A NOTE ON A MOORE BOUND FOR GRAPHS EMBEDDED IN SURFACES

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Abstract. Based on a separator theorem for general surfaces we prove a Moore bound for graphs of given degree and diameter, embedded in a fixed surface.

The problem of determining the largest order (i.e., number of vertices) $n(d,k)$ of a graph of maximum degree at most d and diameter at most k is well known as the *degree-diameter problem*. A spanning tree argument shows that $n(d,k) \leq M(d,k)$, where $M(d,k) = 1 + d + d(d-1) + \dots d(d-1)^{k-1}$ is the *Moore bound*. In particular, for fixed k and $d \to \infty$ we have $n(d,k) < d^k$, and it is believed that d^k is the correct asymptotic order of $n(d,k)$. More exactly, a conjecture of Bollobás [1] claims that for each $\delta > 0$ there exist d_0 and k_0 such that for all $d \geq d_0$ and $k \geq k_0$ we have $n(d,k) > (1-\delta)d^k$. As of now the conjecture has been proved for $k \leq 3$ and $k = 5$ only. For the current state of the degree-diameter problem we refer to a survey article by Miller and Širáň [6].

The degree-diameter problem has often been considered for restricted classes of graphs such as bipartite graphs, Cayley graphs, vertex-transitive graphs, and others. In this article we focus on the degree-diameter problem for graphs embeddable in a fixed surface. Here, by a *surface* we understand a compact, connected 2-manifold without boundary. The classification of surfaces is well known; they split into two classes according to orientability. Each orientable (nonorientable) surface is homeomorphic to a sphere with $g \geq 0$ handles ($h \geq 1$ crosscaps) attached; the *Euler genus* ε of a surface is $\varepsilon = 2g$ or $\varepsilon = h$, respectively.

Given a surface S, let $n_S(d,k)$ denote the maximum order of a graph of largest degree at most d and diameter at most k, embeddable in S. In the case when $S = S_0$ is a sphere, it was shown by Fellows, Hell and Seyffarth [3]

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that $n_{S_0}(d,k) < (6k+3)(2d^{\lfloor k/2 \rfloor}+1)$. On the other hand, for an arbitrary surface S and diameter $k=2$, Knor and Širáň [5] proved that $n_S(d,2) = \lfloor \frac{3}{2}d \rfloor + 1$ for $d \geq d_S$. The two striking features of this result are that it gives the exact value of $n_S(d,2)$ for sufficiently large d, and this value does not depend on the surface at all.

In this note we extend the results of [3, 5] by presenting a general upper bound of asymptotic form $n_S(d,k) < C\ d^{\lfloor k/2 \rfloor}$, valid for all d, k and all surfaces S, where C depends on k and S. Consequently, Bollobás' conjecture cannot be proved by considering graphs embeddable in a fixed surface. To prove the bound we use the idea of the proof of Corollary 14 of [3], which is a combination of a simple counting argument with a planar separator theorem of Lipton and Tarjan [7]. It turns out that it is sufficient to replace the planar separator theorem with the following analogous result for general surfaces.

Proposition 1. *Let G be a graph of order n, cellularly embedded in a surface of Euler genus $\varepsilon > 0$. Let T be a spanning tree of G of radius r rooted at v. Then there exists a partition of $V(G)$ into three subsets A, B, C, such that $|A|, |B| \leq \frac{2}{3}n$, $|C| \leq 2r(\varepsilon+1)+1$, $v \in C$, and there is no edge between A and B.*

Proof. For orientable surfaces this result was proved by Gilbert, Hutchinson and Tarjan [4] and later by Djidjev [2]. It is widely acknowledged that the proof in [4] carries over to nonorientable surfaces. □

With this separator result it is easy to prove a Moore bound for graphs embeddable in general surfaces.

Theorem 1. *Let G be a graph of order n, maximum degree $d \geq 3$, and diameter at most k, cellularly embedded in a surface of Euler genus ε. Then*

$$n \leq (6k(\varepsilon+1)+3)\ \frac{d((d-1)^{\lfloor k/2 \rfloor}-2)}{d-2}\ .$$

Proof. Being of diameter at most k, the graph G contains a spanning tree of radius at most k, rooted at a vertex v. According to Proposition 1 with $r=k$ (and Lemma 2 of [7] for $\varepsilon = 0$), there is a partition of $V(G)$ into subsets A, B, C such that $|A|, |B| \leq \frac{2}{3}n$, $|C| \leq 2k(\varepsilon+1)+1$, with no edge between A and B. From this point on, one may exactly follow the proof of Corollary 14 in [3] to obtain the bound

$$|A| \leq |C|(d + d(d-1) + \cdots + d(d-1)^{\lfloor k/2 \rfloor - 1}) = |C| \, \frac{d((d-1)^{\lfloor k/2 \rfloor} - 1)}{d-2} \, .$$

Since $A = V(G) \setminus (B \cup C)$ and $|B| \leq \frac{2}{3}n$, $|C| \leq 2k(\varepsilon + 1) + 1$, it follows that $|A| \geq n - \frac{2}{3}n - (2k(\varepsilon+1)+1)$. Combining the last four inequalities and solving for n yields

$$n \leq (6k(\varepsilon + 1) + 3) \, \frac{d((d-1)^{\lfloor k/2 \rfloor} - 2)}{d-2}$$

which completes the proof. □

Corollary 1. *For $d \geq 3$, $k \geq 1$, and an arbitrary surface S of Euler genus ε we have*

$$1 + \frac{d((d-1)^{\lfloor k/2 \rfloor} - 2)}{d-2} \leq n_S(d,k) \leq c_{S,k} \, \frac{d((d-1)^{\lfloor k/2 \rfloor} - 2)}{d-2} \, ,$$

where $c_{S,k} = 6k(\varepsilon + 1) + 3$.

Proof. The upper bound follows from the preceding theorem. The lower bound is obtained from the fact that a tree of radius $\lfloor k/2 \rfloor$, maximum degree d, and of order $1 + d + d(d-1) + \cdots + d(d-1)^{\lfloor k/2 \rfloor - 1}$, extended by ε suitable edges joining some of the leaves, yields a graph of maximum degree d and diameter k that embeds (with a single face) on S. □

This corollary suggests that it is worthwhile to study the limit

$$\lim_{d \to \infty} \frac{n_S(d,k)}{d^{\lfloor k/2 \rfloor}} \, .$$

If the limit exists then its value is between 1 and $c_{S,k} = 6k(\varepsilon + 1) + 3$; narrowing this gap is likely to be a hard problem.

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