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SOME PROPERTIES OF COMPOSITION OPERATORS ON THE DIRICHLET SPACE

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Abstract. In this paper we investigate composition operators induced on the Dirichlet space by linear fractional maps. We characterize the essential normality in this setting, obtain conditions for the linear fractional symbols φ and ψ of the unit disc for which $C_\varphi C_\psi^*$ or $C_\psi^* C_\varphi$ is compact, and investigate the shape of the numerical range for linear fractional composition operators induced on the Dirichlet space.

1. Introduction

In a 1988 paper (cf. [**7**]), C. Cowen found a formula expressing the adjoint of a composition operator C_φ induced, on the Hardy space, by a linear fractional transformation of the unit disc, as a product of Toeplitz operators and another linear fractional composition operator. In [**14**] P. Hurst obtained an analogous expression for the adjoint of C_φ acting on $A^2_\alpha(\mathbb{D})$, the weighted Bergman space.

Recently in [**10**] E. Gallardo and A. Montes obtained a formula for the adjoint of a linear fractional composition operator acting on the classical Dirichlet space, as another linear fractional composition operator plus a two rank operator.

In this paper we investigate the composition operators induced, on the classical Dirichlet space, by a linear fractional transformation of the unit disc. In Section 2 we give the notation and preliminary results. In Section 3 we use the E. Gallardo and A. Montes' formula in order to characterize the essentially normal composition operators induced, on the Dirichlet space, by linear fractional maps. In Section 4 we obtain conditions for the symbols φ and ψ, two linear fractional transformations of the unit disc, at which the operator $C_\varphi C_\psi^*$ or $C_\psi^* C_\varphi$ is compact. Finally in Section 5 we investigate the shape of the numerical range for composition operators induced on the Dirichlet space by linear fractional maps.

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2. Preliminaries

A holomorphic function φ that takes the unit open disc $\mathbb{D}$ into itself induces a linear *composition operator* C_φ on the space $\mathrm{Hol}\,(\mathbb{D})$ of all holomorphic functions on $\mathbb{D}$ as follows:

$$C_\varphi f = f \circ \varphi, \qquad (f \in \mathrm{Hol}\,(\mathbb{D})).$$

A lot of work has been done studying composition operators acting on functional Hilbert spaces in $\mathbb{D}$ (and other domains) (cf. [**8**], [**20**], [**15**], and the references therein); in particular on Hardy spaces, Bergman spaces and Dirichlet spaces.

We recall that a functional Hilbert space $\mathcal{H} \neq (0)$ is a Hilbert space of complex-valued functions defined on the set X such that, for each $x \in X$ the point-evaluation functional $f \mapsto f(x)$ is bounded. The Riesz Representation Theorem says that for each $x \in X$ there exists a function $K_x \in \mathcal{H}$ such that $\langle f, K_x \rangle = f(x)$ for each $f \in \mathcal{H}$. The function K_x is called the *reproducing kernel* at x in $\mathcal{H}$.

The Dirichlet space, which we denoted by $\mathcal{D}$, consists of all holomorphic functions f on $\mathbb{D}$ which have finite norm given by

$$\|f\|^2_{\mathcal{D}} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z),$$

where dA is the normalized Lebesgue area measure of the unit disk. The term $|f(0)|^2$ avoids that constant functions have norm zero.

If f is univalent, then $\int_{\mathbb{D}} |f'(z)|^2 \, dA(z)$ is precisely the area of $f(\mathbb{D})$. In general $\int_{\mathbb{D}} |f'(z)|^2 \, dA(z)$ still yields the area of the image of f on $\mathbb{D}$ if one takes multiplicities into account. It is well known that

$$\int_{\mathbb{D}} |f'(z)|^2 \, dA(z) = \sum_{n=1}^{\infty} n |\widehat{f}(n)|^2,$$

where $\widehat{f}(n)$ denotes the n^{th} Taylor coefficient of f.

The Dirichlet space $\mathcal{D}$ is a functional Hilbert space on $\mathbb{D}$ and the function

$$K_w(z) = 1 + \log \frac{1}{1 + \overline{w} z}, \qquad (z \in \mathbb{D}),$$

is the reproducing kernel at w in the Dirichlet space.

An easy calculus shows how C_φ^*, the adjoint operator of C_φ, acts on the reproducing kernel. Indeed, for each $w \in \mathbb{D}$ we have $C_\varphi^* K_w = K_{\varphi(w)}$.

If φ is a holomorphic self-map of the unit disk, the composition operator C_φ induced on $\mathrm{Hol}(\mathbb{D})$, the space of all holomorphic functions on $\mathbb{D}$ endowed with the topology of uniform convergence on compact subsets is continuous. However, in general in the Dirichlet space not all composition operators are bounded. Nevertheless, it is known that if φ is univalent then C_φ is bounded and this is the case we are considering in this paper.

Indeed, we consider composition operators induced by linear fractional maps.

3. Essentially normal composition operators

The goal in the study of composition operators is to understand how properties of composition operators relate to the behavior of their inducing maps. Along this direction, in [**1**] the essentially normal composition operators induced on the Hardy space H^2 by a linear fractional selfmap φ of the unit disc, are characterized.

Recall that an operator T on a Hilbert space is called *normal* if $TT^* = T^*T$, and *essentially normal* if $T^*T - TT^*$ is compact. Compact and normal operators are trivially essentially normal, so we say that an operator is *nontrivially essentially normal* if it is essentially normal, but neither normal nor compact.

For bounded operators A and B on a Hilbert space, we use the notation

$$[A, B] := AB - BA,$$

for the *commutator* of A and B; in particular A is essentially normal if and only if $[A^*, A]$ is compact.

The main result in [**1**] is: *A composition operator induced on H^2 by a linear fractional self-map of the unit disc is nontrivially essentially normal if and only if it is induced by a parabolic non-automorphism one.* Here, by a non-automorphism we mean a linear fractional map which is not an automorphism of the unit disc. The proofs in [**1**] are based on Cowen's adjoint formula.

In [**16**] R. Wier and B. MacCluer study the analogous question in the setting of Bergman spaces. They obtain that the essentially normal linear fractional composition operators on the Bergman spaces are exactly the same as those on the Hardy space: the non trivial ones are precisely those whose symbol is a parabolic non-automorphism one. The generalized expression of [**14**] for the adjoint of C_φ acting on $A^2_\alpha(\mathbb{D})$ is crucial in their work.

In this note we study the question: Which composition operators C_φ induced on $\mathcal{D}$ by a linear fractional selfmap φ of the unit disc, are nontrivially essentially normal? We follow the idea from proofs in [**1**] and use results from the recent paper by E. Gallardo-Gutiérrez, and A. Montes-Rodríguez [**10**], and the ideas in [**13**] in order to obtain the following result:

Main Theorem. *A composition operator C_φ induced on $\mathcal{D}$ by a linear fractional selfmap φ of the unit disc is essentially normal if and only if φ is not a hyperbolic non-automorphism with a fixed point on $\partial\mathbb{D}$.*

In order to prove our result we recall well known facts about linear fractional maps. If a, b, c and d are complex numbers with $ad - bc \neq 0$, then the linear fractional map

$$\varphi(z) = \frac{az + b}{cz + d},$$

is a one-to-one map from the extended complex plane $\widehat{\mathbb{C}}$ onto itself. Indeed we define $\varphi(\infty) = a/c$, and $\varphi(-d/c) = \infty$ if $c \neq 0$, while $\varphi(\infty) = \infty$ if $c = 0$.

A linear fractional map which is not the identity has one or two fixed points in the extended complex plane. Two linear fractional maps φ and ψ are said to be conjugate if there is another linear fractional map T such that $\varphi = T^{-1} \circ \psi \circ T$.

If φ has only one fixed point α, then it is called *parabolic* and it is conjugate under $Tz = 1/(z-\alpha)$ to $\psi(z) = z + \tau$ with $\tau \neq 0$. Observe that the derivative at the fixed point is 1.

If φ has two distinct fixed points α and β, then φ is conjugate under $Tz = (z-\alpha)/(z-\beta)$ to $\psi(z) = \mu z$. In this case, the linear fractional map is called *elliptic* if $|\mu| = 1$; *hyperbolic* if $\mu > 0$ and *loxodromic*, otherwise (see [**20**] for more details). It is not difficult to show that the derivative at the fixed points satisfy $\varphi'(\alpha) = 1/\varphi'(\beta)$.

It is easy to see that if φ is parabolic, then the sequence $\{\varphi_n(z)\}$ converges for every $z \in \mathbb{D}$, uniformly on compact subsets to the fixed point α. In this case, we say that α is an *attractive* fixed point; if φ is hyperbolic or loxodromic, its fixed points are one attractive and one repulsive. When φ is elliptic, its fixed points are neither attractive nor repulsive. For φ loxodromic or hyperbolic the attractive fixed point of φ is the one for which the modulus of the derivative is strictly less than one.

Additionally when we put the condition $\varphi(\mathbb{D}) \subset \mathbb{D}$, we obtain some restrictions on the location of the fixed points of φ as follows.

Proposition 3.1. *(See* [**20**, p. 5]*) If φ is a linear fractional map with $\varphi(\mathbb{D}) \subset \mathbb{D}$ then:*

1. *If φ is parabolic, then its fixed point is on $\partial\mathbb{D}$.*
2. *If φ is hyperbolic, the attractive point is in $\overline{\mathbb{D}}$ and the other fixed point outside of $\mathbb{D}$. Both fixed points are on $\partial\mathbb{D}$ if and only if φ is an automorphism of $\mathbb{D}$.*
3. *If φ is loxodromic or elliptic, one fixed point is in $\mathbb{D}$ and the other fixed point outside of $\overline{\mathbb{D}}$. The elliptic ones are always automorphisms of $\mathbb{D}$. The fixed point in $\mathbb{D}$ for the loxodromic ones is attractive.*

We are interested solely in non-compact operators, so we consider only φ linear fractional maps with $\|\varphi\|_\infty = 1$; this is, with $\sup_{z\in\mathbb{D}} |\varphi(z)| = 1$. Indeed, if $\|\varphi\|_\infty < 1$ is easy to see that C_φ is a Hilbert-Schmidt operator (cf. [**9**, Lemma 2.1]) and so compact.

We make other reduction. Following [**13**] (cf. also [**10**] and [**11**] where the idea in [**13**] is used) we consider $\mathcal{D}_0$, the space of functions in the Dirichlet space $\mathcal{D}$ that vanish at the origin. Since constant functions are invariant under any composition operator, the operator C_φ acting on $\mathcal{D}$ is of the form

$$C_\varphi = \begin{pmatrix} 1 & X \\ 0 & \widetilde{C}_\varphi \end{pmatrix} : \begin{pmatrix} \mathcal{D} \ominus \mathcal{D}_0 \\ \mathcal{D}_0 \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{D} \ominus \mathcal{D}_0 \\ \mathcal{D}_0 \end{pmatrix},$$

where $\widetilde{C}_\varphi$ is the *compression* of C_φ to $\mathcal{D}_0$, i.e. $\widetilde{C}_\varphi = P_{\mathcal{D}_0} C_\varphi|_{\mathcal{D}_0}$, with $P_{\mathcal{D}_0}$ the orthogonal projection onto $\mathcal{D}_0$.

It is easy to see that C_φ is essentially normal if and only if $\widetilde{C}_\varphi$ is essentially normal. So, to prove our results we may and do consider $\widetilde{C}_\varphi$ and $\mathcal{D}_0$. Since there is no risk of confusion, we still denote $\widetilde{C}_\varphi$ by C_φ.

The following result in [**10**] characterizes linear fractional composition operators which are normal on $\mathcal{D}_0$:

Proposition 3.2. [**10**, Th. 4.1] *A linear fractional composition operator C_φ is normal on $\mathcal{D}_0$ if and only if one of the following holds:*

1. *The symbol φ is an automorphism.*
2. *The symbol φ is parabolic.*
3. *The symbol φ has an interior and exterior fixed point and φ is conjugate to $z \mapsto \mu z$ with $0 < |\mu| < 1$.*

From this proposition and the precedent observations the following corollary is clear:

Corollary 3.3. *A linear fractional composition operator C_φ is essentially normal on $\mathcal{D}$ if one of the following holds:*

1. *The symbol φ is an automorphism.*
2. *The symbol φ is parabolic.*
3. *The symbol φ has an interior and exterior fixed point and φ is conjugate to $z \mapsto \mu z$ with $0 < |\mu| < 1$.*

We consider now the remaining case of a linear fractional self-map of $\mathbb{D}$: φ is a hyperbolic non-automorphism with a fixed point on $\partial\mathbb{D}$. We will prove that in this case C_φ is not essentially normal and thus we will characterize the linear fractional composition operators being essentially normal in $\mathcal{D}$.

We use a representation of the adjoint of a linear-fractionally induced composition operator on $\mathcal{D}_0$ obtained in [**10**] analogous to the Cowen's adjoint formula:

Proposition 3.4. [**10**, Th. 3.2] *Let $\varphi(z) = (az+b)/(cz+d)$ be a linear fractional self map of $\mathbb{D}$ and consider C_φ acting on $\mathcal{D}_0$. Then, $C_\varphi^* = C_\psi$ where $\psi(z) = (\overline{a}z - \overline{c})/(-\overline{b}z + \overline{d})$.*

We observe that

$$\psi = \rho \circ \varphi^{-1} \circ \rho, \qquad \text{where} \qquad \rho(z) = 1/\overline{z},$$

(i.e. ρ is the mapping of inversion in the unit circle), and the inverse refers to φ viewed as a univalent mapping of $\widehat{\mathbb{C}}$ onto itself. It also follows from this formula that the fixed points of ψ are the ρ-images of the fixed points of φ; in particular φ and ψ have the same boundary fixed points.

We will also need the following function-theory result.

Lemma 3.5. [**1**, Lemma 5.1] *Suppose that φ is a fractional linear selfmap of $\mathbb{D}$ with a fixed point $\omega \in \partial\mathbb{D}$. Then:*

1. *If φ is not an automorphism then $\psi \circ \varphi$ and $\varphi \circ \psi$ are parabolic (with fixed point at ω).*
2. *$\psi \circ \varphi$ commutes with $\varphi \circ \psi$,*

where ψ is the map that occurs in proposition 3.4.

Finally, we have:

Proposition 3.6. *If φ, a linear fractional self-map of $\mathbb{D}$ is a hyperbolic non-automorphism with a fixed point on $\partial\mathbb{D}$ then C_φ is not essentially normal.*

Proof. By Proposition 3.4 we need only to show that $[C_\psi, C_\varphi]$ (acting in the space $\mathcal{D}_0$) is not compact. Both ψ and φ share a fixed point on $\partial\mathbb{D}$. Since φ is hyperbolic, it has another fixed point p in the Riemann sphere, but not on $\partial\mathbb{D}$ (since φ is not an automorphism of $\mathbb{D}$). Now ψ is also hyperbolic, and its non-boundary fixed point is $\rho(p) \neq p$. Thus ψ does not commute with φ (else $\psi(p)$ would be a fixed point of φ not on $\partial\mathbb{D}$ and not equal to p, thus endowing φ with too many fixed points). It follows that $\gamma := \varphi \circ \psi$ and $\chi := \psi \circ \varphi$ are distinct linear fractional selfmaps of $\mathbb{D}$ with the same boundary fixed point as φ. By Lemma 3.5 γ and χ are both parabolic, and since they have the same fixed point, they commute, and therefore so do the composition operators C_γ and C_χ.

The maps γ and χ are conjugate to the translations $\tau_a(z) = z + a$ and $\tau_b(z) = z + b$ respectively ($a \neq b$, $\operatorname{Im} a > 0$ and $\operatorname{Im} b > 0$). Now, by following the ideas in the proof of [**10**, Th. 4.3] one can easily obtain that $[C_\psi, C_\varphi] = C_\gamma - C_\chi$ is unitarily similar to the multiplication operator $M_\phi : L^2(\mathbb{R}^+, tdt) \to L^2(\mathbb{R}^+, tdt)$ where $\phi(t) = e^{iat} - e^{ibt}$. Thus, $\sigma(C_\gamma - C_\chi)$; the spectrum of $C_\gamma - C_\chi$, is the non-contable set $\{e^{iat} - e^{ibt} : t \geq 0\} \cup \{0\}$ and then $[C_\psi, C_\varphi]$ can not be compact. □

4. The operators $C_\varphi C_\psi^*$ and $C_\psi^* C_\varphi$

The study of compactness of composition operators and related properties is one of the fundamental themes in the theory. Whenever φ is a linear fractional self map of $\mathbb{D}$ and $\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$, the operator C_φ (acting in the Dirichlet space, as in other functional Hilbert spaces) is easily seen to be compact; in the remaining cases $\varphi(\mathbb{D})$ is tangent to the unit circle and the operator is no longer compact (cf. [**23**]). For the sake of completeness we mention the following result.

Theorem 4.1. *Let φ be linear fractional selfmap of $\mathbb{D}$. The following assertions are equivalent:*

1. *$C_\varphi : \mathcal{D} \to \mathcal{D}$ is compact.*
2. *$C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$ is compact.*
3. *$C_\varphi : \mathcal{D} \to \mathcal{D}$ is Hilbert-Schmidt.*
4. *$C_\varphi : H^2 \to H^2$ is compact.*
5. *$C_\varphi : H^2 \to H^2$ is Hilbert-Schmidt.*
6. *$\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$.*

Proof. The proof is immediate from elemental considerations and the preceding observation. □

Let φ and ψ be linear fractional selfmaps of the unit disc. The problem of the compactness of $C_\varphi C_\psi^*$ and $C_\psi^* C_\varphi$ was studied on the Hardy space in [**4**] and on the Bergman space in [**5**].

We obtain in the setting of the Dirichlet space a characterization of the compactness of these operators analogous to that in [**4**]. In this context the proof is simpler, as the adjoint formula is simpler on the Dirichlet space. As it was noted in [**5**] and is easy to see with the followings results, there are non compact composition operators C_φ and C_ψ linear fractionally induced, such that the product $C_\varphi C_\psi^*$ or $C_\psi^* C_\varphi$ is compact. In a similar way we see that the compactness of $C_\varphi C_\psi^*$ is not equivalent to the compactness of $C_\psi^* C_\varphi$.

Theorem 4.2. *(cf.* [**4**, Th. 2.1]*) Suppose that φ and ψ are linear fractional self maps of $\mathbb{D}$. Then $C_\varphi C_\psi^*$ is not compact as operator on $\mathcal{D}$ if and only if there exist points η_1 and η_2 in $\partial\mathbb{D}$ such that $\varphi(\eta_1) = \psi(\eta_2) \in \partial\mathbb{D}$.*

Proof. We write $\sigma = \rho \circ \varphi^{-1} \circ \rho$ and $\gamma = \rho \circ \psi^{-1} \circ \rho$ with $\rho(z) = 1/\overline{z}$, $z \in \widehat{\mathbb{C}}$.

The adjoint formula 3.4 of E. Gallardo and A. Montes says that $C_\varphi^* = C_\sigma$ and $C_\psi^* = C_\gamma$ (as operators on $\mathcal{D}_0$). So, $C_\varphi C_\psi^* : \mathcal{D} \to \mathcal{D}$ is not compact if and only if $C_{\gamma\circ\varphi}$ is not compact. Therefore, $\rho \circ \psi^{-1} \circ \rho \circ \varphi$ maps a point of $\partial\mathbb{D}$ onto $\partial\mathbb{D}$ and this is equivalent to the conclusion of the theorem. ☐

Theorem 4.3. *(cf.* [**4**, Th. 2.2]*) Suppose that φ and ψ are linear fractional self maps of $\mathbb{D}$. Then $C_\psi^* C_\varphi$ is not compact as operator on $\mathcal{D}$ if and only if there exist points ω_1 and ω_2 in $\partial\mathbb{D}$ such that $\varphi^{-1}(\omega_1) = \psi^{-1}(\omega_2) \in \partial\mathbb{D}$.*

Proof. With the notation of the preceding theorem, $C_\psi^* C_\varphi : \mathcal{D} \to \mathcal{D}$ is not compact if and only if $C_{\varphi\circ\gamma}$ is not compact. Hence, $\varphi \circ \rho \circ \psi^{-1} \circ \rho$ maps a point of $\partial\mathbb{D}$ onto $\partial\mathbb{D}$ and this is equivalent to the conclusion of the theorem. ☐

5. Numerical Range of Linear Fractional Composition Operators

For a bounded operator T on a Hilbert space $\mathcal{H}$, the **numerical range** of T is defined as the subset of the complex plane:

$$W(T) := \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

There are some important properties of numerical range that we will use (see [**21**] and [**19**] for example).

Proposition 5.1. *For an operator T on a Hilbert space $\mathcal{H}$:*

1. *$W(T)$ is invariant under unitary similarity.*
2. *$W(T)$ lies in the closed unit disc of radius $\|T\|$ centered at the origin.*
3. *$W(T)$ contains all the eigenvalues of T. Moreover, if T is a unitarily diagonalizable operator, then $W(T)$ is the convex hull of its eigenvalues.*
4. *The spectrum of T belongs to the closure $\overline{W(T)}$ of $W(T)$. Moreover, if T is a normal operator then $\overline{W(T)}$ equals the convex hull of its spectrum.*
5. *Toeplitz-Hausdorff Theorem: $W(T)$ is always convex.*
6. *$W(T^*) = \{\overline{\lambda} : \lambda \in W(T)\}$.*

Due to properties 4. and 5. above, we have that $\overline{W(T)}$ contains the convex hull of the spectrum of T. An important difference between spectrum and numerical range is that while the former is similarity invariant, the latter is not.

There is some work on the study of the shape of the numerical range of composition operators in Hilbert spaces. In particular, there are recent papers of Bourdon and Shapiro [**2**] and Matache [**19**] on this matter. Specifically, in [**2**] the shape of the numerical range of composition operators induced on the Hardy space H^2 by disc automorphisms is studied. In [**19**], the shape of composition operators induced on the same space by monomials is studied. Here we do the same but in the $\mathcal{D}_0$ space. For this we rely on the work by Gallardo and Montes [**10**]. We use several of their results.

Theorem 5.2. [**10**, Th. 4.3] *Let C_φ be a linear fractional composition operator acting on $\mathcal{D}_0$. Then:*

1. *If φ is conjugate to $\eta(z) = \mu z$, with $0 < |\mu| \leq 1$, then C_φ is unitarily similar to a diagonal operator.*
2. *If φ is parabolic which is conjugate to $\tau(z) = z + a$, then C_φ is unitarily similar to multiplication by $\phi(t) := e^{iat}$ on $L^2(\mathbb{R}^+, tdt)$.*
3. *If φ is a hyperbolic automorphism conjugate to $\eta(z) = \lambda z$, then C_φ is unitarily similar to multiplication by $\phi(t) := \lambda^{-it}$ on $L^2(\mathbb{R}, 2\pi dt)$.*
4. *If φ is hyperbolic with just one fixed point on $\partial\mathbb{U}$, then C_φ is unitarily similar to the product of an unitary operator and a normal operator, or viceversa.*

As a consequence of the previous theorem, E. Gallardo and A. Montes obtained:

Theorem 5.3. [**10**, Th. 5.1] *Let C_φ be a linear fractional composition operator acting on $\mathcal{D}_0$. Then:*

1. *If φ is an elliptic automorphism and the derivative $\varphi'(\alpha)$ at its interior fixed point is an n-root of the unity, then $\sigma(C_\varphi)$, the spectrum of C_φ, equals to $\{\varphi'(\alpha)^k : k = 0, 1, \dots n-1\}$.*
2. *If φ is an automorphism which is not conjugate to a rotation through a rational multiple of φ, then $\sigma(C_\varphi) = \partial\mathbb{D}$.*
3. *If φ is a parabolic non-automorphism which is conjugate to $\tau(z) = z + a$, $\mathrm{Im}\, a > 0$, then $\sigma(C_\varphi) = \{e^{iat} : t \geq 0\} \cup \{0\}$.*
4. *If φ is hyperbolic with just one fixed point, then $\sigma(C_\varphi) = \overline{\mathbb{D}}$.*
5. *If φ is not elliptic and has an exterior and an interior fixed point and $\varphi'(\alpha)$ is the derivative at the latter fixed point, then $\sigma(C_\varphi) = \{\varphi'(\alpha)^n : n = 1, 2, \dots\} \cup \{0\}$.*

In order to obtain the preceding theorem E. Gallardo and A. Montes (cf. [**10**]), following and idea in [**13**] consider $\mathcal{D}_\Pi$, the Dirichlet space of the upper half plane consistig of those analytic functions F on Π, the upper half plane, for which the integral

$$\frac{1}{\pi}\int_\Pi |F'(x+iy)|^2\, dx\, dy$$

is finite. If we identify functions that differ by a constant, then $\mathcal{D}_\Pi$ becomes a Hilbert space and it is isometrically isomorphic to $\mathcal{D}_0$. Additionally, the space $\mathcal{D}_\Pi$ is isometrically isomorphic, under the Fourier transform, to $L^2(\mathbb{R}^+, tdt)$.

We will also use the following corollary Theorem 5.2:

Corollary 5.4. [**10**, Cor. 6.1] *Let φ be a linear fractional self-map of $\mathbb{D}$. If φ is elliptic or it has a boundary fixed point, then $\|C_\varphi\|_{\mathcal{D}_0} = 1$.*

5.1. The Numerical Range on $\mathcal{D}_0$

In this section we will show properties of the shape of the numerical range of linear fractional composition operators on the space $\mathcal{D}_0$. For this, we are heavily borrowing from the last two results mentioned before.

Theorem 5.5. *Let C_φ be a linear fractional composition operator acting on $\mathcal{D}_0$. Then,*

1. *If φ is an elliptic automorphism with interior fixed point α and $\varphi'(\alpha)$ is an n-root of the unity, then*
$$W(C_\varphi) = \mathrm{co}\{\varphi'(\alpha)^k : k = 0, 1, \dots n-1\};$$
i.e. $W(C_\varphi)$ is the n-vertex polygonal closed region with vertex in the n-roots of the unity.
2. *If φ is conjugate to a rotation through an irrational multiple of π: $z \mapsto \mu z$, $|\mu| = 1$, then $W(C_\varphi) = \mathbb{D} \cup \{\mu, \mu^2, \dots\}$.*
3. *If φ is a hyperbolic or a parabolic automorphism, then $W(C_\varphi) = \mathbb{D}$.*
4. *If φ is a parabolic non-automorphism, then $\overline{W(C_\varphi)}$ is the convex hull of a spiral joining 1 to 0.*
5. *If φ is hyperbolic with just one boundary fixed point, then $W(C_\varphi) = \mathbb{D}$.*
6. *If φ is not elliptic and has an exterior and an interior fixed point and $\varphi'(\alpha)$ is the derivative at the latter point, then*
$$\overline{W(C_\varphi)} = \mathrm{co}(\{\varphi'(\alpha)^n : n = 1, 2 \dots\} \cup \{0\}).$$

Proof. To prove (1) we observe that, by the first part of Theorem 5.3, we have that C_φ is unitarily similar to a diagonal operator with the family $\{\varphi'(\alpha)^k : k = 0, 1 \dots n-1\}$ as eigenvalues of C_φ (taking $\left\{\frac{z^m}{\sqrt{m}}\right\}_{m=1}^{\infty}$ as an orthonormal basis of $\mathcal{D}_0$). Recall that the set of eigenvalues is invariant under unitary similarity. Then, we use Proposition 5.1 to obtain the result.

For parts 2. and 3. we use the fact that the operator C_φ is normal on $\mathcal{D}_0$ (3.2) and then, by proposition 5.1, we have that $\overline{W(C_\varphi)} = \mathrm{co}(\sigma(C_\varphi))$. But in both cases the symbol φ is an automorphism which is not conjugate to a rotation through a rational multiple of π, and then $\sigma(C_\varphi) = \partial\mathbb{D}$, so $\overline{W(C_\varphi)} = \overline{\mathbb{D}}$.

If φ is as in part (2), we have again that C_φ is unitarily similar to a diagonal operator, but now with the sequence $\{\mu, \mu^2, \dots\}$ as eigenvalues of C_φ. Then $W(C_\varphi) = \mathbb{D} \cup \{\mu, \mu^2, \dots\}$.

Suppose now that φ is a hyperbolic automorphism conjugate to $\eta(z) = \lambda z$, $\lambda > 0$, then it has two fixed points in $\partial\mathbb{D}$, and by Corollary 3.2 we have $\|C_\varphi\|_{\mathcal{D}_0} = 1$. Now if there is a point $z \in W(C_\varphi)$ such that $|z| = 1$, then there exist $f \in \mathcal{D}_0$, $\|f\|_{\mathcal{D}_0} = 1$ such that $\langle C_\varphi f, f\rangle = z$, but by Cauchy-Schwartz inequality:

$$1 = |z| = |\langle C_\varphi f, f\rangle| \leq \|C_\varphi f\|_{\mathcal{D}_0} \leq \|C_\varphi\|_{\mathcal{D}_0} = 1, \tag{1}$$

so there exists $\alpha \in \mathbb{C}$ such that $C_\varphi f = \alpha f$, and therefore, $z = \langle C_\varphi f, f\rangle = \alpha\langle f, f\rangle = \alpha$. Then an element belongs to $W(C_\varphi) \cap \partial\mathbb{D}$ if and only if it is an eigenvalue of C_φ.

Now Theorem 5.2 says that C_φ is unitarily similar to a multiplication operator M_ϕ on $L^2(\mathbb{R}, 2\pi dt)$ induced by the multiplier $\phi : \mathbb{R} \longrightarrow \mathbb{C}$, $t \mapsto \lambda^{-it}$ $(= e^{-it\log\lambda})$. We know that if α is an eigenvalue of C_φ then so is of M_ϕ, and therefore, it must exist $f \in L^2(\mathbb{R}, 2\pi dt)$, $f \neq 0$ such that $e^{-it\log\lambda} f(t) = \alpha f(t)$ for all $t \in \mathbb{R}$. As f must be nonzero in a positive Lebesgue measure set, $e^{-it\log\lambda} = \alpha$ in that set, which is impossible. Therefore, C_φ does not have any eigenvalue and the convexity of $W(C_\varphi)$ ensures that $W(C_\varphi) = \mathbb{D}$.

The case in which the symbol φ is a parabolic automorphism conjugate to $\tau(z) = z + a$, $\mathrm{Im}\, a = 0$, $a \neq 0$, is similar: Again we have that φ has a fixed point in $\partial\mathbb{D}$ and hence $\|C_\varphi\|_{\mathcal{D}_0} = 1$ and the same reasoning about equation 1, brings up that only eigenvalues can belong in $W(C_\varphi) \cap \partial\mathbb{D}$. Now we have, by Theorem 5.2, that C_φ is unitarily similar to a multiplication operator on $L^2(\mathbb{R}^+, tdt)$ with multiplier $\phi : \mathbb{R}^+ \longrightarrow \mathbb{C}$, $t \mapsto e^{iat}$. But the tdt measure is absolutely continuous with respect to the Lebesgue measure, and the reasoning follows as in the latter case.

Cases 4. and 6. follow easily from the fact that C_φ is normal on $\mathcal{D}_0$ (Theorem 3.2) and hence (Proposition 5.1) $\overline{W(C_\varphi)} = \mathrm{co}(\sigma(C_\varphi))$.

To prove case 5. we use the fact that φ has one fixed point in $\partial\mathbb{D}$ and hence $\|C_\varphi\|_{\mathcal{D}_0} = 1$. But since $\sigma(C_\varphi) \subset \overline{W(C_\varphi)}$, then (Theorem 5.3) $\overline{\mathbb{D}} \subset \overline{W(C_\varphi)}$, and again because of Proposition 5.1, we have that $W(C_\varphi)$ lies in the closed unit disc of radius $\|C_\varphi\|$ centered at the origin, thus $\overline{W(C_\varphi)} = \overline{\mathbb{D}}$. Moreover, we can deduce using equation (1) that $\alpha \in W(C_\varphi) \cap \partial\mathbb{D}$ if and only if there is $f \in \mathcal{D}_0$, such that $C_\varphi f = \alpha f$.

Suppose for a moment that another fixed point of φ is interior, then (see [**10**, Th. 4.3]) C_φ is unitarily similar to $C_\psi : \mathcal{D}_\Pi \longrightarrow \mathcal{D}_\Pi$; where $\psi(z) = \lambda z + a$, $\mathrm{Im}\, a > 0$ and $0 < \lambda < 1$. Now, if α is an eigenvalue of C_ψ, it must exist $g \in \mathcal{D}_\Pi$, $g \neq 0$ such that $C_\psi g = \alpha g$, that is,

$$g(\psi(z)) = \alpha g(z) \quad \text{ for all } z \in \Pi.$$

In particular, $g\big(\psi\big(\frac{a}{1-\lambda}\big)\big) = \alpha g\big(\frac{a}{1-\lambda}\big)$ and so, $g\big(\frac{a}{1-\lambda}\big) = \alpha g\big(\frac{a}{1-\lambda}\big)$; therefore, if $g\big(\frac{a}{1-\lambda}\big) \neq 0$ then $\alpha = 1$ and we can choose $z_0 \in \Pi$ such that $g(z_0) \neq 0$ and since $g(\psi_n(z_0)) = \alpha^n g(z_0) = g(z_0)$ for all n (here ψ_n denotes the composition of ψ with itself n-times), g takes the each value non zero in its range infinitely-many times, this contradicts the fact that $g \in \mathcal{D}_\Pi$ and hence $W(C_\varphi) = \mathbb{D}$.

If $g\big(\frac{a}{1-\lambda}\big) = 0$, then choose $z_0 \in \Pi$ such that $g(z_0) \neq 0$; now it is easy to see that $\psi_n(z_0) \xrightarrow{n} \frac{a}{1-\lambda}$ and then

$$\lim_n g(\psi_n(z_0)) = g(\lim_n \psi_n(z_0)) = g\left(\frac{a}{1-\lambda}\right) = 0,$$

but this contradicts the fact that $|g(\psi_n(z_0))| = |\alpha^n g(z_0)| = |g(z_0)| \neq 0$ for all n. Hence $W(C_\varphi) = \mathbb{D}$.

In case that the other fixed point were exterior, then $C_\varphi^* = C_\phi$, with ϕ a linear fractional self map of $\mathbb{D}$ having just one boundary fixed point and an interior fixed point, so $W(C_\phi) = \mathbb{D}$ and by Proposition 5.1 we can conclude that $W(C_\varphi) = \mathbb{D}$. □

5.2. The Essential Numerical Range on $\mathcal{D}$

If $\mathcal{H}$ is a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$, the space of bounded operators on $\mathcal{H}$, let $\mathcal{K}(\mathcal{H})$ be the subspace of $\mathcal{L}(\mathcal{H})$ formed by all compact operators, and $[T]$ be the coset of T in the Calkin algebra, i.e. the quotient space $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

We recall that the essential norm of T is its norm in the Calkin algebra, i.e. $\|T\|_e = \inf\{\|T - K\| : K \in \mathcal{K}(\mathcal{H})\}$, and the **essential numerical range** of T ($W_e(T)$) is the numerical range of the coset $[T]$. We denote by $w_e(T)$ the **essential numerical radius** of T, that is, $w_e(T) = \sup\{|r| : r \in W_e(T)\}$. The notion of essential numerical range was introduced by Stampfli and Williams in [**22**]. It could be seen that $W_e(T) = \bigcap_{K\in\mathcal{K}(\mathcal{H})} \overline{W(T+K)}$ and hence, $W_e(T)$ is a closed subset of $\overline{W(T)}$.

Similarly, the essential spectrum $\sigma_e(T)$ of an operator T is defined to be the spectrum of the coset $[T]$ in the Calkin algebra. The essential spectrum $\sigma_e(T)$ is always a compact subset contained in $\sigma(T)$ (cf [**6**], for example).

We have the following properties on $W_e(T)$ (See [**3**] and [**12**] for example).

Proposition 5.6. *Let $T \in \mathcal{L}(\mathcal{H})$, then:*

1. *$W_e(T)$ is a non-void compact and convex set.*
2. *$W_e(T) = \{0\}$ if and only if T is compact.*
3. *If T is an essentially normal operator, $W_e(T) = \mathrm{co}(\sigma_e(T))$ and $w_e(T) = \|T\|_e$.*
4. *If M is a closed linear subspace of $\mathcal{H}$ such that $M^\perp$ has finite dimension, t hen $W_e(T) = W_e(P_M T|_M)$, where P_M denotes the orthogonal projection onto M.*

In this section, we will find the shape of the essential numerical range of linear fractional composition operators acting on the Dirichlet space $\mathcal{D}$. For this, we will use some results from [**10**].

Proposition 5.7.

1. [**10**, Remark 5.2] *The essential spectrum $\sigma_e(C_\varphi)$ of a linear fractional composition operator C_φ acting on $\mathcal{D}$ coincides with the essential spectrum of C_φ acting on $\mathcal{D}_0$.*
2. [**10**, Cor. 5.2] *Let C_φ be a linear fractional composition operator acting on $\mathcal{D}_0$. Then $\sigma_e(C_\varphi) = \sigma(C_\varphi)$, except if φ is non elliptic and has an exterior and an interior fixed point, in which case $\sigma_e(C_\varphi) = \{0\}$.*

In Section 3 we showed that a composition operator C_φ induced on $\mathcal{D}$ by a linear fractional self-map φ of the unit disc is essentially normal if and only if φ is not a hyperbolic non-automorphism with a fixed point on $\partial\mathbb{D}$. So, we can easily deduce the following result.

Theorem 5.8. *Let C_φ be a linear fractional composition operator acting on $\mathcal{D}$. Then:*

1. *If φ is an elliptic automorphism and the derivative $\varphi'(\alpha)$ at its interior fixed point is an n-root of the unity, then*
$$W_e(C_\varphi) = \operatorname{co}(\{\varphi'(\alpha)^k : k = 0, \dots n-1\}).$$
2. *If φ is an automorphism which is not conjugate to a rotation through a rational multiple of π, then $W_e(C_\varphi) = \overline{\mathbb{D}}$.*
3. *If φ is a parabolic non-automorphism which is conjugate to $\tau(z) = z + a$, then $W_e(C_\varphi) = \operatorname{co}(\{e^{iat} : t \geq 0\} \cup \{0\})$.*
4. *If φ is not elliptic and has an exterior and an interior fixed point, then $W_e(C_\varphi) = \{0\}$.*
5. *If φ is hyperbolic with just one boundary fixed point, then $W_e(C_\varphi) = \overline{\mathbb{D}}$.*

Proof. Cases 1. to 4. follows easily from the last two results mentioned before the Theorem. To prove case 5. we know that $\overline{\mathbb{D}} = \sigma_e(C_\varphi) \subset W_e(C_\varphi)$, but by Proposition 5.7 we have that $W_e(C_\varphi) = W_e(\widetilde{C_\varphi}) \subset \overline{W(\widetilde{C_\varphi})} = \overline{\mathbb{D}}$ and result follows. □

Corollary 5.9. *Let C_φ be a linear fractional composition operator acting on $\mathcal{D}$. Then $\|C_\varphi\|_e = 1$ except if φ is non elliptic and has an exterior and an interior fixed point, in which case $\|C_\varphi\|_e = 0$.*

Corollary 5.10. *Let C_φ be a linear fractional composition operator acting on $\mathcal{D}$. Then C_φ is compact if and only if φ is non elliptic and has an exterior and an interior fixed point.*

5.3. The Numerical Range on the Hardy space of the upper half plane

Let Π denote the upper half plane of the complex plane. The Hardy space of the upper half plane $H^2(\Pi)$ is the space of holomorphic functions on Π for which the norm
$$\|f\|^2_{H^2(\Pi)} = \sup_{y>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x+iy)|^2 dx$$
is finite.

In this space the situation for studying linear fractional composition operators is much simpler than in the Dirichlet space: *only linear fractional transformations $\varphi(z) = az + b$ with $a > 0$ and $\operatorname{Im} b \geq 0$ induce bounded composition operators on* $H^2(\Pi)$ (See [**18**]).

Again in [**10**] we can find a proof for the following result.

Theorem 5.11. *Let $\varphi(z) = az+b$ be such that $a > 0$ and $\operatorname{Im} b \geq 0$ and consider C_φ acting on $H^2(\Pi)$. Then*

1. *If φ is parabolic, then C_φ is unitarily similar to multiplication by e^{ibt} on $L^2(\mathbb{R}^+, dt)$.*
2. *If φ is a hyperbolic automorphism, then C_φ is unitarily similar to multiplication by $z^{-it-1/2}$ on $L^2(\mathbb{R}, 2\pi dt)$.*

In [**10**] and as a consequence of the last result, they prove that for φ as in 5.11, C_φ acting on $H^2(\Pi)$ is normal if and only if φ is an automorphism of Π or φ is parabolic. They also prove that $\|C_\varphi\|_{H^2(\Pi)} = a^{-1/2}$ and obtain the spectrum of C_φ:

Theorem 5.12. *Let $\varphi(z) = az+b$ be such that $a > 0$ and $\operatorname{Im} b \geq 0$ and consider C_φ acting on $H^2(\Pi)$. Then*

1. *If φ is an automorphism, then $\sigma(C_\varphi) = \{z \in \mathbb{C} : .|z| = a^{-1/2}\}$.*
2. *If φ is a parabolic non automorphism, then $\sigma(C_\varphi) = \{e^{ibt} : t \geq 0\} \cup \{0\}$.*
3. *If φ is an hyperbolic non automorphism, then $\sigma(C_\varphi) = \{z \in \mathbb{C} : |z| \leq a^{-1/2}\}$.*

Now with a similar reasoning as in the proof of Theorem 5.5, one can easily see:

Theorem 5.13. *Let $\varphi(z) = az+b$ be such that $a > 0$ and $\operatorname{Im} b \geq 0$ and consider C_φ acting on $H^2(\Pi)$. Then*

1. *If φ is an automorphism, then $W(C_\varphi) = \{z \in \mathbb{C} : .|z| < a^{-1/2}\}$.*
2. *If φ is a parabolic non automorphism, then $\overline{W(C_\varphi)} = \operatorname{co}(\{e^{ibt} : t \geq 0\} \cup \{0\})$.*
3. *If φ is an hyperbolic non automorphism, then $\overline{W(C_\varphi)} = \{z \in \mathbb{C} : |z| \leq a^{-1/2}\}$.*

5.4. Final Remarks

Some work remains to be done in this matter: in cases 1., 2., 3. and 5. of Theorem 5.5 we know exactly the properties of $W(C_\varphi)$ is, but in cases 4. and 6. we just know the shape of $W(C_\varphi)$ and do not know the properties of $\partial W(C_\varphi)$. By using the same reasoning to prove part (3), we can prove that 1 does not belong to $W(C_\varphi)$ when φ is a parabolic non-automorphism, but nothing else. The next step is to calculate the numerical range of linear fractional composition operators acting on $\mathcal{D}$. Some direct consequences from the last results are: since $W_e(C_\varphi) \subset \overline{W(C_\varphi)}$, we can conclude that if φ is hyperbolic with just one boundary fixed point, then $\overline{W(C_\varphi)} = \overline{\mathbb{D}}$. In [**10**] it is shown that a linear fractional composition operator C_φ acting on $\mathcal{D}$ is normal if and only if $\varphi(z) = \mu z$, with $0 < |\mu| \leq 1$, so for this kind of operators $W(C_\varphi) = \operatorname{co}\{\mu^k : k = 0, 1, \dots\}$. It is easy to see that $\|C_\varphi\| = 1$ in $\mathcal{D}$ if φ is a linear fractional self-map of $\mathbb{D}$ that fixes the origin, hence $\overline{W(C_\varphi)} = \overline{\mathbb{D}}$ if φ is an automorphism that fixes the origin and is not conjugate to a rotation through a rational multiple of π.

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