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CERTAIN RESULTS INVOLVING A CLASS OF FUNCTIONS ASSOCIATED WITH THE HURWITZ ZETA FUNCTION

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ABSTRACT. The purpose of this paper is to consider a new generalization of the Hurwitz zeta function. Generating functions, Mellin transform, and a series identity are obtained for this generalized class of functions. Some of the results are used to provide a further generalization of the Lambert transform. Relevance with various known results are depicted invariably. Multivariable extensions are also pointed out briefly.

1. INTRODUCTION AND PRELIMINARIES

The generalized (Hurwitz's) zeta function is defined by ([**1**], [**2**])

$$\zeta(s,a)=\sum_{n=0}^{\infty}(a+n)^{-s}\qquad(a\neq 0,-1,-2,\dots;\ \Re(s)>1), \tag{1.1}$$

so that when $a=1$, we have

$$\zeta(s,1)=\sum_{n=0}^{\infty}n^{-s}=\zeta(s), \tag{1.2}$$

where $\zeta(s)$ is the Riemann zeta function. The function $\Phi(x,s,a)$ extends (1.1) further, and this generalized Hurwitz-Lerch zeta function [**1**, p. 316], is defined by

$$\Phi(x,s,a)=\sum_{n=0}^{\infty}(a+n)^{-s}x^{n}\qquad(a\neq 0,-1,-2,\dots,\ |x|<1;\ \Re(s)>1,\text{when }\ |x|=1). \tag{1.3}$$

Evidently, we have

$$\Phi(1,s,1)=\zeta(\mathrm{s}), \tag{1.4}$$

$$\Phi(1,s,a)=\zeta(s,a). \tag{1.5}$$

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The function $\Phi(x,s,a)$ has the integral representation

$$\Phi(x,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{e}^{-at}(1-x\mathrm{e}^{-t})^{-1}dt. \tag{1.6}$$

$$(\Re(a)>0;\ \text{either } |x|\le 1,\ x\neq 1, \text{and } \Re(s)>0,\ \text{ or } x=1,\ \Re(s)>1).$$

In the present paper we introduce a class of functions $\Theta_b^\lambda(x,\alpha,a,b)$ which is defined by

$$\Theta_\mu^\lambda(x,\alpha,a,b)=\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\mathrm{e}^{-at-bt^{-\lambda}}(1-x\mathrm{e}^{-t})^{-\mu}dt \tag{1.7}$$

$$(\lambda>0,\ \mu\ge 1,\ \Re(a)>0,\ \Re(b)>0\ ;\ \text{when } \Re(b)=0,$$
$$\text{then either } |x|\le 1\ (x\neq 1),\ \Re(\alpha)>0,\ \text{ or } x=1, \Re(\alpha))$$

The various results obtained in this paper include series representation, Mellin transform, and generating functions involving the above class of functions (1.7). Some of the results are used to consider a new generalization of the Lambert transform. Relevance with several new and known results are pointed out. Multivariable extensions are also briefly indicated in the concluding section.

Special Cases of (1.7)

(i) When $\lambda=\mu=1,\ x=1$, we have

$$\begin{aligned}\Theta_1^1(1,\alpha,a,b) &= \zeta_b(\alpha,a)\\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{t^{\alpha-1}\mathrm{e}^{-at-b/t}}{(1-\mathrm{e}^{-t})}dt,\end{aligned} \tag{1.8}$$

$$(0\le a\le 1,\ \Re(b)>0;\ b=0,\ \Re(\alpha)>1)$$

where $\zeta_b(\alpha,a)$ is the extended Hurwitz zeta function defined by [**1**, p. 308].

(ii) When $\lambda=\mu=1,\ b=0$, we have

$$\begin{aligned}\Theta_1^1(x,\alpha,a,0) &= \Phi(x,\alpha,a)\\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{t^{\alpha-1}\mathrm{e}^{-at}}{(1-x\mathrm{e}^{-t})}dt,\end{aligned} \tag{1.9}$$

$$(\Re(a)>0;\ \text{either } |\mathrm{x}|\le 1\ (x\neq 1), \Re(\alpha)>0 \text{ or } x=1,\ \Re(\alpha)>1)$$

where $\Phi(x,\alpha,a)$ is the generalized zeta function defined by (1.6).

(iii) When $\lambda=\mu=1,\ x=-1,\ a=1$, then

$$\begin{aligned}\Theta_1^1(-1,\alpha,1,b) &= (1-2^{1-\alpha})\zeta_b^*(\alpha,a)\\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{t^{\alpha-1}\mathrm{e}^{-at-b/t}}{(1+\mathrm{e}^{-t})}dt,\end{aligned} \tag{1.10}$$

where $\zeta_b^*(\alpha,a)$ is the extended Hurwitz zeta function defined by [**1**, p. 309].

(iv) When $b = 0$, then

$$\Theta_\mu^\lambda(x,\alpha,a,0) = \phi_\mu^*(x,\alpha,a) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{t^{\alpha-1}\mathrm{e}^{-at}}{(1-x\mathrm{e}^{-t})^\mu}dt, \tag{1.11}$$

$(\mu \geq 1,\ \Re(a) > 0\ ;\ \text{either}\ |x| \leq 1\ (x \neq 1),\ \Re(\alpha) > 0,\ \text{or}\ x = 1,\ \Re(\alpha) > \mu)$

which was studied recently by Goyal and Laddha [**3**].

The series representation of (1.11) is given by

$$\Phi_\mu^*(x,\alpha,a) = \sum_{n=0}^\infty \frac{(\mu)_n x^n}{(a+n)^\alpha n!} \tag{1.12}$$

$$(\mu \geq 1, \Re(a) > 0, \Re(\alpha) > 0, |x| \leq 1).$$

2. Series repretentation and Mellin transform

In this section we first find series representation and the Mellin transform of the class of functions $\Theta_b^\lambda(x,\alpha,a,b)$ (defined by (1.7)).

Making use of (1.7), changing the order of integration and summation (under the prescribed conditions stated with (1.7)), we have

$$\begin{aligned}
\Theta_\mu^\lambda(x,\alpha,a,b) &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\mathrm{e}^{-at-bt^{-\lambda}}(1-x\mathrm{e}^{-t})^{-\mu}dt \\
&= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\mathrm{e}^{-at}(1-x\mathrm{e}^{-t})^{-\mu}\left(\sum_{m=0}^\infty \frac{(-b)^m}{m!}t^{-\lambda m}\right)dt \\
&= \frac{1}{\Gamma(\alpha)}\sum_{m=0}^\infty \frac{(-b)^m}{m!}\int_0^\infty t^{\alpha-\lambda m-1}\mathrm{e}^{-at}(1-x\mathrm{e}^{-t})^{-\mu}dt.
\end{aligned}$$

This gives the series representation as

$$\Theta_\mu^\lambda(x,\alpha,a,b) = \frac{1}{\Gamma(\alpha)}\sum_{m=0}^\infty \frac{\Gamma(\alpha-\lambda m)(-b)^m}{m!}\Phi_\mu^*(x,\alpha-\lambda m,a) \tag{2.1}$$

$$(\lambda > 0,\ \mu \geq 1,\ \Re(a) > 0,\ \Re(b) \geq 0,\ \Re(\alpha) \neq \nu\lambda\ (\nu \in \mathbb{N}),\ |x| \leq 1).$$

To find the Mellin transform of the function $\Theta_\mu^\lambda(x,\alpha,a,b)$, we have from the definition of the Mellin transform [**1**, p. 10]:

$$\begin{aligned}
\mathfrak{m}_s\{\Theta_\mu^\lambda(x,\alpha,a,b)\} &= \int_0^\infty b^{s-1}\Theta_\mu^\lambda(x,\alpha,a,b)db \\
&= \frac{1}{\Gamma(\alpha)}\int_0^\infty b^{s-1}\left(\int_0^\infty \frac{t^{\alpha-1}\mathrm{e}^{-at-bt^{-\lambda}}}{(1-x\mathrm{e}^{-t})^\mu}dt\right)db
\end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1} e^{-at}}{(1-xe^{-t})^\mu} \left(\int_0^\infty b^{s-1} e^{-bt^{-\lambda}} db \right) dt,$$

$$= \frac{\Gamma(s)}{\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha+\lambda s-1} e^{-at}}{(1-xe^{-t})^\mu} dt.$$

Thus

$$\mathfrak{m}_s \left\{ \Theta_\mu^\lambda(x, \alpha, a, b) \right\} = \frac{\Gamma(s)\Gamma(\alpha+\lambda s)}{\Gamma(\alpha)} \Phi_\mu^*(x, \alpha+\lambda s, a). \tag{2.2}$$

$$(\lambda > 0,\ \mu \geq 1,\ \Re(s) > 0,\ \Re(\mathrm{a}) > 0;\ \text{ either } |x| \leq 1\ (x \neq 1),$$
$$\Re(\alpha) > -\lambda \Re(s),\ \text{ or } x = 1,\ \Re(\alpha) > 1 - \lambda\Re(s))$$

For $s = 1$ in (2.2), we at once have

$$\int_0^\infty \Theta_\mu^\lambda(x, \alpha, a, b) db = \frac{\Gamma(\alpha+\lambda)}{\Gamma(\alpha)} \Phi_\mu^*(x, \alpha+\lambda, a). \tag{2.3}$$

3. Generating relations

Using (2.1) and (1.12), we have

$$\sum_{k=0}^\infty \binom{\alpha+k-1}{k} \Theta_\mu^\lambda(x, \alpha+k, a, b) t^k$$

$$= \sum_{k=0}^\infty \binom{\alpha+k-1}{k} t^k \frac{1}{\Gamma(\alpha+k)} \sum_{m=0}^\infty \frac{-b^m}{m!} \Gamma(\alpha+k-\lambda m) \Phi_\mu^*(x, \alpha+k-\lambda m, a)$$

$$= \sum_{m=0}^\infty \frac{(-b)^m}{m!} \sum_{k=0}^\infty \binom{\alpha+k-1}{k} \frac{\Gamma(\alpha+k-\lambda m)}{\Gamma(\alpha+k)} \Phi_\mu^*(x, \alpha+k-\lambda m, a) \mathrm{t}^k$$

$$= \sum_{m=0}^\infty \frac{(-b)^m}{m!} \sum_{k=0}^\infty \binom{\alpha+k-1}{k} \frac{\Gamma(\alpha+k-\lambda m)}{\Gamma(\alpha+k)} t^k \sum_{n=0}^\infty \frac{(\mu)_n x^n}{(a+n)^{\alpha-\lambda m+k} n!}$$

$$= \sum_{m=0}^\infty \frac{(-b)^m}{m!} \sum_{n=0}^\infty \frac{(\mu)_n x^n}{(a+n)^{\alpha-\lambda m} n!} \sum_{k=0}^\infty \binom{\alpha+k-1}{k} \frac{\Gamma(\alpha+k-\lambda m) t^k}{\Gamma(\alpha+k)(a+n)^k}$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{m=0}^\infty \frac{\Gamma(\alpha-\lambda m)(-b)^m}{m!} \sum_{n=0}^\infty \frac{(\mu)_n x^n}{(a+n)^{\alpha-\lambda m} n!} \left(1 - \frac{t}{a+n}\right)^{-(\alpha-\lambda m)}$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{m=0}^\infty \frac{\Gamma(\alpha-\lambda m)(-b)^m}{m!} \sum_{n=0}^\infty \frac{(\mu)_n x^n}{(a+n-t)^{\alpha-\lambda m} n!}$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{m=0}^\infty \frac{\Gamma(\alpha-\lambda m)(-b)^m}{m!} \ \Phi_\mu^*(x, \alpha-\lambda m, a-t)$$

Hence

$$\sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}\Theta_{\mu}^{\lambda}(x,\alpha+k,a,b)t^{k} = \frac{1}{\Gamma(\alpha)}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha-\lambda m)\ (\text{ - b})^{m}}{m!}\Phi_{\mu}^{*}(x,\alpha-\lambda m,a-t) \tag{3.1}$$

$(\lambda>0,\ \mu\geq 1,\ \Re(a)>0,\ \Re(b)>0,\ \Re(\alpha)\neq\nu\Re(\lambda)\ (\nu\in\mathbb{N}),\ |x|\leq 1,\ |t|<|a|)$.

If we put $\lambda=\mu=1$, $x=1$ in (3.1), then in view of (1.7), we have

$$\sum_{\text{k = 0}}^{\infty}\binom{\alpha+k-1}{k}\zeta_b(\alpha+k,a)t^{k} = \sum_{m=0}^{\infty}\frac{\Gamma(\alpha-m)(-b)^{m}}{m!}\zeta(\alpha-m,a-t), \tag{3.2}$$

where $\zeta_b(x,a)$ is the extended Hurwitz zeta function defined by (1.8). The result [**1**, p. 321, Eqn. 7.220] corresponds to (3.2), when $b=0$.

On the other hand, when $\lambda=\mu=1$, $b=0$ in (3.1), then in view of (1.9) we receive

$$\sum_{\text{k = 0}}^{\infty}\binom{\alpha+k-1}{k}\Phi(x,\alpha+k,a)t^{k}=\Phi(x,\alpha,a-t) \tag{3.3}$$

$$(\alpha\neq 1,\quad |t|<|a|).$$

This result was given by Raina and Srivastava [**6**, p. 302].

For $\lambda=\mu=1$, and $x=-1$ (3.1) in view of (1.10) and (1.11) gives

$$\sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}\zeta_b^{*}(\alpha+k,a) = \frac{1}{1-2^{1-\alpha}}\sum_{m=0}^{\infty}\frac{b^{m}}{(1-\alpha)_m m!}\ \Phi(-1,\alpha-m,a-t). \tag{3.4}$$

Next, using (2.1) and (1.12), we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}\binom{\alpha+2k-1}{2k}\Theta_{\mu}^{\lambda}(x,\alpha+2k,a,b)t^{2k}\\
&= \sum_{k=0}^{\infty}\binom{\alpha+2k-1}{2k}t^{2k}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha+2k-\lambda m)(-b)^{m}}{\Gamma(\alpha+2k)m!}\ \Phi_{\mu}^{*}(x,\alpha+2k-\lambda m,a)\\
&= \frac{1}{\Gamma(\alpha)}\sum_{m=0}^{\infty}\frac{(-b)^{m}}{m!}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+2k-\lambda m)}{(2k)!}t^{2k}\sum_{i=0}^{\infty}\frac{(\mu)_i x^{i}}{(a+i)^{\alpha+2k-\lambda m}i!}\\
&= \frac{1}{\Gamma(\alpha)}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha-\lambda m)(-b)^{m}}{m!}\sum_{i=0}^{\infty}\frac{(\mu)_i x^{i}}{(a+i)^{\alpha-\lambda m}i!}\sum_{k=0}^{\infty}\frac{(\alpha-\lambda m)_{2k}t^{2k}}{(a+i)^{2k}(2k)!}
\end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha-\lambda m)(-b)^m}{m!} \sum_{i=0}^{\infty} \frac{(\mu)_i x^i}{2(a+i)^{\alpha-\lambda m} i!} \cdot \left[\left(1+\frac{t}{a+i}\right)^{-(\alpha-\lambda m)} + \left(1-\frac{t}{a+i}\right)^{-(\alpha-\lambda m)}\right]$$

$$= \frac{1}{2\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha-\lambda m)(-b)^m}{m!} \sum_{i=0}^{\infty} \frac{(\mu)_i x^i}{i!} \cdot \left[(a+i+t)^{-(\alpha-\lambda m)} + (1+i-t)^{-(\alpha-\lambda m)}\right]$$

$$= \frac{1}{2\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha-\lambda m)(-b)^m}{m!} \left\{\Phi_\mu^*(x,\alpha-\lambda m, a-t) + \Phi_\mu^*(x,\alpha-\lambda m, a+t)\right\}.$$

Thus

$$\text{(3.5)} \quad \sum_{k=0}^{\infty} \binom{\alpha+2k-1}{2k} \Theta_\mu^\lambda(x,\alpha+2k,a,b) t^{2k} = \frac{1}{2\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha-\lambda m)(-b)^m}{m!} \left[\Phi_\mu^*(x,\alpha-\lambda m, a-t) + \Phi_\mu^*(x,\alpha-\lambda m, a+t)\right],$$

under the same conditions as stated with (3.1).

Setting $\lambda = \mu = 1$, $x = 1$ in (3.5), then in view of (1.8) and (1.12), we have

$$\text{(3.6)} \quad \sum_{k=0}^{\infty} \binom{\alpha+2k-1}{2k} \zeta_b(\alpha+2k,a) t^{2k} = \frac{1}{2\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha-m)(-b)^m}{m!} \left[\zeta(\alpha-m,a-t) + \zeta(\alpha-m,a+t)\right].$$

The result [**1**, p. 32, Eqn. (7.223)] corresponds to (3.6) when $a = 1$, $b = 0$.

If we put $\lambda = \mu = 1$ and $x = -1$ in (3.5), then in view of (1.10) and (1.11), we have

$$\text{(3.7)} \quad \sum_{k=0}^{\infty} \binom{\alpha+2k-1}{2k} \zeta_b^*(\alpha+2k,a) t^{2k} = \frac{1}{2\Gamma(\alpha)(1-2^{1-\alpha})} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha-m)(-b)^m(1-2^{1-\alpha+m})}{m!} \cdot \left[\zeta_0^*(\alpha-m,a-t) + \zeta_0^*(\alpha-m,a+t)\right].$$

Similarly we can establish the relation:

$$\begin{aligned}(3.8)\qquad &\sum_{k=0}^{\infty}\binom{\alpha+2k}{2k+1}\Theta_{\mu}^{\lambda}(x,\alpha+2k+1,a,b)t^{2k+1}\\ &= \frac{1}{2\Gamma(\alpha)}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha-\lambda m)(-b)^m}{m!}\\ &\qquad\cdot\left[\Phi_{\mu}^{*}(x,\alpha-\lambda m,a-t)-\Phi_{\mu}^{*}(x,\alpha-\lambda m,a+t)\right],\end{aligned}$$

under the same condition as stated with (3.1).

Now we turn again to (2.1), and find a generating function which involves additional parameters. We have

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\theta)_n(\varphi)_n}{(\upsilon)_n}\Theta_{\mu}^{\lambda}(x,\theta+\varphi-\upsilon+n,a,b)\frac{t^n}{n!}\\ =\ &\sum_{n=0}^{\infty}\frac{(\theta)_n(\varphi)_n t^n}{(\upsilon)_n n!}\sum_{m=0}^{\infty}\frac{\Gamma(\theta+\varphi-\upsilon-\lambda m)(-b)^m}{\Gamma(\theta+\varphi-\upsilon+n)m!}\phi_{\mu}^{*}(x,\theta+\varphi-\upsilon+n-\lambda m,a)\\ =\ &\sum_{n=0}^{\infty}\frac{(\theta)_n(\varphi)_n t^n}{(\upsilon)_n\Gamma(\theta+\varphi-\upsilon+n)n!}\sum_{m=0}^{\infty}\frac{\Gamma(\theta+\varphi-\upsilon+n-\lambda m)(-b)^m}{m!}\\ &\cdot\sum_{k=0}^{\infty}\frac{(\mu)_k x^k}{k!(a+k)^{\theta+\varphi-\upsilon+n-\lambda m}}\\ =\ &\sum_{k=0}^{\infty}\frac{(\mu)_k x^k}{k!}\sum_{m=0}^{\infty}\frac{\Gamma(\theta+\varphi-\upsilon-\lambda m)(-b)^m}{\Gamma(\theta+\varphi-\upsilon)m!(a+k)^{\theta+\varphi-\upsilon-\lambda m}}\\ &\cdot\sum_{n=0}^{\infty}\frac{(\theta)_n(\varphi)_n(\theta+\varphi-\upsilon-\lambda m)_n \mathrm{t}^n}{(\upsilon)_n(\theta+\varphi-\upsilon)_n(a+k)^n n!}\\ =\ &\sum_{m,k=0}^{\infty}\frac{\Gamma(\theta+\varphi-\upsilon-\lambda m)(\mu)_k(-b)^m x^k}{\Gamma(\theta+\varphi-\upsilon)(a+k)^{\theta+\varphi-\upsilon-\lambda m}m!k!}\ {}_3F_2\left({\theta,\varphi,\theta+\varphi-\upsilon-\lambda m;\atop \upsilon,\theta+\varphi-\upsilon\quad ;}\ t/a+k\right).\end{aligned}$$

Hence

$$\begin{aligned}(3.9)\qquad &\sum_{n=0}^{\infty}\frac{(\theta)_n(\varphi)_n}{(\upsilon)_n}\Theta_{\mu}^{\lambda}(x,\theta+\varphi-\upsilon+n,a,b)\frac{t^n}{n!}\\ &= \frac{1}{\Gamma(\theta+\varphi-\upsilon)}\sum_{m,k=0}^{\infty}\frac{\Gamma(\theta+\varphi-\upsilon-\lambda m)(\mu)_k(-b)^m x^k}{(a+k)^{\theta+\varphi-\upsilon-\lambda m}m!k!}\\ &\quad\cdot{}_3F_2\left({\theta,\varphi,\theta+\varphi-\upsilon-\lambda m;\atop \upsilon,\theta+\varphi-\upsilon\quad ;}\ t/a+k\right)\end{aligned}$$

$$(\Re(\theta+\varphi)>\Re(\upsilon)>0,\ \Re(\theta+\varphi-\upsilon)\neq r\lambda(r\in\mathbb{N}),\ |t/a|<1).$$

Substituting $\lambda=\mu=1$, $x=1$ in (3.9), then in view of (1.9) we have

$$\text{(3.10)}\qquad \sum_{n=0}^{\infty}\frac{(\theta)_n(\varphi)_n}{(\upsilon)_n}\varsigma_b(\theta+\varphi-\upsilon+n,a,)\frac{t^n}{n!}$$
$$=\sum_{m,k=0}^{\infty}\frac{(\theta+\varphi-\upsilon)_m(-b)^m}{(a+k)^{\theta+\varphi-\upsilon-m}m!}\ {}_3F_2\left({}^{\theta,\varphi,\theta+\varphi-\upsilon-m;}_{\upsilon,\theta+\varphi-\upsilon\quad ;}\ t/a+k\right)$$
$$(\Re(\theta+\varphi)>\Re(\upsilon)>0,\ |t/a|<1).$$

For $\lambda=\nu=1$, $b=0$, (3.9) in view of (1.10) yields

$$\text{(3.11)}\qquad \sum_{n=0}^{\infty}\frac{(\theta)_n(\varphi)_n}{(\upsilon)_n}\Phi(x,\theta+\varphi-\upsilon+n,a)\frac{t^n}{n!}$$
$$=\sum_{k=0}^{\infty}\frac{x^k}{(a+k)^{\theta+\varphi-\upsilon}}\ {}_2F_1\left({}^{\theta,\varphi;}_{\upsilon\ ;}\ t/a+k\right),$$

which is due to Raina and Srivastava [**6**, p. 302].

Again using (2.1), we can express

$$\text{(3.12)}\qquad \Theta_\mu^\lambda(x,\alpha,a,b)$$
$$=\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha-\lambda m)(-b)^m}{m!}\Phi_\mu^*(x,\alpha-\lambda m,a)$$
$$=\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha-\lambda m)(-b)^m}{m!}\sum_{n=1}^{\infty}\frac{(\mu)_{n-1}x^{n-1}}{(a+n-1)^{\alpha-\lambda m}(n-1)!}.$$

Appealing to the series identity of Srivastva [**5**] (see also [**1**, p. 316, Eqn. 7.176])

$$\sum_{\text{k}\,=\,1}^{\infty}f(k)=\sum_{j=1}^{q}\sum_{k=0}^{\infty}f(qk+j)\quad(q\in\mathbb{N}),$$

then (3.12) gives

$$\Theta_\mu^\lambda(x,\alpha,a,b)$$
$$=\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha-\lambda m)(-b)^m}{m!}\sum_{j=1}^{q}\sum_{n=0}^{\infty}\frac{(\mu)_{nq+j-1}}{\Gamma(nq+j)}\frac{x^{nq+j-1}}{(a+nq+j-1)^{\alpha-\lambda m}}$$
$$=\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha-\lambda m)(-b)^m q^{\lambda m-\alpha}}{m!}\sum_{j=1}^{q}\sum_{n=0}^{\infty}\frac{(\mu)_{nq+j-1}x^{nq+j-1}}{\Gamma(nq+j)\left(a+\frac{nq+j-1}{q}\right)^{\alpha-\lambda m}},$$

which yields

$$\text{(3.13)}\qquad \Theta_\mu^\lambda(x,\alpha,a,b)$$
$$=\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{\infty}\sum_{j=1}^{q}\frac{\Gamma(\alpha-\lambda m)(-b)^m q^{\lambda m-\alpha}}{m!}\Phi_\mu^*\left(x^q,\alpha-\lambda m,\frac{a+j-1}{q}\right)x^{j-1}.$$

For $b = 0$, (3.13) reduces to the identity

$$\Theta_{\mu}^{\lambda}(x, \alpha, a, 0) = q^{-\alpha} \sum_{j=1}^{q} \Phi_{\mu}^{*} \left(x^{q}, \alpha, \frac{a+j-1}{q} \right) x^{j-1}, \tag{3.14}$$

which when $x = \mu = 1$ reduces to the following known identity ([**5**], see also [**1**, p. 317])

$$\Theta_{\mu}^{\lambda}(1, \alpha, a, 0) = \zeta(\alpha, a) = q^{-\alpha} \sum_{j=1}^{q} \zeta \left(\alpha, \frac{a+j-1}{q} \right). \tag{3.15}$$

4. Generalized Lambert Transform

Suppose $f(t)$ $(\forall t \in [0, \infty))$ is a continuous function such that

$$f(t) = \mathrm{O}(\mathrm{e}^{kt}) \qquad (t \to \infty), \tag{4.1}$$

then, the Lambert transform of $f(t)$ is defined by

$$F(s) = \mathrm{LM}\{f(t)\} = \int_{0}^{\infty} \frac{st}{(\mathrm{e}^{st} - 1)} f(t) dt \qquad (\Re(s) > 0). \tag{4.2}$$

In the papers [**3**], [**4**] and [**6**], various generalizations of the transform (4.2) were given.

We consider here a new generalization of (4.2) in the following form:

$$\mathcal{G}^{*} \{f(t)\} = \mathrm{GLM}_{b}\{f(t)\} = \int_{0}^{\infty} \frac{(st)^{k} \, \mathrm{e}^{-b(st)^{\lambda}}}{(\mathrm{e}^{st} - x)^{\mu}} f(t) dt, \tag{4.3}$$

$$(\lambda > 0, \ \mu \geq 1, \ \Re(s) > 0, \ f(t) \in \mathcal{A}, \ \Re(b) > 0; \text{ when } b = 0,$$
$$\text{either } |x| \leq 1 \ (x \neq 1), \ \Re(k + \gamma) > -1, \text{ or } x = 1, \ \Re(k + \gamma) > \mu - 1)$$

where $\mathcal{A}$ denotes the class of functions $f(t)$ which are continuous for $t > 0$, and satisfy the order estimates:

$$f(t) = \begin{cases} \mathrm{O}(t^{\gamma}), \text{ if } t \to 0_{+}, \\ \mathrm{O}(t^{\delta}), \text{ if } t \to \infty. \end{cases} \tag{4.4}$$

Obviously for $b = 0$ and $\mu = 1$, (4.3) becomes

$$\mathrm{GLM}\{f(t)\} = \int_{0}^{\infty} \frac{(st)^{k}}{(\mathrm{e}^{st} - x)^{\mu}} f(t) dt, \tag{4.5}$$

$$(\mu \geq 1, \ \Re(s) > 0, \ f(t) \in \mathcal{A}; \text{ either } |x| \leq 1 \ (x \neq 1),$$
$$\Re(\gamma + k) > -1 \text{ or} x = 1, \ \Re(\gamma + k) > \mu - 1)$$

which was studied in [**6**].

Put $f(t) = t^{\alpha-1}\mathrm{e}^{-ast}$ in (4.3), using (1.7), we have

$$\begin{aligned}(4.6)\qquad \mathcal{G}^*\{t^{\alpha-1}\mathrm{e}^{-ast}\} &= s^k\int_0^\infty t^{\alpha+k-1}\mathrm{e}^{-ast-b(st)^{-\lambda}}(1-x\mathrm{e}^{-st})^{-\mu}dt\\ &= \frac{\Gamma(\alpha+k)}{s^{\alpha+k-1}}\Theta_\mu^\lambda(x,\alpha+k,a+\mu,b)\end{aligned}$$

$(\lambda > 0, \mu \geq 1,\ \Re(a) > 0,\ \Re(s) > 0,\ \Re(b) > 0;$ when $b = 0$, then either $|x| \leq 1$ $(x \neq 1),\ \Re(\alpha + k + \gamma) > -1,$ or $x = 1,\ \Re(\alpha + k + \gamma) > \mu - 1)$.

Inversion formula for the transform (4.3)

On applying the Mellin transform [**7**, p. 46], (4.3) gives

$$\begin{aligned}\phi(m) &= \int_0^\infty s^{-m-1}\left(\int_0^\infty (st)^k\,\mathrm{e}^{-b(st)^{-\lambda}}(\mathrm{e}^{st}-x)^{-\mu}f(t)dt\right)ds\\ &= \int_0^\infty t^k f(t)\left(\int_0^\infty s^{k-m-1}\mathrm{e}^{-\mu st-b(st)^{-\lambda}}(1-x\mathrm{e}^{-st})^{-\mu}ds\right)dt.\end{aligned}$$

In view of (1.7), this gives

$$(4.7)\qquad \phi(m) = \Gamma(k-m)\Theta_\mu^\lambda(x,k-m,\mu,b)\int_0^\infty t^{k+m-1}f(t)dt,$$

provided that $\Re(m) < k$, and the existence and convergence conditions stated with (4.3) hold true.

By the Mellin inversion [**7**, p. 46] theorem, we obtain the following inversion formula for the integral transform (7.3):

$$\begin{aligned}&\frac{1}{2}[f(t+0)+f(t-0)]\\ (4.8)\qquad &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\Gamma(k-m)\Theta_\mu^\lambda(x,k-m,\mu,b)\right\}^{-1}t^{-m}\phi(m)dm,\end{aligned}$$

provided that $y^m f(y) \in L(0,\infty)$, $f(y)$ is of bounded variation in the neighbourhood of the point $y = t$, $\sigma > 1/2$, $\Re(k-m) > 0$, and $\phi(m)$ is given by (4.7).

5. Concluding Remarks

In this concluding section we find it worthwhile to mention briefly here a multivariable extension of the class of functions $\Theta_\mu^\lambda(x,\alpha,a,b)$. This multivariable function $\Theta_{(\mu_i)}^{(\lambda,p_i)}(x_1,\ldots,x_n;\alpha;a,b)$ can be defined by

$$\begin{aligned}&\Theta_{(\mu_i)}^{(\lambda,p_i)}(x_1,...,x_n;\alpha;a,b)\\ (5.1)\quad &= \frac{1}{\Gamma(\alpha)}\sum_{k=0}^\infty\frac{\Gamma(\alpha-\lambda k)\,(-b)^k}{k!}\sum_{m_1,\ldots,m_n=0}^\infty(a+\Omega)^{-(\alpha-\lambda k)}\prod_{i=1}^n\left\{\frac{(\mu_i)_{m_i}x_i^{m_i}}{m_i!}\right\},\end{aligned}$$

$$(\lambda > 0,\ \mu_i \geq 1(i = 1, \ldots, n),\ \Re(a) > 0,\ \Re(b) \geq 0,$$
$$\Re(\alpha) \neq \nu\lambda\ (\nu \in \mathbb{N}),\ \max_{1 \leq i \leq n}(|x_i|) \leq 1,\ \Omega = \sum_{i=1}^{n} p_i m_i).$$

Equivalently, the integral representation of $\Theta_{(\mu_i)}^{(\lambda,p_i)}(x_1, ..., x_n; \alpha; a, b)$ is given by

$$\Theta_{(p_i)}^{(\lambda,\mu_i)}(x_1, \ldots, x_n; \alpha; a, b) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \mathrm{e}^{-at-bt^{-\lambda}} \prod_{i=1}^{n} (1 - x_i \mathrm{e}^{-p_i t})^{\mu_i} dt \tag{5.2}$$

$$(\lambda > 0, \mu_i \geq 1,\ \Re(p_i) > 0\ (i = 1, \ldots, n),\ \Re(a) > 0,\ \Re(b) > 0;$$
$$\text{when } b = 0, \text{ then either, } \max_{1 \leq i \leq n}(|x_i|) < 1\ (x_i \neq 1),\ \Re(\alpha) > 0,$$
$$\text{or } x_i = 1\ (i = 1, \ldots, n),\ \Re(\alpha) > \max_{1 \leq i \leq n}(\mu_i))$$

Corresponding to the above multivariable extension (5.2), we may also define a generalized Lambert transform in the form

$$\mathcal{H}^*\{f(t)\} = \int_0^\infty \frac{(st)^k \mathrm{e}^{-b(st)^\lambda}}{\prod_{i=1}^{n} (\mathrm{e}^{p_i st} - x_i)^{\mu_i}} f(t) dt, \tag{5.3}$$

$$(\lambda > 0,\ \mu_i \geq 1,\ \Re(p_i) > 0\ (i = 1, \ldots, n), \Re(s) > 0,\ f(t) \in \mathcal{A}, \Re(b) > 0;$$
$$\text{when } b = 0, \text{ either } \max_{1 \leq i \leq n}(|x_i|) \leq 1\ (x_i \neq 1)\ (i = 1, \ldots, n), \Re(k + \gamma) > -1,$$
$$\text{or } x_i = 1\ (i = 1, \ldots, n),\ \Re(\alpha) > \max_{1 \leq i \leq n}(\mu_i - 1))$$

These obvious extensions can be manipulated in several ways and various results can be obtained by following the same procedures as mentioned above (see also [**4**]). We do not pursue further, and skip further details in this regard.

References

1. Chaudhry M. Aslam and Zubair Syed M., *On a Class of Incomplete Gamma Functions with Applications*, Chapman & Hall/CRC (Boca Raton/London/NewYork/Washington D.C.) 2000.
2. Erdélyi A. and Magnus W., Oberhettinger F. and Tricomi F. G., *Higher Transcendental Functions*, Vol. 1, McGraw–Hill, NewYork, Toronto and London, 1953.
3. Goyal S. P. and Laddha R. K., *On the generalized Riemann zeta functions and the generalized Lambert transform*, Ganita Sandesh, **11**(1997), 99–108.
4. Raina R. K and Nahar T. S., *A note on certain class of functions related to Hurwitz zeta function and Lambert transform*, Tamkang J. Math. **31** (2000), 49–55.
5. Srivastava H. M., *Some formula for the Bernoulli and Euler polynomials at rational arguments,* Math, Proc. Comb. Phil. Soc.**129** (2000), 77–84.
6. Raina R. K. and Srivastava H. M., *Certain results associated with the generalized Riemann zeta functions,* Rev. Tec. Ing. Univ. Zulia, **18 (3)**(1995), 301–304.

7. Titchmarsh E. C., *Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford, 1948.

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