$ we denote the subclass of the Carathéodory class $\mathcal{P}$, consisting of functions p, analytic in $\mathcal{U}$, $p(0) = 1$, $\text{Re}\, p(z) > 0$ in $\mathcal{U}$, and such that $p \prec p_k$ in $\mathcal{U}$. Observe that when k varies, $\mathcal{P}(p_k)$ generate a number of subclasses of the class $\mathcal{P}$.

The aim of this paper is to present some properties of the class $\mathcal{P}(p_k)$. In Section 2 we prove the continuity of functions "extremal" in $\mathcal{P}(p_k)$ as regards the parameter k. Some coefficients problems are treated in Section 3, in particular we obtain the sharp bound on the coefficient functional $|b_2 - \mu b_1^2|$ $(-\infty < \mu < \infty)$.

2. General properties of the family $\mathcal{P}(p_k)$

We recall some notation and properties of Jacobi elliptic functions which will be used in next theorems (cf. e.g. [**1**], [**4**]).

The *elliptic integral* (or *normal elliptic integral*) *of the first kind* has been defined at (1.5). By $\mathcal{K}(t)$ we denote the *complete elliptic integral of the first kind*

$$\mathcal{K}(t) = \mathcal{F}(1, t), \quad \text{and let} \quad \mathcal{K}'(t) = \mathcal{K}(t'), \quad t' = \sqrt{1 - t^2}, \quad t \in (0, 1).$$

Let $\mathcal{E}(w,t)$ denote the *elliptic integral of the second kind*, e.g.

$$(2.1) \qquad \mathcal{E}(w,t) = \int_0^w \sqrt{\frac{1-t^2x^2}{1-t^2}}\,dx,$$

and let $\mathcal{E}(t) = \mathcal{E}(1,t)$ be the *complete elliptic integral of the second kind*, $t \in (0,1)$. Also, set $\mathcal{E}'(t) = \mathcal{E}(t')$. Changing the variable by $x = \sin\theta$ integrals (1.5) and (2.1) reduce to the Legendre form

$$\mathcal{F}(\varphi,t) = \int_0^\varphi (1-t^2\sin^2\theta)^{-1/2}d\theta,$$

$$\mathcal{E}(\varphi,t) = \int_0^\varphi \sqrt{1-t^2\sin^2\theta}\,d\theta.$$

The equation $z = \mathcal{F}(\varphi,t)$, where z is assumed to be real, defines φ as a function of z which has been called by Jacobi the *amplitude* of z and denoted $\varphi = \mathrm{am}(z,t)$. Further Jacobi introduced $\sin(\mathrm{amz})$, and $\cos(\mathrm{amz})$ (*sinus* and *cosinus amplitudinus*) that have several applications in geometry and mechanics. Among numerous interesting properties of elliptic functions, the following will be used in the proof:

$$(2.2) \qquad \lim_{t\to 0^+}\mathcal{K}(t) = \lim_{t\to 0^+}\mathcal{E}(t) = \lim_{t\to 1^-}\mathcal{K}'(t) = \frac{\pi}{2},$$

$$(2.3) \qquad \lim_{t\to 0^+}\mathcal{K}'(t) = \infty, \quad \lim_{t\to 0^+}\mathcal{E}'(t) = 1,$$

$$(2.4) \qquad \begin{aligned} &\lim_{t\to 1^-}\mathcal{K}(t) = \infty, & &\lim_{t\to 1^-}\mathcal{E}(t) = 1, \\ &\lim_{t\to 1^-}(1-t^2)\mathcal{K}(t) = 0, & &\lim_{t\to 1^-}\frac{\mathcal{K}'(t)}{\mathcal{K}(t)} = 0, \end{aligned}$$

$$(2.5) \qquad \lim_{t\to 0^+}\frac{\mathcal{K}'(t)}{\mathcal{K}(t)} = \infty, \quad \text{so that} \quad \lim_{t\to 0^+}\sinh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) = \infty.$$

Functions $\mathcal{K},\mathcal{K}',\mathcal{E},\mathcal{E}'$ are continuous and differentiable on $(0,1)$, and

$$(2.6) \qquad \frac{d\mathcal{K}(t)}{dt} = \frac{\mathcal{E}(t)-(1-t^2)\mathcal{K}(t)}{t(1-t^2)}, \qquad \frac{d\mathcal{E}(t)}{dt} = \frac{\mathcal{E}(t)-\mathcal{K}(t)}{t},$$

$$\frac{d\mathcal{K}'(t)}{dt} = \frac{t^2\mathcal{K}'(t)-\mathcal{E}(t)}{t(1-t^2)},$$

from which the Legendre identity can be derived (cf. [**1**, p. 112])

$$\mathcal{E}(t)\mathcal{K}'(t) + \mathcal{E}'(t)\mathcal{K}(t) - \mathcal{K}(t)\mathcal{K}'(t) = \frac{\pi}{2}.$$

Further, (2.6) and the above identity yields the result

$$(2.7) \qquad \begin{aligned} \frac{d[\mathcal{K}'(t)/\mathcal{K}(t)]}{dt} &= \frac{\mathcal{K}(t)\mathcal{K}'(t)-\mathcal{E}(t)\mathcal{K}'(t)-\mathcal{E}'(t)\mathcal{K}(t)}{t(1-t^2)\mathcal{K}^2(t)} \\ &= -\frac{\pi}{2t(1-t^2)\mathcal{K}^2(t)}. \end{aligned}$$

Moreover

(2.8) $$\frac{\pi}{1 + \sqrt{1 - t^2}} \leq \mathcal{K}(t) \leq \frac{\pi}{2\sqrt{1 - t^2}},$$

(c.f. [**2**], see also [**3**]).

Finally, by a simple computation we arrive at

(2.9) $$\lim_{t \to 1^-} \mathcal{F}(\sqrt{z/t}, t) = \operatorname{arctanh}\sqrt{z}.$$

We now return to functions "extremal" in the class $\mathcal{P}(p_k)$.

Theorem 2.1. *Functions p_k is continuous as regards the parameter $k \in [0, \infty)$.*

Proof. First we observe that

$$\begin{aligned}
\lim_{k \to 0^+} p_k(z) &= \lim_{k \to 0^+} \frac{2}{1 - k^2} \sinh^2\left(A(k)\operatorname{arctanh}\sqrt{z}\right) + 1 \\
&= 2\sinh^2\left(\operatorname{arcsinh}\frac{\sqrt{z}}{\sqrt{1 - z}}\right) + 1 = p_0(z),
\end{aligned}$$

since $\lim_{k \to 0^+} A(k) = \lim_{k \to 0^+} (2/\pi)\arccos k = 1$.

Simultaneously, by (1.4) and setting $t = 2\operatorname{arctanh}\sqrt{z}$ one gets

$$\begin{aligned}
\lim_{k \to 1^-} p_k(z) &= \lim_{k \to 1^-}\left(1 + \frac{2}{1 - k^2}\sinh^2\left(A(k)\operatorname{arctanh}\sqrt{z}\right)\right) \\
&= 1 + 2\lim_{k \to 1^-}\frac{\sinh^2\frac{A(k)t}{2}}{1 - k^2} \\
&= 1 + \frac{t^2}{2}\lim_{k \to 1^-}\left(\frac{A(k)}{\sqrt{1 - k^2}}\right)^2\left(\frac{\sinh\frac{A(k)t}{2}}{\frac{A(k)t}{2}}\right)^2 \\
&= 1 + \frac{t^2}{2}\left(\lim_{k \to 1^-}\frac{\frac{-2}{\pi\sqrt{1-k^2}}}{\frac{-k}{\sqrt{1-k^2}}}\right)^2\left(\lim_{k \to 1^-}\frac{\sinh\frac{A(k)t}{2}}{\frac{A(k)t}{2}}\right)^2 \\
&= 1 + \frac{2}{\pi^2}t^2 = p_1(z).
\end{aligned}$$

Here, $A(k) = (2/\pi)\arccos k \to 0^+$ as $k \to 1^-$.

Finally, we will prove the right-hand continuity of p_k at $k = 1$ if we show that

(2.10) $$\lim_{k \to 1^+}\frac{\sin\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t}, t)\right)}{\sqrt{k^2 - 1}} = \frac{2}{\pi}\operatorname{arctanh}\sqrt{z}.$$

Note that $k = \cosh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)$, so that $\sqrt{k^2 - 1} = \sinh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)$ and if $k \to 1^+$ then $t \to 1^-$, thus (2.10) is equivalent to

(2.11) $$\lim_{t \to 1^-}\frac{\sin\left(\frac{\pi}{4\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t}, t)\right)}{\sinh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)} = \frac{2}{\pi}\operatorname{arctanh}\sqrt{z}.$$

Since by (2.4) and (2.9) both, the numerator and the denominator tend to 0 we need to prove, by the l'Hospital rule, that there exists the limit of the quotient of derivatives of (2.11), or equivalently

$$
(2.12) \quad \lim_{t \to 1^-} \left(\cos\left(\frac{\pi}{2\mathcal{K}(t)} \mathcal{F}(\sqrt{z/t}, t) \right) \cdot \right.
$$
$$
\left. \cdot \frac{\frac{\pi}{2} \left[\frac{\mathcal{E}(t) - (1 - t^2)\mathcal{K}(t)}{t(1 - t^2)\mathcal{K}^2(t)} \mathcal{F}(\sqrt{z/t}, t) - \frac{1}{\mathcal{K}(t)} \frac{d[\mathcal{F}(\sqrt{z/t}, t)]}{dt} \right]}{\cosh\left(\frac{\pi \mathcal{K}'(t)}{2\mathcal{K}(t)} \right) \frac{\pi^2}{4} \frac{1}{t(1 - t^2)\mathcal{K}^2(t)}} \right).
$$

Set

$$
D(z, t) = \left(\mathcal{E}(t) - (1 - t^2)\mathcal{K}(t) \right) \mathcal{F}(\sqrt{z/t}, t) - t(1 - t^2)\mathcal{K}(t) \frac{d[\mathcal{F}(\sqrt{z/t}, t)]}{dt}.
$$

Then (2.12) reduces to

$$
\frac{2}{\pi} \lim_{t \to 1^-} \frac{\cos\left(\frac{\pi}{2\mathcal{K}(t)} \mathcal{F}(\sqrt{z/t}, t) \right)}{\cosh\left(\frac{\pi \mathcal{K}'(t)}{2\mathcal{K}(t)} \right)} D(z, t).
$$

Differentiating with respect to t we obtain from (1.5)

$$
\frac{d[\mathcal{F}(\sqrt{z/t}, t)]}{dt} = \int_0^{\sqrt{z/t}} \frac{t x^2}{\sqrt{(1 - x^2)(1 - t^2 x^2)^3}} dx - \frac{\sqrt{z}}{2t \sqrt{t - z} \sqrt{1 - tz}}
$$

so that

$$
\lim_{t \to 1^-} \frac{d[\mathcal{F}(\sqrt{z/t}, t)]}{dt} = \frac{1}{4} \log \frac{1 - \sqrt{z}}{1 + \sqrt{z}} - \frac{1}{2(1 - \sqrt{z})}.
$$

Since, by (2.4)

$$
\lim_{t \to 1^-} t(1 - t^2)\mathcal{K}(t) = 0, \quad \text{and} \quad \lim_{t \to 1^-} \mathcal{E}(t) = 1
$$

then using (2.9) we obtain

$$
\lim_{t \to 1^-} D(z, t) = \operatorname{arctanh}\sqrt{z}.
$$

Therefore, the above and first and fourth relation from (2.4) finally yield

$$
\frac{2}{\pi} \lim_{t \to 1^-} \frac{\cos\left(\frac{\pi}{2\mathcal{K}(t)} \mathcal{F}(\sqrt{z/t}, t) \right)}{\cosh\left(\frac{\pi \mathcal{K}'(t)}{2\mathcal{K}(t)} \right)} D(z, t) = \frac{2}{\pi} \operatorname{arctanh}\sqrt{z},
$$

that is equivalent to (2.11). It completes the proof. □

Theorem 2.2. *Let $k \in [0, \infty)$ be fixed. The function $p_k(z)$ has the positive Taylor coefficients around the origin.*

The proof of Theorem 2.2 appeared complicated for the case $k \in (1, \infty)$ and has been proved in [11] using the theory of continued fractions. We quote it here for the sake of completeness. Applying Theorem 2.2 estimates of the modulus and the real part of $p \in \mathcal{P}(p_k)$ were derived [11].

3. Coefficient bounds

Now, we find some bounds in the family $\mathcal{P}(p_k)$. The first problem, we discus, is the Fekete-Szegö-Goluzin's problem in the class $\mathcal{P}(p_k)$. We begin by proving the theorem that is itself interesting, since it improves the Livingston result in $\mathcal{P}(p_k)$ [12]. For fixed k, set

$$p_k(z) = 1 + P_1(k)z + P_2(k)z^2 + \cdots, \quad z \in \mathcal{U}.$$

Theorem 3.1. *Let* $0 \leq k < \infty$ *be fixed. Then*

$$(3.1) \qquad\qquad |P_1^2(k) - P_2(k)| \leq P_1(k).$$

Proof. The inequality (3.1) is obvious for $k = 0$, by Livingston result [12], therefore we assume $k > 0$. We consider separately cases:

1. $k \in (0, 1)$,
2. $k = 1$,
3. $k > 1$.

Case 1. By virtue of (1.4) a precise form of coefficients of p_k were derived (cf. [5], [9])

$$P_1(k) = \frac{2A^2(k)}{1 - k^2}, \qquad P_2(k) = \frac{2A^2(k)(A^2(k) + 2)}{3(1 - k^2)} = P_1(k)\frac{A^2(k) + 2}{3},$$

$$A(k) = \frac{2}{\pi} \arccos k.$$

Since $P_1(k)$ is positive for $k \in (0, 1)$, the inequality (3.1) reduces to proving $|P_1(k) - (A^2(k) + 2)/3| \leq 1$. Note that

$$P_1(k) - \frac{A^2(k) + 2}{3} = \frac{A^2(k)(5 + k^2) + 1 - k^2}{3(1 - k^2)} > 0$$

for $k \in (0, 1)$, then it suffices to show the inequality $P_1(k) - (A^2(k) + 2)/3 \leq 1$, or equivalently

$$\frac{5(1 - k^2)}{k^2 + 5} - \frac{4}{\pi^2} \arccos^2 k \geq 0.$$

Set

$$h(k) := \sqrt{\frac{5(1 - k^2)}{k^2 + 5}} - \frac{2}{\pi} \arccos k.$$

Then functions h is well defined on a closed interval $[0, 1]$ and

$$h'(k) = \frac{2}{\pi\sqrt{1 - k^2}} \left[1 - \frac{3\sqrt{5}\pi k}{(k^2 + 5)^{3/2}} \right].$$

Note that $h'(0) > 0, h'(1^-) < 0$. Since $g(k) = \frac{3\sqrt{5}\pi k}{(k^2+5)^{3/2}}$ is monotone then there exists the only point $k_0 \in (0,1)$ such that $h'(k_0) = 0$. Then $h'(k) > 0$ in $0 < k < k_0$ and $h'(k) < 0$ in $0 < k_0 < k < 1$. Therefore $h(k) \geq \min\{h(0), h(1)\} = 0$ in $0 \leq k < 1$, so that the proof of the case 1. is complete.

Case 2. In this instance $P_1(k) = 8/\pi^2$ and $P_2(k) = 16/(3\pi^2) = 2P_1(k)/3$ (cf. [**14**], [**15**]). The inequality (3.1) now follows immediately by means of the relation $-1/3 < P_1(k) < 5/3$.

Case 3. To this purpose we use the following form of p_k for $k > 1$

$$(3.2) \qquad p_k(z) = \frac{1}{k^2-1}\sin\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(u(z)/\sqrt{t}, t)\right) + \frac{k^2}{k^2-1},$$

(cf. [**5**, p. 20]), where $u(z) = (z-\sqrt{t})/(1-\sqrt{t}z)$ and $k = \cosh(\mu(t)/2)$ for $t \in (0,1)$. In view of [**5**]

$$P_1(k) = \frac{\pi^2}{4(k^2-1)\mathcal{K}^2(t)\sqrt{t}(1+t)},$$

$$P_2(k) = P_1(k)\frac{4\mathcal{K}^2(t)(t^2+6t+1) - \pi^2}{24\mathcal{K}^2(t)\sqrt{t}(1+t)} =: P_1(k)D(k),$$

see also [**9**]. Since $P_1(k)$ is positive for all $t \in (0,1)$, the inequality (3.1) will hold if $|P_1(k) - D(k)| \leq 1$, equivalently $P_1(k) \leq D(k)+1$ and $D(k) \leq P_1(k)+1$. Now, we will show that the inequality $D(k) \leq P_1(k) + 1$ holds. We rewrite the inequality $D(k) \leq P_1(k) + 1$ into the form

$$(3.3) \qquad \frac{4\mathcal{K}^2(t)(t^2+6t+1) - \pi^2}{4\mathcal{K}^2(t)} \leq \frac{3\pi^2}{2(k^2-1)\mathcal{K}^2(t)} + 6\sqrt{t}(1+t).$$

Observing that $k^2 - 1 = \sinh^2(\pi\mathcal{K}'(t)/(4\mathcal{K}(t)))$, the relation (3.3) becomes

$$(3.4) \qquad t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - \frac{\pi^2}{4\mathcal{K}^2(t)} \leq \frac{3\pi^2}{2\mathcal{K}^2(t)\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)}.$$

Observe next, that if $k \to 1^+$ then $t \to 1^-$ and the case $k \to \infty$ corresponds to the case $t \to 0^+$. Thus, we may study the inequality (3.4) as regards $t \in (0,1)$. The left-hand side function satisfies

$$(3.5) \qquad t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - \frac{\pi^2}{4\mathcal{K}^2(t)} \leq t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - (1-t^2).$$

by means of the right-hand estimation in (2.8). Set

$$w(t) := t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - (1-t^2) = 2t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t}.$$

The function $w(t)$ is defined on the closed interval $[0,1]$ and it decreases continuously from $w(0) = 0$ to $w(1) = -4$. Indeed $w'(t) = 4t - 9\sqrt{t} + 6 - 3/\sqrt{t} = (t-3) + 3(\sqrt{t}-1)^3/\sqrt{t} < 0$ on $(0,1)$. Therefore $w(t) < 0$ on $(0,1)$.

Now we show that the right-hand function of (3.4) is positive in $(0,1)$ and increases from 0 to $96/\pi^2$. Let

$$W(t) := \frac{3\pi^2}{2\mathcal{K}^2(t)\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)}.$$

Making use the first relation in (2.2) and the last relation in (2.5) we find that

$$\lim_{t\to 0^+} W(t) = 0.$$

Also, after necessary transformations, we obtain

$$\lim_{t\to 1^-} W(t) = \lim_{t\to 1^-} \frac{24\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)^2}{(\mathcal{K}')^2(t)\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)} = \frac{96}{\pi^2},$$

because of the last formula in (2.4) and the fact that $\lim_{x\to 0}\sinh x/x = 1$. Now, we will show that $W(t)$ is increasing. Differentiating $W(t)$ and using (2.6) and (2.7) one gets

$$W'(t) = -\frac{3\pi^2\left[(\mathcal{E}(t) - (1-t^2)\mathcal{K}(t))\sinh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - \frac{\pi^2}{8\mathcal{K}(t)}\cosh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)\right]}{t(1-t^2)\mathcal{K}^3(t)\sinh^3\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)}.$$

In order to show that $W'(t) > 0$ it suffices to prove that the expression in the square brackets of $W'(t)$ is negative for $t \in (0,1)$. Such relation may be rewritten in the form

$$(\mathcal{E}(t) - (1-t^2)\mathcal{K}(t))\sinh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) < \frac{\pi^2}{8\mathcal{K}(t)}\cosh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)$$

or

(3.6) $$\frac{8}{\pi^2}\left[\frac{\mathcal{E}(t) - (1-t^2)\mathcal{K}(t)}{\mathcal{K}(t)}\right] < \coth\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right).$$

Set

$$\phi(t) = \mathcal{E}(t) - (1-t^2)\mathcal{K}(t).$$

Then, in view of (2.6) we have

$$\frac{d\phi(t)}{dt} = \frac{\mathcal{E}(t) - \mathcal{K}(t)}{t} - \frac{-2t^2\mathcal{K}(t) + \mathcal{E}(t) - \mathcal{K}(t) + t^2\mathcal{K}(t)}{t} = t\mathcal{K}(t) > 0$$

in $(0,1)$. Moreover $\phi(0^+) = 0$ and $\phi(1^-) = 1$ by the second and third relation in (2.4). Thus $0 < \phi(t) < 1$ in $(0,1)$. Note also that $\mathcal{K}(t)$ is increasing from $\pi/2$ to ∞, when $t \in (0,1)$. Therefore

$$\frac{8}{\pi^2}\left[\frac{\mathcal{E}(t) - (1-t^2)\mathcal{K}(t)}{\mathcal{K}(t)}\right] < \frac{8}{\pi^2}\frac{1}{\pi/2} = \frac{16}{\pi^3} < 1,$$

whereas the right-hand side of (3.6) is greater than 1 since $\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) > 0$. Then the inequality (3.6) holds, equivalently $W'(t) > 0$ on $(0,1)$ so that W increases

on $(0, 1)$. Thus, having in view properties of $w(t)$ and $W(t)$ we conclude that $w(t) \leq W(t)$ for all $t \in (0, 1)$, so that (3.4) is satisfied.

Next, we will show that $P_1(k) \leq D(k) + 1$ holds for $t \in (0, 1)$, or equivalently

$$\frac{3\pi^2}{2\mathcal{K}^2(t) \sinh^2\left(\frac{\pi \mathcal{K}'(t)}{4\mathcal{K}(t)}\right)} \leq t^2 + 6t\sqrt{t} + 6t + 6\sqrt{t} + 1 - \frac{\pi^2}{2\mathcal{K}^2(t)},$$

by reversing the inequality in (3.4). Since $\mathcal{K}(t) > \frac{\pi}{2}$ we have $-\frac{\pi^2}{4\mathcal{K}^2(t)} > -1$ so that it suffices to show that

$$(3.7) \qquad \frac{3\pi^2}{2\mathcal{K}^2(t) \sinh^2\left(\frac{\pi \mathcal{K}'(t)}{2\mathcal{K}(t)}\right)} \leq t^2 + 6t\sqrt{t} + 6t + 6\sqrt{t}.$$

Set

$$r(t) := t^2 + 6t\sqrt{t} + 6t + 6\sqrt{t}.$$

Then $r'(t) = 2t + 9\sqrt{t} + 6 + 3/\sqrt{t} > 0$ on $(0, 1)$ and $r(0) = 0$, $r(1) = 19$. Moreover $r(1/\sqrt{2}) \approx 14.158379 > \frac{96}{\pi^2}$ whereas the value $\frac{96}{\pi^2}$ is the supremum of the left hand side of (3.7) as was shown in the first part of the proof of that case. Thus, it suffices to show (3.7) for $t \in (0, 1/\sqrt{2})$. Since $\mathcal{K}(t) > \frac{\pi}{2}$ then

$$\frac{3\pi^2}{2\mathcal{K}^2(t) \sinh^2\left(\frac{\pi \mathcal{K}'(t)}{4\mathcal{K}(t)}\right)} < \frac{6}{\sinh^2\left(\frac{\pi \mathcal{K}'(t)}{4\mathcal{K}(t)}\right)}.$$

Now, we will show that

$$\frac{6}{\sinh^2\left(\frac{\pi \mathcal{K}'(t)}{4\mathcal{K}(t)}\right)} \leq 6(t + \sqrt{t}) \leq r(t)$$

for $t \in (0, 1/\sqrt{2}]$, which concludes the desired result. The last inequality is obvious therefore it suffices to show that

$$(3.8) \qquad (t + \sqrt{t}) \sinh^2\left(\frac{\pi \mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - 1 \geq 0.$$

Let

$$s(t) := (t + \sqrt{t}) \sinh^2\left(\frac{\pi \mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - 1.$$

Now we prove that $s(t)$ decreases in $(0, 1/\sqrt{2})$ to $s(1/\sqrt{2}) > 0$. Differentiating, we obtain

$$s'(t) = \frac{1}{2} \sinh\left(\frac{\pi \mathcal{K}'(t)}{2\mathcal{K}(t)}\right) \left[\left(1 + \frac{1}{2\sqrt{t}}\right) \tanh\left(\frac{\pi \mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - \frac{\pi^2(\sqrt{t} + 1)}{4\sqrt{t}(1 - t^2)\mathcal{K}^2(t)}\right].$$

Since $\sinh\left(\frac{\pi \mathcal{K}'(t)}{2\mathcal{K}(t)}\right)$ is positive for $t \in (0, 1/\sqrt{2}]$ then $s'(t) < 0$ if and only if the expression in square brackets of $s'(t)$ is negative, or equivalently

$$\left(1 + \frac{1}{2\sqrt{t}}\right) \tanh\left(\frac{\pi \mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - \frac{\pi^2(\sqrt{t} + 1)}{4\sqrt{t}(1 - t^2)\mathcal{K}^2(t)} < 0.$$

The above will be fulfilled if

$$\frac{2\sqrt{t}+1}{2(\sqrt{t}+1)}\tanh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - \frac{\pi^2}{4(1-t^2)\mathcal{K}^2(t)} < 0,$$

or, by means of the relation $\tanh x < 1$, when the inequality

$$\frac{2\sqrt{t}+1}{2(\sqrt{t}+1)} - \frac{\pi^2}{4(1-t^2)\mathcal{K}^2(t)} < 0$$

holds. It is easy to see that $b(t) = \frac{2\sqrt{t}+1}{2(1+\sqrt{t})}$ is increasing on $(0, 1/\sqrt{2})$ with the maximal value $b(1/\sqrt{2}) \approx 0.73$. Let

$$c(t) := \frac{\pi^2}{4(1-t^2)\mathcal{K}^2(t)}.$$

Since, by (2.6),

$$c'(t) = \frac{\pi^2}{2}\frac{\mathcal{K}(t) - \mathcal{E}(t)}{t(1-t^2)^2\mathcal{K}^3(t)}$$

and $\mathcal{K}(t) > \mathcal{E}(t)$ on $(0,1)$, then $c'(t) > 0$ on $(0,1)$ so does on $(0, 1/\sqrt{2})$ and therefore $c(t) > c(0^+) = 1$ for all $t \in (0, 1/\sqrt{2})$. Thus $b(t) - c(t) < 0$ for all $t \in (0, 1/\sqrt{2})$ and hence $s(t)$ decreases on $(0, 1/\sqrt{2})$.

Next, we show that $s(0^+) = \infty$. Note that

$$\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) = \left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)^2\left[1 + \frac{1}{3!}\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)^2 + \cdots\right]^2.$$

Then

$$\lim_{t\to 0^+}(t+\sqrt{t})\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)$$

$$= \lim_{t\to 0^+}\frac{\pi^2(1+\sqrt{t})}{4\mathcal{K}^2(t)}\sqrt{t}\,(\mathcal{K}'(t))^2\left[1 + \frac{1}{3!}\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)^2 + \cdots\right]^2.$$

By properties of $\mathcal{K}(t)$ at 0^+ (the relation (2.2)) we have

$$\lim_{t\to 0^+}\frac{\pi^2(1+\sqrt{t})}{4\mathcal{K}^2(t)} = 1,$$

so that we need to calculate the limit

$$\lim_{t\to 0^+}\sqrt{t}\,(\mathcal{K}'(t))^2.$$

Applying (2.8) to the value $\sqrt{1-t^2}$ we obtain

(3.9) $$\frac{\pi}{1+t} \leq \mathcal{K}(\sqrt{1-t^2}) = \mathcal{K}'(t) \leq \frac{\pi}{2t},$$

and since $\sqrt{t}/(1+t)^2$ and $\sqrt{t}/t^2$ tend to ∞, as $t \to 0^+$, we conclude that

(3.10) $$\lim_{t\to 0^+}\sqrt{t}\,(\mathcal{K}'(t))^2 = \infty.$$

Thus also $\lim_{t\to 0^+} s(t) = \infty$. Moreover $s(1/\sqrt{2}) \approx 0.9614 > 0$ so that we obtain the desired result. Hence the proof of the case 3. is complete. $\qquad\square$

Theorem 3.2. *Let $0 \le k < \infty$ be fixed, and let a function $p \in \mathcal{P}(p_k)$ be such that $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$. Then*

$$(3.11) \qquad\qquad |b_1^2 - b_2| \le P_1(k).$$

The equality holds if $p(z)$ is $p_k(z^2)$ or one of its rotation.

Proof. Since $p \prec p_k$ then, in view of a definition of the subordination, there exists a function $\omega(z) = \alpha_1 z + \alpha_2 z^2 + \cdots$, $|\omega(z)| < 1$ such that $p(z) = p_k(\omega(z))$, therefore

$$1 + b_1 z + b_2 z^2 + \cdots = 1 + P_1(k)\alpha_1 z + z^2 (P_1(k)\alpha_2 + P_2(k)\alpha_1^2) + \cdots.$$

Comparing the coefficients of z and z^2 we have $b_1 = P_1(k)\alpha_1$ and $b_2 = P_1(k)\alpha_2 + P_2(k)\alpha_1^2$, thus

$$\begin{aligned} |b_1^2 - b_2| &= |P_1^2(k)\alpha_1^2 - P_1(k)\alpha_2 - P_2(k)\alpha_1^2| \\ &= |\alpha_1^2(P_1^2(k) - P_2(k)) - P_1(k)\alpha_2| \\ &\le |\alpha_1|^2 |P_1^2(k) - P_2(k)| + |P_1(k)||\alpha_2|. \end{aligned}$$

For the Schwarz' function ω the classical inequality $|\alpha_2| \le 1 - |\alpha_1|^2$ holds then, on account (3.1), we conclude

$$\begin{aligned} |b_1^2 - b_2| &\le |\alpha_1|^2 |P_1^2(k) - P_2(k)| + |P_1(k)|(1 - |\alpha_1|^2) \\ &= |\alpha_1|^2 \left[|P_1^2(k) - P_2(k)| - P_1(k) \right] + P_1(k) \\ &\le P_1(k), \end{aligned}$$

and the proof of the inequality of (3.11) is complete.

The equality in (3.11) holds if $|b_1| = 0$ and $|b_2| = P_1(k)$, or equivalently, $p(z)$ is $p_k(z^2)$ or one of its rotations. $\qquad\square$

Remark. Observe that the bound like (3.11) can be used in those subclasses of Carathéodory class for which the inequality similar to (3.1) holds. Let

$$(3.12) \qquad\qquad q(z) = 1 + c_1 z + c_2 z^2 + \cdots,$$

be such that

$$(3.13) \qquad\qquad |c_1^2 - c_2| \le c_1, \quad \text{with} \quad c_1 \ge 0,$$

it means q satisfies the bounds similar to (3.1). Then, reasoning along the same line as in Theorem 3.2 we may prove that for $p \in \mathcal{P}(q)$, $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$ the inequality

$$(3.14) \qquad\qquad |b_1^2 - b_2| \le c_1,$$

is satisfied.

For instance, if $0 < \gamma \le 1$ then the function $\varphi(z) = [(1 + z)/(1 - z)]^\gamma$ maps the unit disk onto an angle, symmetric with respect to real axis, of width $\gamma\pi$ and contained in the right half-plane. Moreover, $\varphi(z) = 1 + 2\gamma z + 2\gamma^2 z^2 + \cdots$. Then, $|c_1^2 - c_2| = 2\gamma^2 \le 2\gamma = c_1$ therefore (3.13) is satisfied, so that (3.14) applies.

Concluding, if $p \in \mathcal{P}(\varphi) = \{q : q \prec \varphi\}$ and $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$, then $|b_1^2 - b_2| \leq 2\gamma$. This inequality remarkable improves the result $|b_1^2 - b_2| \leq 2$, due to Ma and Minda [13]. Similarly, the family $\mathcal{P}((1 + (1 - 2\beta)z)/(1 - z))$ can be treated. In this instance we obtain the inequality $|b_1^2 - b_2| \leq 2(1 - \beta)$ for $p \in \mathcal{P}((1 + (1 - 2\beta)z)/(1 - z))$.

Theorem 3.3. *Let $0 \leq k < \infty$ be fixed, and let a function $p \in \mathcal{P}(p_k)$ be of the form $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$. Then*

$$(3.15) \qquad |b_2 - \mu b_1^2| \leq \begin{cases} P_1(k) - \mu P_1^2(k) & \mu \leq 0, \\ P_1(k) & \mu \in (0, 1], \\ P_1(k) + (\mu - 1)P_1^2(k) & \mu \geq 1. \end{cases}$$

When $\mu < 0$ or $\mu > 1$, the equality holds if $p(z)$ is $p_k(z)$ or one of its rotations. If $0 < \mu < 1$ then the equality holds if $p(z) = p_k(z^2)$ or one of its rotation.

Proof. Since $p \prec p_k$ then by Rogosinski Subordination Theorem we have $|b_n| \leq P_1(k)$ for $n \geq 1$ and each fixed $k \in [0, \infty)$ First assume that $\mu \geq 1$. In view of Theorem 3.2, we have $|b_2 - b_1^2| \leq P_1(k)$ therefore we obtain

$$|b_2 - \mu b_1^2| \leq |b_1^2 - b_2| + (\mu - 1)|b_1|^2 \leq P_1(k) + (\mu - 1)P_1^2(k).$$

Next, suppose that $\mu \leq 0$. Then

$$|b_2 - \mu b_1^2| \leq |b_2| + (-\mu)|b_1|^2 \leq P_1(k) - \mu P_1^2(k).$$

Finally, if $0 < \mu \leq 1$ then $\mu = 1/t$ with $t \geq 1$. Hence one gets

$$\begin{aligned} |b_2 - \mu b_1^2| &= |b_2 - \frac{1}{t}b_1^2| = \frac{1}{t}|tb_2 - b_1^2| = \frac{1}{t}|(t-1)b_2 + b_2 - b_1^2| \\ &\leq \frac{1}{t}[(t-1)|b_2| + |b_2 - b_1^2|] \\ &\leq \frac{1}{t}[(t-1)P_1(k) + P_1(k)] = P_1(k), \end{aligned}$$

and the proof of all cases of (3.15) is complete.

When $\mu < 0$ or $\mu > 1$, the equality holds if and only if $|b_1| = P_1(k)$, that is, $p(z) = p_k(z)$ or one of its rotation. If $0 < \mu < 1$ then the equality holds if $|b_1| = 0$ and $|b_2| = P_1(k)$, or equivalently, $p(z)$ is $p_k(z^2)$ or one of its rotations. $\qquad \square$

Remark. In the paper [13] Ma and Minda proved similar bounds in the class $\mathcal{P}$. For instance, when $\mu \leq 0$ authors obtained the estimate $|b_2 - \mu b_1^2| \leq 2 - 4\mu$. Observe that in the case of $\mathcal{P}(p_k)$ the result is far better; the same it holds in the remaining range of the constant μ.

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