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ON (m, n)-QUASI-INJECTIVE MODULES

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Abstract. Let R be a ring. For two fixed positive integers m and n, an R-module M is called (m, n)***-quasi-injective*** if each R-homomorphism from an n-generated submodule of M^m to M extends to one from M^m to M. It is showed that M_R is (m, n)-quasi-injective if and only if the right $R^{n\times n}$-module $M^{m\times n}$ is principally quasi-injective. Many properties of (m, n)-injective rings and principally quasi-injective modules are extended to these modules. Moreover, some properties of (m, n)-quasi-injective Kasch modules are investigated.

Throughout this paper R and S are associative rings with identities, and all modules are unitary. Unless specified otherwise, m and n will be two fixed positive integers. For an Abelian group G, we write $G^{m\times n}$ for the set of all formal $m \times n$--matrices with entries in G, and write G^n (resp. G_n) for $G^{1\times n}$(resp. for $G^{n\times 1}$). Multiplication maps $x \mapsto ax$ and $x \mapsto xa$ will be denoted by $a\cdot$ and $\cdot a$, respectively. For $A = (a_{ij})_{m\times n} \in G^{m\times n}$ (resp. $a = (a_1, \dots, a_n)^T \in G_n$), we write $\pi_{ij}(A)$ (resp. $\pi_i(a)$) for a_{ij} (resp. a_i). For any $x \in G$, we write $l_{ij}(x)$ (resp. $l_i(x)$)for the $m \times n$-matrices (resp. the $m \times 1$-matrices) whose (i, j) entry (resp. i-th entry) is x and the others are 0's. Let ${}_SM_R$ be a bimodule. For $x \in M^{m\times n}$, $u \in S^{l\times m}$ and $v \in R^{n\times k}$, under the usual multiplication of matrices, ux (resp. xv) is a well-defined element in $M^{l\times n}$ (resp. $M^{m\times k}$). If $X \subseteq M^{l\times n}$, $U \subseteq S^{l\times m}$ and $V \subseteq R^{n\times k}$, define

$$\begin{aligned} r_{R^{n\times k}}(X) &= \left\{v \in R^{n\times k} \mid xv = 0, \forall\, x \in X\right\},\\ l_{S^{m\times l}}(X) &= \left\{u \in S^{m\times l} \mid ux = 0, \forall\, x \in X\right\},\\ r_{M^{m\times n}}(U) &= \left\{y \in M^{m\times n} \mid uy = 0, \forall\, u \in U\right\},\\ l_{M^{m\times n}}(V) &= \left\{z \in M^{m\times n} \mid zv = 0, \forall\, v \in V\right\}. \end{aligned}$$

1. Characterizations of (m, n)-quasi-injective modules

Firstly, we recall some concepts. A right R-module M_R is called ***principally quasi-injective*** (or ***PQ-injective*** in brief) [**5**] if each R-homomorphism from a cyclic submodule of M to M can be extended to an endomorphism of M. A ring R is said to be ***right*** (m, n)***-injective*** [**3**] in case each right R-homomorphism

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from an n-generated submodule of R^m to R extends to one from R^m to R. A right R-module M_R is said to be ***finitely quasi-injective*** [**8**] if each R-homomorphism from a finitely generated submodule of M to M extends to an endomorphism of M. Motivated by these concepts, we introduce the following definition.

Definition 1.1. An R-module M is called (m,n)***-quasi-injective*** in case each R-homomorphism from an n-generated submodule of M^m to M extends to one from M^m to M. An R-module M is called n***-quasi-injective*** if it is $(1,n)$-quasi-injective.

Examples. (1) Every quasi-injective module is (m,n)-quasi-injective for all positive integers m and n [**2**, Proposition 16.13(2)].

(2) R is right (m,n)-injective if and only if R_R is (m,n)-quasi-injective.

(3) M_R is PQ-injective if and only if M_R is $(1,1)$-quasi-injective.

(4) M_R is finitely quasi-injective if and only if M_R is n-quasi-injective for all positive integers n.

It is easy to see that M_R is (m,n)-quasi-injective if and only if M_R is (l,k)-quasi-injective for all $1 \le l \le m$ and $1 \le k \le n$.

Definition 1.2. A bimodule ${}_SM_R$ is called ***left balanced*** in case every right R-endomorphism of M is left multiplication by an element of S.

Remark. (1) ${}_{\mathrm{End}(M_R)}M_R$ is left balanced for every right R-module M_R.

(2) Given a module ${}_SM$, then the bimodule ${}_SM_{\mathrm{End}({}_SM)}$ is left balanced if and only if ${}_SM_{\mathrm{End}({}_SM)}$ is balanced [**2**, p. 60].

Theorem 1.3. *Let ${}_SM_R$ be a left balanced bimodule, then the following statements are equivalent:*

(1) *M_R is (m,n)-quasi-injective.*

(2) *$l_{M^n}r_{R_n}\{\alpha_1,\alpha_2,\cdots,\alpha_m\} = S\alpha_1+S\alpha_2+\cdots+S\alpha_m$ for any m-element subset $\{\alpha_1,\alpha_2,\cdots,\alpha_m\}$ of M^n.*

(2)′ *$l_{M^n}r_{R_n}(A) = S^mA$ for all $A \in M^{m\times n}$.*

(3) *If $r_{R_n}(A) \subseteq r_{R_n}(B)$ where $A,\ B \in M^{m\times n}$, then $S^mB \subseteq S^mA$.*

(4) *If $z \in M^n$ and $A \in M^{m\times n}$ satisfy $r_{R_n}(A) \subseteq r_{R_n}(z)$, then $z \in S^mA$.*

(5) *$l_{M^l}[CR_n \cap r_{R_l}(A)] = l_{M^l}(C) + S^mA$ for all positive integers l, $A \in M^{m\times l}$ and $C \in R^{l\times n}$.*

(5)′ *$l_{M^n}[CR_n \cap r_{R_n}(A)] = l_{M^n}(C) + S^mA$ for all $A \in M^{m\times n}$ and $C \in R^{n\times n}$.*

(6) *The right R-module M^m (or M_m) is n-quasi-injective.*

Proof. (1) $\Leftrightarrow$ (6), (2) $\Leftrightarrow$ (2)′ and (5) $\Rightarrow$ (5)′ $\Rightarrow$ (2)′ $\Rightarrow$ (3) are trivial.

(1) $\Leftrightarrow$ (2). Argue as the proof of [**3**, Theorem 2.4].

(3) $\Rightarrow$ (4). Let $B = \begin{pmatrix} z \\ 0 \end{pmatrix} \in M^{m\times n}$. Then $r_{R_n}(A) \subseteq r_{R_n}(z) = r_{R_n}(B)$ and $S^mB = Sz$. By (3), we have $Sz = S^mB \subseteq S^mA$. Therefore $z \in S^mA$.

(4) $\Rightarrow$ (5). Let $x \in l_{M^l}[CR_n \cap r_{R_l}(A)]$. For all $y \in r_{R_n}(AC)$, $ACy = 0$ implies that $Cy \in CR_n \cap r_{R_l}(A)$. Hence $xCy = 0$, i.e., $y \in r_{R_n}(xC)$. Thus

$$r_{R_n}(AC) \subseteq r_{R_n}(xC).$$

By (4), $xC = uAC$ for some $u \in S^m$. So

$$x = (x - uA) + uA \in l_{M^l}(C) + S^m A.$$

Therefore,

$$l_{M^l}[CR_n \cap r_{R_l}(A)] \subseteq l_{M^l}(C) + S^m A.$$

The inverse inclusion is clear. □

Corollary 1.4. *Let ${}_SM_R$ be a left balanced bimodule. Then*

(1) *M_R is PQ-injective if and only if $l_M r_R(a) = Sa$ for any $a \in M$ if and only if $r_R(x) \subseteq r_R(y)$ where $x, y \in M$ implies $y \in Sx$;*
(2) *M_R is n-quasi-injective if and only if $l_{M^n} r_{R_n}(\alpha) = S\alpha$ for any $\alpha \in M^n$ if and only if $r_{R_n}(A) \subseteq r_{R_n}(B)$ where $A, B \in M^n$ implies $B \in SA$;*
(3) *M_R is $(m,1)$-quasi-injective if and only if $M^m_R (or (M_m)_R)$ is PQ-injective if and only if $l_M r_R(N) = N$ for any m-generated submodule N of ${}_SM$;*
(4) *M_R is finitely-quasi-injective if and only if $l_{M^n} r_{R_n}(\alpha) = S\alpha$ for all positive integers n and any $\alpha \in M^n$ if and only if $r_{R_n}(A) \subseteq r_{R_n}(B)$ where $A, B \in M^n$ implies $B \in SA$ for all positive integers n.*

Theorem 1.5. *Let ${}_SM_R$ be a left balanced bimodule. Then the following conditions are equivalent.*

(1) *M_R is (m,n)-quasi-injective.*
(2) *M_R is $(m,1)$-quasi-injective and $l_{S^m}(I \cap K) = l_{S^m}(I) + l_{S^m}(K)$, where I, K are submodules of $(M_m)_R$ such that $I + K$ is n-generated.*
(3) *M_R is $(m,1)$-quasi-injective and $l_{S^m}(I \cap K) = l_{S^m}(I) + l_{S^m}(K)$, where I, K are submodules of $(M_m)_R$ such that I is cyclic and K is $(n-1)$-generated ($K = 0$ if $n = 1$).*

Proof. (1) ⇒ (2). It is obvious that M_R is $(m,1)$-quasi-injective and $l_{S^m}(I \cap K) \supseteq l_{S^m}(I) + l_{S^m}(K)$. Conversely, let $x \in l_{S^m}(I \cap K)$ and define $f : I + K \to M$ by $f(c + b) = xc$ for all $c \in I$ and $b \in K$. Then f is a right R-homomorphism. Since M_R is (m,n)-quasi-injective and ${}_SM_R$ is left balanced, $f = y\cdot$ for some $y \in S^m$. Therefore, for any $c \in I$ and $b \in K$, we have $yc = f(c) = xc$ and $yb = f(b) = 0$. This means that

$$x = (x - y) + y \in l_{S^m}(I) + l_{S^m}(K).$$

(2) ⇒ (3) is obvious.

(3) ⇒ (1). We proceed by induction on n. Let $K = \alpha_1 R + \alpha_2 R + \cdots + \alpha_n R$ be an n-generated submodule of $(M_m)_R$ and $f : K \to M$ be a right R-homomorphism. Write $K_1 = \alpha_1 R$, $K_2 = \alpha_2 R + \cdots + \alpha_n R$. By induction hypothesis, $f|_{K_1} = y_1\cdot$ and $f|_{K_2} = y_2\cdot$ for some $y_1, y_2 \in S^m$. Clearly,

$$y_1 - y_2 \in l_{S^m}(K_1 \cap K_2) = l_{S^m}(K_1) + l_{S^m}(K_2).$$

Suppose $y_1 - y_2 = z_1 + z_2$ with $z_i \in l_{S^m}(K_i)$ $(i = 1, 2)$ and let $y = y_1 - z_1 = y_2 + z_2$. Then for any $x = x_1 + x_2 \in K$ with $x_i \in K_i$ $(i = 1, 2)$,

$$f(x) = f(x_1) + f(x_2) = y_1 x_1 + y_2 x_2 = (y_1 - z_1)x_1 + (y_2 + z_2)x_2 = y(x_1 + x_2) = yx.$$

So $f = y\cdot$ and (1) follows. □

Corollary 1.6. *Given a left balanced bimodule* ${}_SM_R$.

(1) *The following statements are equivalent:*
   (i) M_R *is n-quasi-injective.*
   (ii) M_R *is PQ-injective and* $l_S(I \cap K) = l_S(I) + l_S(K)$, *where* I, K *are submodule of* M_R *and* $I + K$ *is n-generated.*
   (iii) M_R *is PQ-injective and* $l_S(I \cap K) = l_S(I) + l_S(K)$, *where* I *is a cyclic submodules of* M_R *and* K *is an* $(n-1)$*-generated submodule of* M_R.

(2) M_R *is finitely quasi-injective if and only if* $l_M r_R(x) = Sx$ *for all* $x \in M$ *and* $l_S(I \cap K) = l_S(I) + l_S(K)$ *for any finitely generated submodules* I *and* K *of* M_R.

(3) M_R *is* $(m, 2)$*-quasi-injective if and only if* $(M_m)_R$ *is PQ-injective and*

$$l_{S^m}(\alpha R \cap \beta R) = l_{S^m}(\alpha) + l_{S^m}(\beta)$$

for all $\alpha, \beta \in M_m$. *In particular,* M_R *is* 2*-quasi-injective if and only if* M_R *is PQ-injective and*

$$l_S(xR \cap yR) = l_S(x) + l_S(y)$$

for all $x, y \in M$.

Lemma 1.7. *Let* M *be a right* R*-module. If* $f \in \mathrm{End}(M^{m\times n}_{R^{n\times n}})$, *then*

(1) $\pi_{ij} f(X) = \pi_{ij} f\,(\sum_{k=1}^{m} l_{kj}(x_{kj}))$ *for each* $X = (x_{ij}) \in M^{m\times n}$ *and all* $1 \leq i \leq m$, $1 \leq j \leq n$.

(2) $\pi_{ij} f l_{kj} = \pi_{i1} f l_{k1}$ *for all* $1 \leq i \leq m$, $1 \leq j \leq n$ *and* $1 \leq k \leq m$.

Proof. (1) Since

$$f\left(\sum_{k=1}^{m} l_{kt}(x_{kt})\right) = f(XE_{tt}) = f(X)E_{tt} = \sum_{k=1}^{m} l_{kt}(\pi_{kt} f(X)),$$

we have $\pi_{ij} f\left(\sum\limits_{k=1}^{m} l_{kt}(x_{kt})\right) = 0$ in case $t \neq j$. Thus

$$\pi_{ij} f(X) = \pi_{ij}\left[\sum_{t=1}^{n} f(\sum_{k=1}^{m} l_{kt}(x_{kt}))\right] = \pi_{ij} f\left(\sum_{k=1}^{m} l_{kj}(x_{kj})\right).$$

(2) For any $x \in M$,

$$\pi_{ij} f l_{kj}(x) = \pi_{ij} f(l_{k1}(x)P(1,j)) = \pi_{ij}[f(l_{k1}(x))P(1,j)] = \pi_{i1} f l_{k1}(x).$$

So

$$\pi_{ij} f l_{kj} = \pi_{i1} f l_{k1}.$$

□

Corollary 1.8. *Given a module* M_R *with* $S = \mathrm{End}(M_R)$. *Then a map* $f : M^{m\times n} \to M^{m\times n}$ *is a right* $R^{n\times n}$*-homomorphism if and only if* $f = C\cdot$ *for some* $C \in S^{m\times m}$.

Proof. ($\Rightarrow$) Suppose $f \in \mathrm{End}(M^{m\times n}_{R^{n\times n}})$ and take $C = (\pi_{i1} f l_{k1})_{m\times m} \in S^{m\times m}$. Then for each $X = (x_{ij})_{m\times n} \in M^{m\times n}$ and all $1 \le i \le m$, $1 \le j \le n$, by Lemma 1.7, we have

$$\pi_{ij} f(X) = \pi_{ij} f\left(\sum_{k=1}^{m} l_{kj}(x_{kj})\right) = \sum_{k=1}^{m} \pi_{ij} f l_{kj}(x_{kj}) = \sum_{k=1}^{m} \pi_{i1} f l_{k1}(x_{kj}) = \pi_{ij}(CX).$$

Therefore

$$f(X) = CX.$$

($\Leftarrow$) It is clear. □

Theorem 1.9. *Given a module M_R with $S = \mathrm{End}(M_R)$. M_R is (m,n)-quasi-injective if and only if the right $R^{n\times n}$-module $M^{m\times n}$ is PQ-injective.*

Proof. ($\Rightarrow$). Let $A, B \in M^{m\times n}$ with $r_{R^{n\times n}}(A) \subseteq r_{R^{n\times n}}(B)$ and write

$$B = \begin{pmatrix} B_1 \\ \vdots \\ B_m \end{pmatrix}.$$

Then for each $i = 1, 2, \cdots, m$, $r_{R^{n\times n}}(A) \subseteq r_{R^{n\times n}}(B_i)$. Consequently $r_{R_n}(A) \subseteq r_{R_n}(B_i)$. Since M_R is (m,n)-quasi-injective, by Theorem 1.3(4), $B_i \in S^m A$ ($i = 1, 2, \cdots, m$). So $B = CA$ for some $C \in S^{m\times m}$. Now we define $f : M^{m\times n} \to M^{m\times n}$ by $f(X) = CX$. Then $f \in \mathrm{End}(M^{m\times n}_{R^{n\times n}})$ and $B = f(A)$, whence $M^{m\times n}_{R^{n\times n}}$ is PQ-injective by Corollary 1.4(1).

($\Leftarrow$) Suppose $z \in M^n$, $A \in M^{m\times n}$ and $r_{R_n}(A) \subseteq r_{R_n}(z)$. Let $B = \begin{pmatrix} z \\ 0 \end{pmatrix} \in M^{m\times n}$. Then $r_{R^{n\times n}}(A) \subseteq r_{R^{n\times n}}(B)$. Since $M^{m\times n}_{R^{n\times n}}$ is PQ-injective, $B = CA$ for some $C \in S^{m\times m}$ by Corollary 1.4(1) and Corollary 1.8. It follows that $z \in S^m A$. By Theorem 1.3(4), we see that M_R is (m,n)-quasi-injective. □

Corollary 1.10. *A ring R is right (m,n)-injective if and only if the right $R^{n\times n}$-module $R^{m\times n}$ is PQ-injective. In particular, R is right (n,n)-injective if and only if $M_n(R)$ is P-injective.*

By Theorem 1.9, Corollary 1.4 and Corollary 1.8, we have

Corollary 1.11. *M_R is finitely quasi-injective if and only if the right $R^{n\times n}$-module M^n is PQ-injective for all positive integers n if and only if $l_{M^n} r_{R^{n\times n}}(x) = Sx$ for all positive integers n and all $x \in M^n$, where $S = \mathrm{End}(M_R)$.*

2. Properties of (m,n)-quasi-injective modules

In this section, some known results on PQ-injective modules and principally injective rings are extended to (m,n)-quasi-injective modules.

We begin with the following theorem, which extends [**5**, Proposition 1.2].

Theorem 2.1. *Given a left balanced bimodule ${}_SM_R$ with M_R (m,n)-quasi-injective and $A, B \in M^{m\times n}$.*

(1) *If* $(BR_n)_R$ *embeds in* $(AR_n)_R$*, then* ${}_S(S^mB)$ *is an image of* ${}_S(S^mA)$*.*
(2) *If* $(AR_n)_R$ *is an image of* $(BR_n)_R$*, then* ${}_S(S^mA)$ *embeds in* ${}_S(S^mB)$*.*
(3) *If* $(BR_n)_R \cong (AR_n)_R$*, then* ${}_S(S^mA) \cong {}_S(S^mB)$*.*

Proof. If $\sigma : BR_n \to AR_n$ is a right R-homomorphism, then the (m, n)-quasi-injectivity of M_R forces $\sigma = g|_{BR_n}$ for some $g \in \mathrm{End}((M_m)_R)$. Let $D=(\pi_i g l_j)_{m\times m}$. Then $g = D\cdot$. But ${}_SM_R$ is let balanced, so $g = C\cdot$ for some $C \in S^{m\times m}$. Choose $u_1, u_2, \cdots, u_n \in R_n$ such that $\sigma(Be_i)=Au_i$, where $e_i=(0, \cdots, 0, 1, 0, \cdots, 0)^T \in R_n$ (with 1 in the ith position and 0's in all the other positions), $i = 1, 2, \cdots, n$. Let $U = (u_1, u_2, \cdots, u_n)$. Then

$$\begin{aligned} AU = (Au_1, Au_2, \cdots, Au_n) &= (\sigma(Be_1), \sigma(Be_2), \cdots, \sigma(Be_n)) \\ &= (CBe_1, CBe_2, \cdots, CBe_n) = CB. \end{aligned}$$

Now we define $\varphi : S^mA \to S^mB$ by $yA \mapsto yAU$. Then φ is a left S-homomorphism.

(1) If σ is a monomorphism, then for any $x = (x_1, x_2, \cdots, x_n)^T \in r_{R_n}(AU)$,

$$\sigma(Bx) = \sigma\left(\sum_{i=1}^{n} Be_i x_i\right) = \sum_{i=1}^{n} \sigma(Be_i)x_i = \sum_{i=1}^{n}(Au_i)x_i = 0$$

follows that

$$Bx = 0.$$

Thus $r_{R_n}(AU) \subseteq r_{R_n}(B)$. By Theorem 1.3(3), $S^mB \subseteq S^mAU$. But $S^mAU = S^mCB \subseteq S^mB$, so $S^mB = S^mAU$. Hence φ is an epimorphism.

(2) Suppose σ is an epimorphism. Let $Ae_i = \sigma(Bv_i)$, $v_i \in R_n$, $i = 1, 2, \cdots, n$, and write $V=(v_1, v_2, \cdots, v_n)$. Then $V \in R^{n\times n}$ and $A=CBV$. Thus, if $\varphi(yA) = 0$, then $yAU = 0$, i.e., $yCB = 0$, whence $yA = yCBV = 0$. Therefore φ is a monomorphism.

(3) By (1) and (2). □

The next theorem extends [**5**, Lemma 1.2].

Theorem 2.2. *Suppose that* ${}_SM_R$ *is left balanced and* M_R *is* (m, n)*-quasi-injective. Then*

$$l_{S^k}[r_{M_k}(A) \cap BR_n] = S^mA + l_{S^k}(B)$$

for all positive integers k*,* $A \in S^{m\times k}$ *and* $B \in M^{k\times n}$*.*

Proof. Let $x \in l_{S^k}[r_{M_k}(A) \cap BR_n]$. For all $y \in r_{R_n}(AB)$, we have $ABy = 0$. This implies that $By \in r_{M_k}(A) \cap BR_n$. So $xBy = 0$, i.e., $y \in r_{R_n}(xB)$. Thus $r_{R_n}(AB) \subseteq r_{R_n}(xB)$. Since M_R is (m, n)-quasi-injective, by Theorem 1.3(4), $xB = u(AB)$ for some $u \in S^m$. Then $x - uA \in l_{S^k}(B)$. Hence

$$x = uA + (x - uA) \in S^mA + l_{S^k}(B).$$

Therefore

$$l_{S^k}[r_{M_k}(A) \cap BR_n] \subseteq S^mA + l_{S^k}(B).$$

The inverse inclusion is obvious. □

Corollary 2.3. *Let M_R be (m,n)-quasi-injective. If $\alpha_1,\alpha_2,\dots,\alpha_m \in S = \mathrm{End}(M_R)$, $x_1,x_2,\cdots,x_n \in M$, then*

$$l_S\left[(\bigcap_{i=1}^{m}\operatorname{Ker}\alpha_i)\cap\sum_{j=1}^{n}x_jR)\right]=\sum_{i=1}^{m}S\alpha_i+\bigcap_{j=1}^{n}l_S(x_j).$$

Proof. Take $k=1$, $A=(\alpha_1,\dots,\alpha_m)^T$ and $B=(x_1,x_2,\cdots,x_n)$ in Theorem 2.2 and then the result follows. □

Corollary 2.4. *Let M_R be an n-generated (m,n)-quasi-injective module with $S=\mathrm{End}(M_R)$. Then*

(1) $l_S\left(\bigcap_{i=1}^{m}\operatorname{Ker}\alpha_i\right)=\sum_{i=1}^{m}S\alpha_i$ *for any $\alpha_1,\alpha_2,\dots,\alpha_m\in S$.*

(2) *If $\alpha_i,\beta_i \in S$ $(i=1,2,\cdots,m)$ satisfy $\bigcap_{i=1}^{m}\operatorname{Ker}\alpha_i \subseteq \bigcap_{i=1}^{m}\operatorname{Ker}\beta_i$, then $\beta_i\in\sum_{i=1}^{m}S\alpha_i$ $(i=1,2,\cdots,m)$.*

Take $M_R=xR$, $k=n$, $A=\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_m\end{pmatrix}$ and $B=\begin{pmatrix}x & & \\ & \ddots & \\ & & x\end{pmatrix}_{n\times n}$ in Theorem 2.2. Then we have the following corollary.

Corollary 2.5. *Let M_R be a cyclic (m,n)-quasi-injective module with $S=\mathrm{End}(M_R)$. Then*

$$l_{S^n}r_{M_n}\{\alpha_1,\alpha_2,\cdots,\alpha_m\}=\sum_{i=1}^{m}S\alpha_i$$

for any α_1, α_2, $\cdots,\alpha_m\in S^n$.

Let M_R be a module with $S=\mathrm{End}(M_R)$, write $W(S)=\{w\in S|\operatorname{Ker}(w)\trianglelefteq M\}$. Then $W(S)=J(S)$ in case M_R is a cyclic PQ-injective module [**5**, Proposition 2.4]. For the case of n-quasi-injective modules, we have

Lemma 2.6. *If M_R is n-quasi-injective and n-generated, then $W(S)=J(S)$, where $S=\mathrm{End}(M_R)$.*

Proof. If $a\in W(S)$, then $r_M(a)=\operatorname{Ker}a\trianglelefteq M$, and this forces $r_M(1-a)=0$, i.e., $l_Sr_M(1-a)=S$. Since M_R is n-quasi-injective and n-generated, we have $S(1-a)=S$ by Corollary 2.4. This means that $W(S)\subseteq J(S)$. Conversely, let $a\in J(S)$. For any $x\in M$, if $r_M(a)\cap xR=0$, then $l_S[r_M(a)\cap xR]=S$. So we have $Sa+l_S(x)=S$ by Corollary 2.3. It follows that $l_S(x)=S$, i.e., $x=0$. Therefore $r_M(a)\trianglelefteq M$, that is, $a\in W(S)$. □

Given a module M_R. We call $U(\neq 0)\in M^{m\times n}$ a ***right uniform element*** if UR_n is a uniform submodule of $(M_m)_R$, and write $M_U=\{x\in S^m|r_{M_m}(x)\cap UR_n\neq 0\}$.

Lemma 2.7. *Let M_R be (m,n)-quasi-injective with $S = \text{End}(M_R)$. If $U \in M^{m\times n}$ is a right uniform element, then M_U is the unique maximal submodule of ${}_SS^m$ which contains $l_{S^m}(U)$.*

Proof. Since UR_n is a uniform submodule of $(M_m)_R$, M_U is a submodule of ${}_SS^m$. It is easy to see that $l_{S^m}(U) \subseteq M_U \neq S^m$. If $A \in S^m \setminus M_U$, then $r_{M_m}(A) \cap UR_n = 0$. So $l_{S^m}(r_{M_m}(A) \cap UR_n) = S^m$. Let $\overline{A} = \begin{pmatrix} A \\ 0 \end{pmatrix} \in S^{m\times m}$. Then $r_{M_m}(\overline{A}) = r_{M_m}(A)$ and $S^m\overline{A} = SA$. But M_R is (m,n)-quasi-injective, by Theorem 2.2, $SA + l_{S^m}(U) = S^m$. Hence $SA + M_U = S^m$. Therefore M_U is a maximal submodule of ${}_SS^m$ which contains $l_{S^m}(U)$. Now, if $l_{S^m}(U) \subseteq {}_SL \subsetneqq S^m$, then $L \subseteq M_U$ (otherwise, if $A \in L \setminus M_U$, then $l_{S^m}(U) + SA = S^m$ as before. So we have $L = S^m$, a contradiction). This completes the proof. $\square$

Lemma 2.8. *Let M_R be (m,n)-quasi-injective with $S = \text{End}(M_R)$ and $W = U_1R_n \oplus \cdots \oplus U_tR_n$, where $U_i \in M^{m\times n}$ are right uniform elements, $i = 1, 2, \cdots, t$. If ${}_SL$ is a maximal submodule of ${}_SS^m$ not of the form M_U for any right uniform element $U \in M^{m\times n}$, then $r_{M_m}(E_m - A) \cap W \trianglelefteq W$ for some $A \in L_m$.*

Proof. Since $L \neq M_{U_1}$, so $r_{M_m}(x) \cap U_1R_n = 0$ for some $x \in L$, thus $r_{R_n}(xU_1) \subseteq r_{R_n}(U_1)$. Let $B = (xU_1, 0)^T \in M^{m\times n}$. Then $r_{R_n}(B) = r_{R_n}(xU_1) \subseteq r_{R_n}(U_1)$. Since M_R is (m,n)-quasi-injective, $S^mU_1 \subseteq S^mB$ by Theorem 1.3(3). Let $\varepsilon_1 = (1, 0, \cdots, 0)$, $\varepsilon_2 = (0, 1, 0, \cdots, 0)$, $\cdots$, $\varepsilon_m = (0, \cdots, 0, 1) \in S^m$ and suppose $\varepsilon_iU_1 = s_ixU_1$ for some $s_i \in S$ $(i = 1, 2, \cdots, m)$. Write $A_1 = (s_1x, \ldots, s_mx)^T$. Then $A_1 \in L_m$ and $(E_m - A_1)U_1 = 0$. So $r_{M_m}(E_m - A_1) \cap U_1R_n \neq 0$. If $r_{M_m}(E_m - A_1) \cap U_2R_n = 0$, then $(E_m - A_1)U_2R_n \cong U_2R_n$ is a unform right R-module. Hence $(E_m - A_2)(E_m - A_1)U_2 = 0$ for some $A_2 \in L_m$. Let $A_3 = A_1 + A_2 - A_2A_1$. Then $(E_m - A_3)U_1 = (E_m - A_3)U_2 = 0$. Thus $r_{M_m}(E_m - A_3) \cap U_iR_n \neq 0$, $i = 1, 2$. Continue in this way to obtain $A \in L_m$ such that $r_{M_m}(E_m - A) \cap W \trianglelefteq W$. $\square$

The following theorem extends [**6**, Theorem 3.3]. We complete this section with it and two corollaries.

Theorem 2.9. *Let M_R be an n-generated n-quasi-injective and finite dimensional module with $S = \text{End}(M_R)$.*

(1) *If $L \subseteq S$ is a maximal left ideal, then $L = M_U$ for some right uniform element $U \in M^n$.*
(2) *$S/J(S)$ is semisimple artinian.*

Proof. Since M_R is finite dimensional, we may assume $W = U_1R_n \oplus \cdots \oplus U_tR_n \trianglelefteq M_R$, where $U_1, \cdots, U_t \in M^n$ and each U_iR_n is uniform [**4**, Proposition 3.19]. If ${}_SL$ is a maximal left ideal of ${}_SS$ not of the form M_U for any right uniform element $U \in M^n$, then $r_M(1 - a) \cap W \trianglelefteq W$ for some $a \in L$ by Lemma 2.8. So $1 - a \in J(S) \subseteq L$ by Lemma 2.6, a contradiction. Thus (1) follows. As to (2), if $a \in M_{U_1} \cap M_{U_2} \cap \cdots \cap M_{U_t}$, then $r_M(a) \cap U_iR_n \neq 0$, $i = 1, 2, \cdots, t$. Hence

$$\bigoplus_{i=1}^{t} [r_M(a) \cap U_iR_n] \trianglelefteq M_R$$

because each U_iR_n is uniform. This means $r_M(a) \trianglelefteq M_R$. By Lemma 2.6, $a \in J(S)$. But each M_{U_i} is maximal in ${}_SS$ by Lemma 2.7, so

$$J(S) = M_{U_1} \cap M_{U_2} \cap \cdots \cap M_{U_t}.$$

Therefore $S/J(S)$ is semisimple artinian. □

Corollary 2.10. *If M_R is finitely quasi-injective finite dimensional and finitely generated, then $S/J(S)$ is semisimple artinian, where $S = \mathrm{End}(M_R)$.*

Corollary 2.11. *If M_R is an n-quasi-injective and n-generated uniform module, then $S = \mathrm{End}(M_R)$ is local.*

3. (m,n)-QUASI-INJECTIVE KASCH MODULES

Following Albu and Wisbauer [**1**], a right R-module M_R is called a ***Kasch*** module if any simple module in $\sigma[M_R]$ embeds in M_R, where $\sigma[M]$ is the category consisting of all M-subgenerated right R-modules [**9**, p. 118]. In this section, we study some properties of (m,n)-quasi-injective (in particular, n-quasi-injective) Kasch modules.

Recall that a bimodule ${}_SM_R$ is said to be ***faithfully balanced*** [**2**] in case the canonical ring homomorphisms $\lambda : S \to \mathrm{End}(M_R)$ and $\rho : R \to \mathrm{End}({}_SM)$ are isomorphisms.

Proposition 3.1. *If ${}_SM_R$ is faithfully balanced and M_R is an $(n, m+1)$-quasi-injective Kasch module, then ${}_SM$ is (m,n)-quasi-injective.*

Proof. Let $\alpha_1, \alpha_2, \cdots, \alpha_m \in M_n$. Then

$$N = \alpha_1 R + \cdots + \alpha_m R \subseteq r_{M_n} l_{S^n}\{\alpha_1, \ldots, \alpha_m\}.$$

Assume $\beta \in r_{M_n} l_{S^n}\{\alpha_1, \ldots, \alpha_m\}$ but $\beta \overline{\in} N$. Then $N_R \subseteq L_R$ for some maximal submodule L_R of $\beta R + N_R$. Since $(\beta R + N)/L$ is a simple module in $\sigma[M_R]$, there exists a monomorphism $\delta : (\beta R + N)/L \to M_R$. Define $f : \beta R + N \to M_R$ by $f(x) = \delta(x + L)$. Then $f(\alpha_i) = 0$ for all $i = 1, 2, \cdots, m$, but $f(\beta) \neq 0$. Note that M_R is $(n, m+1)$-quasi-injective and $\beta R + N$ is an $(m+1)$-generated submodule of $(M_n)_R$, so $f(x) = ux$ for some $u \in (\mathrm{End}(M_R))^n$. And hence there exists $v \in S^n$ such that $f(x) = vx$ for ${}_SM_R$ is balanced. Thus $v\alpha_i = 0$, $i = 1, 2, \cdots, m$, i.e., $v \in l_{S^n}\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$. This implies that $f(\beta) = v\beta = 0$, a contradiction. So $N = r_{M_n} l_{S^n}\{\alpha_1, \cdots, \alpha_m\}$, whence ${}_SM$ is (m,n)-quasi-injective. □

Corollary 3.2. [**3**, Theorem 2.7] *If R is right Kasch and right $(n, m+1)$--injective, then R is left (m,n)-injective.*

Our next theorem extends [**6**, Lemma 2.3].

Theorem 3.3. *Given a left balanced bimodule ${}_SM_R$. If M_R is l-generated and ln-quasi-injective and Kasch, then $l_{S^n}(J_n) \trianglelefteq_S S^n$, where $J = \mathrm{Rad}(M_R)$.*

Proof. If $0 \neq a \in S^n$, then choose a maximal submodule A of the right R-module aM_n. Let $\sigma : aM_n/A \to M_R$ be a monomorphism and define $\alpha : aM_n \to M_R$

by $\alpha(x) = \sigma(x + A)$. Since aM_n is an ln-generated submodule of the ln-quasi-injective module M_R, α extends to an endomorphism of M. Then $\alpha = s_0\cdot$ for some $s_0 \in S$ because ${}_SM_R$ is left balanced. Choose $y \in M_n$ such that $ay\overline{\in}A$. Then $s_0ay = \alpha(ay) = \sigma(ay + A) \neq 0$. So $s_0a \neq 0$. If $aJ_n \nsubseteq A$, then $aT_n + A = aM_n$. Now, let $a = (s_1, \cdots, s_n)$. Then $s_i(\mathrm{Rad}(M_R)) \ll s_iM$ $(i = 1, 2, \cdots, n)$ for M_R is finitely generated. This follows that

$$\sum_{i=1}^{n} s_i(\mathrm{Rad}\, M_R) \ll \sum_{i=1}^{n} s_i(M_R), \quad \text{i.e.,} \quad aJ_n \ll aM_n.$$

Hence $A = aM_n$, a contradiction. Thus $aJ_n \subseteq A$ and it implies that

$$(s_0a)J_n = \alpha(aJ_n) = \sigma(0) = 0.$$

So $0 \neq s_0a \in Sa \cap l_{S^n}(J_n)$. Therefore $l_{S^n}(J_n) \trianglelefteq_S S^n$. □

Corollary 3.4. *Given a cyclic module M_R with $S = \mathrm{End}(M_R)$, if M_R is PQ--injective and Kasch, then $l_S(J) \trianglelefteq_S S$, where $J = \mathrm{Rad}(M_R)$.*

Corollary 3.5. *Given a finitely generated module M_R with $S = \mathrm{End}(M_R)$. If M_R is finitely quasi-injective and Kasch, then $l_{S^n}(J_n) \trianglelefteq_S S^n$ for all positive integers n, where $J = \mathrm{Rad}(M_R)$.*

Lemma 3.6. *Given a module M_R with $S = \mathrm{End}(M_R)$. If $\mathrm{Rad}(M_R) \neq M_R$ and consider the following conditions:*

(1) *M_R is a Kasch module.*
(2) *$l_{S^n}(T) \neq 0$ for all positive integers n and for any maximal submodule T of $(M_n)_R$.*
(3) *$l_{S^n}(T) \neq 0$ for some positive integer n and for any maximal submodule T of $(M_n)_R$.*
(4) *$l_S(T) \neq 0$ for any maximal submodule T of M_R.*

Then we always have the following implications:

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).$$

If M_R generates all simple modules in $\sigma[M]$ (in particular, if M_R is a generator in $\sigma[M]$), then we have (4) $\Rightarrow$ (1).

Proof. Since $\mathrm{Rad}(M) \neq M$, so M (and hence M_n) has maximal submodules.

(1) $\Rightarrow$ (2). Let $\varphi : M_n/T \to M_R$ be a monomorphism, define $f : M_n \to M$ by $x \mapsto \varphi(x + T)$, and write $a = (fl_1, fl_2, \cdots, fl_n)$. Then $0 \neq a \in S^n$ and $aT = f(T) = 0$. So $l_{S^n}(T) \neq 0$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4). If $n = 1$, the implication holds. Now we assume $n > 1$. Let T be any maximal submodule of M, write $K = \begin{pmatrix} T \\ M_{n-1} \end{pmatrix}$, and define $\varphi : M_n/K \to M/T$ via $\begin{pmatrix} x \\ y \end{pmatrix} + K \mapsto x + T$, where $x \in M$, $y \in M_{n-1}$. Then φ is a right R-isomorphism. This means that K is a maximal submodule of M_n. Hence

$l_{S^n}(K) \neq 0$. Suppose $0 \neq (u,v) \in l_{S^n}(K)$, where $u \in S$ and $v \in S^{n-1}$. Then $0 \neq u \in l_S(T)$.

Lastly, assume M generates all simple R-modules in $\sigma[M]$ and (4) holds. Then for every simple module A_R in $\sigma[M]$, there exists a maximal submodule T of M such that $A \cong M/T$. Suppose $0 \neq s_0 \in l_S(T)$. Then $T \subseteq r_M(s_0) \neq M$. Hence $T = r_M(s_0)$. Now we define $\varphi : M/T \to M$ by $x + T \mapsto s_0 x$. Then it is easy to see that φ is an R-monomorphism. □

The following theorem is an extension of [**7**, Theorem 1.2].

Theorem 3.7. *Let M_R be an n-quasi-injective cyclic Kasch module with $S =$ End(M_R). Then the map $K \mapsto r_{M_n}(K)$ and $T \mapsto l_{S^n}(T)$ are mutually inverse bijections between the set of all minimal submodules of ${}_SS^n$ and the set of all maximal submodules of $(M_n)_R$. In particular,*

(1) *$l_{S^n} r_{M_n}(K) = K$ for all minimal submodules K of ${}_SS^n$.*
(2) *$r_{M_n} l_{S^n}(T) = T$ for all maximal submodules T of $(M_n)_R$.*

Proof. (1) follows from Corollary 2.5. As to (2), observe that $T \subseteq r_{M_n} l_{S^n}(T)$ and that $r_{M_n} l_{S^n}(T) \neq M_n$ by Lemma 3.6. The proof is completed by establishing the following claims. □

Claim 1. *$r_{M_n}(K)$ is a maximal submodule of $(M_n)_R$ for each minimal submodule K of ${}_SS^n$.*

Proof. Let $r_{M_n}(K) \subseteq T$, where T is a maximal submodule of M_n. Then $0 \neq l_{S^n}(T) \subseteq l_{S^n} r_{M_n}(K) = K$ by (1). So $l_{S^n}(T) = K$ because K is minimal in ${}_SS^n$. Hence $r_{M_n}(K) = r_{M_n} l_{S^n}(T) = T$ by (2). □

Claim 2. *$l_{S^n}(T)$ is a minimal submodule of ${}_SS^n$ for all maximal submodules T of $(M_n)_R$.*

Proof. Since M_R is Kasch, by Lemma 3.6(2), we may choose $0 \neq x \in l_{S^n}(T)$. Then $T \subseteq r_{M_n}(x) \neq M_n$, whence $T = r_{M_n}(x)$. As M_R is n-quasi-injective and cyclic, this gives $l_{S^n}(T) = l_{S^n} r_{M_n}(x) = Sx$ by Corollary 2.5 and it follows that $l_{S^n}(T)$ is a minimal submodule of ${}_SS^n$. □

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References

1. Albu T. and Wisbauer R., *Kasch Modules, in Advances in Ring Theory*, edited: Jain, S. K. and Rizvi, S. T., Birkhäuser, 1997, 1–16.
2. Anderson F. W. and Fuller K. R., *Rings and Categories of Modules*, GTM13, Springer-Verlag, New York, 1974.
3. Chen J. L., Ding N. Q., Li Y. L. and Zhou Y. Q., *On (m,n)-injectivity of Modules*. Comm. Algebra **29**(12) (2001), 5589–5603.
4. Goodearl K. R., *Ring Theory: Nonsingular Rings and Modules*, Marcel Dekker, 1976.

5. Nicholson W. K., Park J. K. and Yousif M. F., *Principally Quasi-injective Modules*, Comm. Algebra. **27**(4) (1999), 1683–1693.

6. Nicholson W. K. and Yousif M. F., *Principally Injective Rings*, J. Algebra. **174** (1995), 7–93.

7. ______, *On a Theorem of Camillo*, Comm. Algebra. **23**(14) (1995), 5309–5314.

8. Ramamurthi V. S. and Rangaswamy K. M., *On Finitely Injective Modules*, J. Austral. Math. Soc. **16** (1973), 239–248.

9. Wisbauer R., *Foundation of Module and Ring Theory*, Gordon and Breach, Reading, 1991.

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