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A GENERALIZATION OF BAIRE CATEGORY IN A CONTINUOUS SET

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ABSTRACT. The following discusses a generalization of Baire category in a continuous set. The objective is to provide a meaningful classification of subsets of a continuous set as *"large"* or *"small"* sets in linearly ordered continuous sets. In particular, for cardinal number κ, the continuous ordered set ${}^{\kappa}2_*$ a subset of the set of dyadic sequences of length κ is discussed. We establish that this space, and its Cartesian square is not the union of $\mathrm{cf}(\kappa)$ many nowhere dense sets. Further we provide comparative results between Baire category in $\mathbf{R}$ and "generalized Baire category" in ${}^{\kappa}2_*$ as well as some of the significant differences concerning Baire category in $\mathbf{R}$ and κ-category in ${}^{\kappa}2_*$. For example we have shown that a residual set in ${}^{\kappa}2_*$ need not contain a perfect set and that there exist perfect sets of cardinality $|{}^{<\kappa}2_*|$.

1. Introduction

The goal of this paper is to discuss a generalization of Baire category in an ordered continuous set and its Cartesian square. By a continuous set, we are referring to a linearly ordered Dedekind complete set, a set for which every partition into nonempty initial and remainder parts produces a unique element. (We will use the terms continuous set and Dedekind complete set interchangeably.) The underlying concept of Baire category in $\mathbf{R}$, the set of real numbers, is that it classifies subsets of $\mathbf{R}$ as either *"large"* (second category) or *"small"* (first category). Our goal is to generalize the notion of Baire category to other linearly ordered continuous sets. Fundamental to the success of this generalization is the assumption of certain prescribed properties on our continuous set. Two principles one would want to establish in a generalization, would be:

(a) that a nonempty open set is classified as *"large"*, and

(b) that a *"large"* set M is somewhere *"everywhere large"* (there exists some open set for which every open subset intersects M in a set classified as *"large"*).

In addition to developing this generalization of Baire category, we provide examples of sets which satisfy our generalization of a "Baire set", and discuss the

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similarities and differences between this "generalized Baire set" and the set of real numbers **R**.

2. Definitions and Notation

Terminology and notation will be consistent with that which is used in Jech [**5**]. The power of ordinal number κ will be denoted by $|\kappa|$. An ordinal number κ may be referred as a cardinal number if for all ordinals β with $\beta < \kappa$ we have $|\beta| < |\kappa|$. At the same time we may use ω_α to denote the smallest ordinal of power $\aleph_\alpha$. For cardinal κ, let κ^+ denote the successor cardinal to κ. For limit ordinal κ, $\mathrm{cf}(\kappa)$ will denote the smallest ordinal with which κ is cofinal with (i.e. there exists an increasing sequence of ordinals $\{\kappa_\xi\}_{\xi<\mathrm{cf}(\kappa)}$ such that $\kappa = \lim\limits_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi$). A ordinal number κ is said to be a regular ordinal provided $\kappa = \mathrm{cf}(\kappa)$. For ordinals ν, κ we will use ${}^\kappa\nu$ to denote the set of all sequences of length κ formed from terms in ν (i.e. $(x_\xi)_{\xi<\kappa}$ such that x_ξ is an ordinal less than ν). In particular, ${}^\kappa 2$ will denote the set all dyadic sequences of length κ. We will let 2^κ denote the power of the set ${}^\kappa 2$.

Let $(C, <)$ denote a linear ordered Dedekind complete set, throughout this paper we will assume that C has the order topology. Let C^2 denote the Cartesian square of C and give C^2 the product topology. For each $x \in C$, the character of x, denoted by $\mathrm{char}(x)$, is the ordered pair (λ, τ) where λ is the smallest ordinal for which the initial segment determined by x, i.e. the set $\{y \in C : y < x\}$ is cofinal with λ, and τ is the smallest ordinal for which the remaining segment determined by x. Hence $\{y \in C : x < y\}$ is coinitial with τ. When the $\mathrm{char}(x)$ is (λ, τ), then λ may be referred to as the left character of x, and τ as the right character of x.

An ordered set M is said to be an η_α-set if M is cofinal and coinitial with cardinals $\geq \omega_\alpha$, and for every pair A, B of neighboring subsets either A or B is of power $\geq \omega_\alpha$. Historically Hausdorff [**3**] described the η_α-set as a generalization of the set of rational numbers, which has been called the η-set. Of course if a continuous set contains an η_α-set, then for all x, $\mathrm{char}(x) = (\lambda, \tau)$ where at least one of $\lambda, \tau \geq \omega_\alpha$.

2.1. Previous work in generalizing Baire category

In literature there exists several generalizations of Baire category. In [**1**], Folley defined the notion of *ω_α-category* in a continuous set C. His work is based on the assumption of an ordered continuous set containing a dense η_α-subset of cardinality ω_α.

In [**4**], Heckler discussed his notion of "ω_α-category" called "*-category" in a topological space $\mathcal{T} = (T, \mathcal{O})$. The weight of $\mathcal{T}$ (denoted by $\mathrm{wt}(\mathcal{T})$), is the least cardinal λ such that $\mathcal{O}$ admits a basis of cardinality λ. A set is called an *f-set* if it is the union of $\mathrm{wt}(\mathcal{T})$ many nowhere dense sets. Otherwise it is called an *s-set*. However, without making any cardinal assumptions, if one was to use the weight of a space, in classifying sets as *"large"* and *"small"*, it is possible that the space itself could be classified as an f-set ("small").

In [**7**], Milner and Prikry discussed their notion of "ω_α-category" called *μ-category* in a partially ordered set $(P, <)$. The *depth* of a partially ordered set P is the smallest ordinal number γ such that P does not contain a reverse well-ordered subset of length γ. Here they defined a set to be of the *first (second) μ-category* if it is (is not) the union of fewer than μ many nowhere dense sets. A set C has the *μ-Baire property* if every open set of C is of the second μ-category. Milner and Prikry established that if $\nu \geq \omega$, and $\kappa \geq 2$ then ${}^\nu\kappa$, the lexicographically ordered set consisting of sequences of length ν with terms in κ, is a $\mathrm{cf}(\nu)^+$-Baire set.

3. Definition of ω_α category

We define the notion ω_α-category for a topological space T as follows: M is said to be a set of *first ω_α-category* if it is the union of at most ω_α many nowhere dense sets. M is said to be a set of *second ω_α-category* if it is not a set of first ω_α-category. Thus if M is a set of first ω_α-category, and $\tau \geq \alpha$ then M is a set of first ω_τ-category. Consequently if one wants to use "ω_α-category" to classify subsets as "large" or "small", then one would want to choose the most efficient α. Efficiency will be measured along the lines when both (a) and (b) can be achieved. We say T is an *ω_α-Baire space* if every open subset of T is a set of second ω_α-category. In an ω_α-Baire space, a subset M is called an *ω_α-residual set* if its complement $\widetilde{M}$ is a set of first ω_α-category. In an ω_α-Baire space, a subset M is said to be *everywhere of second ω_α-category in open set O*, if for all nonempty open set $U \subset O$, $U \cap M$ is a set of second ω_α-category.

To ensure that when using ω_α-category as a means of classifying subsets of C and C^2 as "large" or "small" sets that properties (a) and (b) are realized, we will assume that continuous set C is *ω_α-good*, where we define *ω_α-good* as follows.

Definition 3.1. *Let ω_α be a regular ordinal and C a linearly ordered Dedekind complete set. Then C has the ω_α-*good *property provided:*

(i) *for each $x \in C$ $\mathrm{char}(x) = (\lambda, \tau)$ implies that $\lambda \geq \omega_\alpha$ and/or $\tau \geq \omega_\alpha$, and*
(ii) *there exists a set $Q \subset C$, such that Q is a set of first ω_α-category, and there exists a system of open sets $\mathcal{U}$ such that $\mathcal{U} = \bigcup_{\xi<\omega_\alpha} \mathcal{U}_\xi$ where*
(ii-1) *for each $\xi < \omega_\alpha$, $O_1, O_2 \in \mathcal{U}_\xi$ if $O_1 \neq O_2$ then $O_1 \cap O_2 = \emptyset$, and*
(ii-2) *for all $x \in C \setminus Q$, the set $\{O \in \mathcal{U} : x \in O\}$ forms a basis for the open sets containing x.*

Observe that if C possesses a dense η_α-subset, then C possesses property (i). The following argument demonstrates that the the set of real numbers $\mathbf{R}$ is ω_0*-good*. Let Q be any subset of $\mathbf{R}$ of the first category. The set of rational intervals, open intervals with rational end points, is a countable collection of open intervals. Thus we can represent the set of rational intervals as $\mathcal{U} = \bigcup_{\xi<\omega} \mathcal{U}_\xi$, where for each ξ, $\mathcal{U}_\xi$ contains only one rational interval. It immediately follows that (ii-1) and (ii-2) hold and so $\mathbf{R}$ is ω_0*-good*.

Now suppose C possesses property (ii). Define

$$\mathcal{U}^{(2)} = \bigcup_{\mu<\omega_\alpha} \left(\bigcup_{\xi<\omega_\alpha} \{O_1 \times O_2 : O_1 \in \mathcal{U}_\mu, O_2 \in \mathcal{U}_\xi\} \right).$$

Note $\mathcal{U}^{(2)}$ can be written as the union of ω_α many collections, i.e.

$$\mathcal{U}^{(2)} = \bigcup_{\tau<\omega_\alpha} \mathcal{U}^{(2)}_\tau$$

where for each τ there exists $\xi, \mu < \omega_\alpha$ such that

$$\mathcal{U}^{(2)}_\tau = \{O_1 \times O_2 : O_1 \in \mathcal{U}_\mu, O_2 \in \mathcal{U}_\xi\}.$$

Now for all $U_1, U_2 \in \mathcal{U}^{(2)}$ if $U_1 \neq U_2$ then $U_1 \cap U_2 = \emptyset$. Further, if N is nowhere dense in C, and A is any subset of C, then both $N \times A$ and $A \times N$ are nowhere dense in C^2. Therefore if B is a set of first ω_α-category in C, and A is any subset of C then both $B \times A$ and $A \times B$ are sets of first ω_α-category in C^2. Thus the set $Q^{(2)} = (Q \times C) \cup (C \times Q)$ is a set of first ω_α-category in C^2. Consequently, for each $\zeta \in C^2 \setminus Q^{(2)}$, the collection $\{U \in \mathcal{U}^{(2)} : \zeta \in U\}$ forms a basis for the open sets which contains ζ. And so we find that C^2 possesses property (ii).

4. ω_α CATEGORY IN ω_α-GOOD SETS

We assume throughout this section that C is ω_α*-good*. The theorems in this section, unless explicitly stated otherwise, are valid for both the linear space C and its Cartesian square C^2.

Theorem 4.1. *Let C be ω_α-good, then every nonempty open set of C and of C^2 is a set of second ω_α-category. Thus both C and C^2 are ω_α-Baire sets.*

The above result implies that an ω_α-residual set is everywhere dense. This result Theorem 4.1 is established for an ordered space in each of the versions [**1, 3, 7**]. In each case the proof is pretty much the same, and analogous to the proof of the Baire category version performed in **R**. What is essential in the linear case is that no point has character (τ, γ) where both τ and γ are less than ω_α.

The next two theorem are essentially results by Folley. Although Folley assumed the existence of an ordered continuous set containing a dense η_α-subset of cardinality ω_α, it is clear that property (i) of ω_α*-good* is the only assumption that is required in Folley's argument to establish these results. We state these results and omit the proofs. Again they are stated for both C and C^2.

Theorem 4.2. [**1**] *Let λ be a cardinal number less than ω_α, then the union of λ many nowhere dense sets is a nowhere dense set.*

Theorem 4.3. [**1**] *Let C be an ω_α-good set then the cardinality of an ω_α-residual set of C is greater than or equal to 2^{ω_α}.*

By an *isomorphism* between two ordered sets, we are referring to a 1-1 map $f : A \longrightarrow B$ between the two ordered sets A and B, such that for all $a_1, a_2 \in A$

with $a_1 < a_2$ we have $f(a_1) < f(a_2)$. The next result is explicitly stated for subsets A of the linear space C. The Baire category in $\mathbf{R}$ version of this result was established in [**6**]. We provide the proof to the generalization, but this proof is merely a generalization of the proof in [**6**].

Theorem 4.4. [**6**] *Let C be an ω_α-good set. If A is a set of first ω_α-category such that A is a dense subset of open set O, then every subset of C isomorphic to $A \cap O$ is a set of first ω_α-category.*

Proof. Let $B \simeq A \cap O$, and f a isomorphism mapping $A \cap O$ onto B. Thus $B = \bigcup_{\xi<\omega_\alpha} f(A_\xi)$, where A_ξ is nowhere dense. So we are left to show $f(A_\xi)$ is nowhere dense. Let U be a nonempty open set. Then there exists an open interval $I \subset U$. If $|I \cap f(A_\xi)| \leq 1$, then we are done, so assume $b_1, b_2 \in I \cap f(A_\xi)$ with $b_1 < b_2$. Thus $a_1 < a_2$, where $f(a_i) = b_i$. As A_ξ is nowhere dense, there exists an open subset of $(a_1, a_2) \subset O$ which does not intersect A_ξ. Since A is dense in O, there exists an interval (a_3, a_4) with end points in A which is a subset of this open set. Thus $(a_3, a_4) \cap A_\xi = \emptyset$, and $(a_3, a_4) \subset (a_1, a_2)$. Let $b_3 = f(a_3)$ and $b_4 = f(a_4)$. Then $(b_3, b_4) \subseteq (b_1, b_2) \subset I$ which is a subset of U. Further $f(A_\xi) \cap (b_3, b_4) = \emptyset$. Therefore B is of first ω_α-category. □

Of course this result holds true for sets of first category in $\mathbf{R}$. The property that A is dense is required as we illustrate. Consider $(0,1) \subset \mathbf{R}$. Let $C \subset (0,1)$ be a Cantor set. Then there exists an isomorphism h such that the interval $(0,1) = h(C_1)$ where $C_1 \subset C$. Obviously C_1 is of 1st category (since it is nowhere dense), but C_1 is isomorphic to $(0,1)$, a set of 2nd category.

The following result is true for ω_α-category in both C and C^2. We will provide a proof only for the linear space C, the proof for C^2 is analogous to what is done here. What is essential in the following proof is property (ii).

Theorem 4.5. *Let C be an ω_α-good set. If A is a set of the second ω_α-category, then there exists an open set O such that A is everywhere of second ω_α-category in O.*

Proof. Suppose the contrary. Since Q is a set of first ω_α-category, it intersects the set A in a set of first ω_α-category, so we may assume without loss of generality that $A \cap Q = \emptyset$.

For each $\xi < \omega_\alpha$, let $K_\xi = \{O \in \mathcal{U}_{C,\xi} : A \cap O \textit{ is a set of first } \omega_\alpha\textit{-category }\}$. Thus, for all $O \in K_\xi$, $A \cap O = \bigcup_{\xi<\omega_\alpha} A_\xi^O$ where A_ξ^O is nowhere dense. Let $\tau < \omega_\alpha$. Set

$$A_\tau = \bigcup_{O\in K_\tau} (A \cap O) = \bigcup_{O \in K_\tau} \left(\bigcup_{\xi<\omega_\alpha} A_\xi^O \right) = \bigcup_{\xi<\omega_\alpha} \left(\bigcup_{O\in K_\tau} A_\xi^O \right).$$

We find that for each $\xi < \omega_\alpha$, the set $\bigcup_{O\in K_\tau} A_\xi^O$ is nowhere dense. For if U is any nonempty open set, and $U \cap \bigcup_{O\in K_\tau} A_\xi^O \neq \emptyset$, then $U \cap A_\xi^{O_1} \neq \emptyset$ for some $O_1 \in K_\tau$. As $A_\xi^{O_1}$ is nowhere dense, there exists open set $V \subset U \cap O_1$ such that $V \cap A_\xi^{O_1} = \emptyset$. Now V is an open subset of $U \cap O_1$ such that $V \cap \bigcup_{O\in K_\tau} A_\xi^O = \emptyset$. Thus A_τ is a set of first ω_α-category.

Let $N = A \setminus \left(\bigcup_{\tau<\omega_\alpha} A_\tau\right)$. Since $\bigcup_{\tau<\omega_\alpha} A_\tau$ is a set of first ω_α-category, we must have that N is of second ω_α-category. If there would exist an open set U such that N is everywhere of second ω_α-category in U, then A would be everywhere of second ω_α-category in U. So we assume that for each open set U, there exists an open subset $U_1 \subset U$ such that $N \cap U_1$ is of first ω_α-category. We will show that this assumption implies that N is nowhere dense.

Let U be a nonempty open set such that $U \cap N \neq \emptyset$ and let $\zeta \in U \cap N$. As $N \subset (C \setminus Q)$, there exists $O_1 \in \mathcal{U}_C$ such that $\zeta \in O_1 \subseteq U$. Since N is not everywhere of second ω_α-category in O_1, there exists a nonempty open set $U_1 \subseteq O_1$ such that $U_1 \cap N$ is a set of first ω_α-category. As our goal is to establish that N is a nowhere dense set, so we will assume the case that $U_1 \cap N$ is nonempty. Let $\zeta_1 \in U_1 \cap N$. Then since $\zeta_1 \notin Q$, there exists $U_2 \in \mathcal{U}_C$ such that $\zeta_1 \in U_2 \subseteq U_1$. Now $U_2 \cap A = (U_2 \cap (A \setminus N)) \cup (U_2 \cap N)$. Thus $U_2 \cap A$ is a set of first ω_α-category. Consequently there exists a $\xi < \omega_\alpha$, such that $U_2 \in K_\xi$. Hence $U_2 \cap A \subseteq (A \setminus N)$ contrary to the fact $\zeta_1 \in U_2 \cap N$. And so we find that $U_1 \cap N = \emptyset$, implying that N is nowhere dense.

As N is nowhere dense, the set $A = (A \setminus N) \cup N$ is a set of first ω_α-category which is a contradiction. Consequently we find that there must exist an open set O such that A is everywhere of second ω_α-category in O. □

5. ω_α-category in a continuous set consisting of dyadic sequences

Definition 5.1. *For ordinal number κ, we will use ${}^\kappa 2$ to represent the lexicographically ordered set consisting of all dyadic sequences of length κ. Let ${}^\kappa 2_* = \{f \in {}^\kappa 2 : f = (f_\xi)_{\xi<\kappa}$ such that there exists a $\tau < \kappa$ with $f_\tau = 0$, and $\tau' < \kappa$ such that $f_{\tau'} = 1$ and for all ξ with $f_\xi = 0$ there exists ξ' with $\xi < \xi' < \kappa$ and $f_{\xi'} = 0\}$.*

If κ is a limit ordinal then ${}^\kappa 2_*$ is a continuous set without the first and the last element. Let ${}^{<\kappa} 2_*$ denote all dyadic sequences of length $< \kappa$ which have a final term that is 1. Then ${}^{<\kappa} 2_*$ is a dense subset of ${}^\kappa 2_*$. Further ${}^\kappa 2_*$ possesses property (i), where $\omega_\alpha = \mathrm{cf}(\kappa)$. Thus the Dedekind completion of ${}^{<\kappa} 2_*$ is ${}^\kappa 2_*$. Of course ${}^\omega 2_*$ is isomorphic to the set of real numbers $\mathbf{R}$, and ${}^{<\omega} 2_*$ is isomorphic to the set of rational numbers.

We will assume that κ is an initial ordinal number, i.e. κ is a cardinal number. For such an assumption, due to Harzheim [**2**], we have the following:

1. $|{}^\kappa 2_*| = 2^\kappa$,
2. $|{}^{<\kappa} 2_*| = \sum_{\xi<\kappa} 2^\xi$,
3. every pair of intervals in ${}^\kappa 2_*$ determined by end points in ${}^{<\kappa} 2_*$ are isomorphic,
4. ${}^{<\kappa} 2_*$ is an $\eta_{\mathrm{cf}(\kappa)}$-set,
5. every element of ${}^{<\kappa} 2_*$, as a member of ${}^\kappa 2_*$, has character $(\mathrm{cf}(\kappa), \mathrm{cf}(\kappa))$, and
6. every gap in ${}^{<\kappa} 2_*$ has character (λ, τ) where both λ and $\tau \le \kappa$, and is occupied by an element of ${}^\kappa 2_*$.

Let $Q_\kappa = \{f \in {}^\kappa 2_* : f$ *has character different than* $(\mathrm{cf}(\kappa), \mathrm{cf}(\kappa))\}$, then Q_κ is a dense subset of ${}^\kappa 2_*$, and of cardinality $\sum_{\xi<\kappa} 2^\xi$. Observe that $|Q_\kappa| = |{}^{<\kappa} 2_*|$.

Hausdorff [**3**] has shown that any two η_α-sets of cardinality $\aleph_\alpha$ are isomorphic. Hence their Dedekind completions are isomorphic, as well. Further an η_α-set is a universal set for the cardinal $\aleph_\alpha$, i.e. for each ordered set B of cardinality $\leq \aleph_\alpha$, an η_α-set contains a subset isomorphic to B. In addition, it has been established that every η_α-set contains a subset isomorphic to ${}^{<\omega_\alpha}2_*$. Thus Folley's assumption of a continuous set containing an η_α-set of cardinality $\aleph_\alpha$ is equivalent to assuming that he was working in the ordered space ${}^{\omega_\alpha}2_*$, for regular α, and assuming that $|{}^{<\omega_\alpha}2_*| = \aleph_\alpha$.

An ordered set M is said to be *κ-free* if it contains neither a subset of type κ nor κ^*. If the ordered set M is κ-free, then the depth ρ of M is such that $\rho < \kappa$. Harzheim in [**2**] has established that ${}^{<\kappa}2_*$ is the union of κ many κ-free sets. Thus ${}^{<\kappa}2_* = \bigcup_{\xi<\kappa} H_\xi$ where each H_ξ is κ-free. Further, by examining Harzheim's construction of this decomposition, using the fact that the union of less than κ many κ-free sets is itself a κ-free set, and by observing that for all $\beta < \kappa$, ${}^{<\beta}2_* \subset {}^{<\kappa}2_*$, we find that by taking appropriate unions we may assume that the sets H_ξ satisfy the following properties:

1. H_ξ is an infinite set without the first and the last element, and there exists a regular ordinal β, with $\xi \leq \beta < \kappa$ such that H_ξ is a β-free set,
2. for all $a, b \in H_\xi$, $|(a,b) \cap H_\xi| = |H_\xi|$,
3. for all $\xi < \mu < \kappa$, H_ξ is a subset of H_μ, and
4. for all ξ, μ with $\xi < \mu < \kappa$ and every partition $\{I, R\}$ of H_ξ into initial and remainder parts, there exists a $c \in H_\mu$ such that $I < c < R$, (one may choose I or R to be empty).

Observe that if $\beta < \kappa$ and M a β-free set then M is nowhere dense in ${}^\kappa 2_*$. As $\kappa = \lim_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi$, for some increasing sequence of ordinals $\{\kappa_\xi\}_{\xi<\mathrm{cf}(\kappa)}$, we see that ${}^{<\kappa}2_* = \bigcup_{\xi<\mathrm{cf}(\kappa)} H_{\kappa_\xi}$. Consequently, ${}^{<\kappa}2_*$ is a set of first $\mathrm{cf}(\kappa)$-category.

Theorem 5.1. *Q_κ is a set of first* $\mathrm{cf}(\kappa)$*-category.*

Proof. Due to Harzheim [**2**], we have ${}^{<\kappa}2_* = \bigcup_{\xi<\mathrm{cf}(\kappa)} H_{\kappa_\xi}$ where H_ξ is a β-free set for some $\beta < \kappa$. Thus each H_ξ is nowhere dense. Let

$$A = \bigcup_{\tau<\mathrm{cf}(\kappa)} \overline{\left(\bigcup_{\xi\leq\tau} H_{\kappa_\xi}\right)} = \bigcup_{\tau<\mathrm{cf}(\kappa)} \overline{H_{\kappa_\tau}}.$$

Noting that if a set is β-free, then its closure is as well β-free, we find that for each $\tau < \mathrm{cf}(\kappa)$, there exists a $\beta < \kappa$ such that $\bigcup_{\tau<\mathrm{cf}(\kappa)} \overline{H_{\kappa_\tau}}$ is β-free. Hence it is nowhere dense. Consequently A is a set of first $\mathrm{cf}(\kappa)$-category.

We claim that $Q_\kappa \subset A$. Let $p \in Q_\kappa$, then p has character different than $(\mathrm{cf}(\kappa), \mathrm{cf}(\kappa))$, let us assume that p has left character ρ, so $\rho < \kappa$. As ${}^{<\kappa}2_*$ is dense in ${}^\kappa 2_*$, there exists an increasing sequence $\{x_\xi\}_{\xi<\rho}$ formed out of ${}^{<\kappa}2_*$ with limit p. Since $\mathrm{cf}(\kappa)$ is a regular ordinal and $\rho \neq \mathrm{cf}(\kappa)$, there exists a $\phi < \mathrm{cf}(\kappa)$ such that $\{x_\xi : \xi < \rho\} \subset \bigcup_{\xi\leq\phi} H_{\kappa_\xi} = H_{\kappa_\phi}$. Thus $p \in A$. □

Lemma 5.1. *If $A \subset {}^\kappa 2_*$, and $|A| < |\,{}^{<\kappa}2_*|$, then A is nowhere dense.*

Proof. In [**8**], Rotman established that if an ordered set A contains no η_α-set then the Dedekind completion of A contains no η_α-set. Therefore if A is a dense subset of ${}^{\kappa}2_*$, for each $\xi < \mathrm{cf}(\kappa)$, A must contain a η_{κ_ξ}-set A_{κ_ξ}. This set A_{κ_ξ} contains a set isomorphic to ${}^{<\kappa_\xi}2_*$. Consequently $|A| \geq |{}^{<\kappa_\xi}2_*|$ for all $\xi < \mathrm{cf}(\kappa)$. Hence $|A| \geq \sum_{\xi<\mathrm{cf}(\kappa)} |{}^{<\kappa_\xi}2_*|$, and so $|A| \geq |{}^{<\kappa}2_*|$, which would lead to a contradiction. □

Thus we see that any set of cardinality less that $|{}^{<\kappa}2_*|$ is nowhere dense. This property is comparable to a property in $\mathbf{R}$ regarding the set of rationals. That is, any set of cardinality less than the cardinality of the rationals is nowhere dense in $\mathbf{R}$. Recall that Theorem 4.4 stated that every set isomorphic to ${}^{<\kappa}2_*$, is a set of first $\mathrm{cf}(\kappa)$-category, the following theorem strengthens this result.

Theorem 5.2. *If $A \subset {}^{\kappa}2_*$ and is isomorphic to a subset of $Q_\kappa \cup {}^{<\kappa}2_*$, then A is a set of first* $\mathrm{cf}(\kappa)$*-category.*

Proof. Clearly $Q_\kappa \cup {}^{<\kappa}2_* \subseteq \bigcup_{\tau<\mathrm{cf}(\kappa)} \overline{H_{\kappa_\tau}}$. Suppose A is isomorphic to D where $D \subseteq Q_\kappa \cup {}^{<\kappa}2_*$. Let f be an isomorphism from D to A. Since $D = \cup_{\tau<\mathrm{cf}(\kappa)}(D \cap \overline{(H_{\kappa_\tau})})$, and as the property that a β-free is preserved under isomorphisms, we find that for each $\tau < \mathrm{cf}(\kappa)$, $f(D \cap \overline{H_{\kappa_\tau}})$ is nowhere dense. Hence A is the union of $\mathrm{cf}(\kappa)$ many nowhere dense sets, and so it is of first $\mathrm{cf}(\kappa)$-category. □

Notice that when one considers $\mathrm{cf}(\kappa)$-category in ${}^{\kappa}2_*$, and compares a set to $Q_\kappa \cup {}^{<\kappa}2_*$, the discussion has been limited to isomorphisms. That is, if one establishes an isomorphism between a set A and a subset of $Q_\kappa \cup {}^{<\kappa}2_*$, then it is of first $\mathrm{cf}(\kappa)$-category. However in $\mathbf{R}$, one only needs to establish a 1-1 correspondence between A and a subset of the rationals to show that it is of first category. (It is true that any countable set will be isomorphic to some subset of the rationals). The question whether one can establish the property that a cardinality equivalence of a set with $Q_\kappa \cup {}^{<\kappa}2_*$ will establish first $\mathrm{cf}(\kappa)$-category is unlikely. For it is unknown whether there exists a regular cardinal number κ such that $|{}^{\kappa}2_*| = |{}^{<\kappa}2_*|$. Of course if $|{}^{\kappa}2_*|$ satisfied this property then there exists sets of second $\mathrm{cf}(\kappa)$-category of this power.

Definition 5.2. *A set M is dense-in-itself provided that it is nonempty and that every element of M is a limit point of M. A set P is a perfect set provided that it is closed and dense-in-itself.*

Theorem 5.3. *For each $\xi < \mathrm{cf}(\kappa)$, $P_\xi = \overline{H_{\kappa_\xi}} \setminus H_{\kappa_\xi}$ is a perfect set.*

Proof. We claim that $P_\xi \neq \emptyset$. Suppose $P_\xi = \emptyset$, then $\overline{H_{\kappa_\xi}} = H_{\kappa_\xi}$. For each $x \in H_{\kappa_\xi}$, let $x^+ = \inf_{\substack{y\in H_{\kappa_\xi}\\ y>x}} y$ and $x^- = \sup_{\substack{y\in H_{\kappa_\xi}\\ y<x}} y$. So $x^- \leq x \leq x^+$ and $x^-, x^+, x \in H_{\kappa_\xi}$ (since we are assuming $\overline{H_{\kappa_\xi}} = H_{\kappa_\xi}$). Clearly $x^- \not< x$ and $x \not< x^+$, because $|(a,b) \cap H_{\kappa_\xi}| = |H_{\kappa_\xi}|$ for all $a, b \in H_{\kappa_\xi}$ with $a < b$. Hence $x^- = x^+ = x$. Consider the following partition of H_{κ_ξ}, let $I = \{y \in H_{\kappa_\xi} : y < x\}$ and $R = H_{\kappa_\xi} \setminus I$. Then $\{I, R\}$ is a partition of H_{κ_ξ} into nonempty initial and remainder parts. Thus for all $\tau < \mathrm{cf}(\kappa)$, with $\xi < \tau$ there exists a $c \in H_{\kappa_\tau}$ such that $I < c < R$. But

$x = \sup_{\substack{y \in H_{\kappa_\xi} \\ y < x}} y$, where $x \in R$ and $\{y \in H_{\kappa_\xi} : y < x\} \subseteq I$. This would imply that $x^- \leq c < x$, which contradicts that $x^- = x$. Hence $P_\xi \neq \emptyset$.

We now show that $\widetilde{P_\xi}$ is an open set. Let $q \in \widetilde{P_\xi}$. As $\widetilde{\overline{H_{\kappa_\xi}}}$ is an open set where $\widetilde{\overline{H_{\kappa_\xi}}} \subset \widetilde{P_\xi}$, we shall assume that $q \notin \widetilde{\overline{H_{\kappa_\xi}}}$. Since $q \in \widetilde{P_\xi}$ where $P_\xi = \overline{H_{\kappa_\xi}} \setminus H_{\kappa_\xi}$, we have $q \notin P_\xi$ and $q \in \overline{H_{\kappa_\xi}}$. Consequently we have $q \in H_{\kappa_\xi}$. Let $q^- = \sup_{\substack{t \in H_{\kappa_\xi} \\ t < q}} t$ and $q^+ = \inf_{\substack{t \in H_{\kappa_\xi} \\ t > q}} t$. By an argument analogous to what occurred above, we find that both q^- and $q^+ \notin H_{\kappa_\xi}$. Thus $q^- < q < q^+$, and so $q \in (q^-, q^+) \subset \widetilde{P_\xi}$. Hence P_ξ is closed.

Now to show P_ξ is dense-in-itself. Let $q \in P_\xi$ and O an open interval containing q, there exists either an increasing or a decreasing sequence $\{z_\mu\}_{\mu<\rho}$ formed out of H_{κ_ξ} for which q is a limit point. Without loss of generality, let us assume it is increasing. Fix a $\mu < \rho$ such that $z_\mu \in O$, then $|(z_\mu, z_{\mu+1}) \cap H_{\kappa_\xi}| = |H_{\kappa_\xi}|$. It follows that $P_\xi \cap (z_\mu, z_{\mu+1}) \neq \emptyset$. As $(z_\mu, z_{\mu+1}) \subset O$, we have established that P_ξ is dense-in-itself. Hence P_ξ is a perfect set. □

Theorem 5.4. *Suppose κ is a regular ordinal. For all $\xi < \kappa$, (i) $P_\xi \subset Q_\kappa$ and (ii) $P_\xi \cap {}^{<\kappa}2_* = \emptyset$*

Proof. If $p \in P_\xi$ then p is a limit point of H_{κ_ξ}, and thus p is the limit of an increasing sequence or a decreasing sequence formed out of H_{κ_ξ}. As the closure of a β-free set is as well a β-free set, we find that either the left or the right character of p is $\leq \beta$ which is less than κ. Thus if κ is regular then $\bigcup_{\xi<\mathrm{cf}(\kappa)} P_\xi = Q_\kappa$, and so $P_\xi \subset Q_\kappa$.

We claim that $P_\xi \cap {}^{<\kappa}2_* = \emptyset$. Let $p \in P_\xi = \overline{H_{\kappa_\xi}} \setminus H_{\kappa_\xi}$, and $p \in {}^{<\kappa}2_*$, then there exists a τ with $\xi < \tau < \mathrm{cf}(\kappa)$ such that $p \in H_{\kappa_\tau}$. As C is a continuous set, and as $p \in \overline{H_{\kappa_\xi}} \setminus H_{\kappa_\xi}$, there exists a monotonic sequence $\{z_\mu\}_{\mu<\rho}$ formed out of H_{κ_ξ} with a limit point of p. Let us assume without loss of generality that this sequence is increasing. Thus $p = \sup_{\mu<\rho} z_\mu$. We define a partition in H_{κ_τ} by setting $I = \{y \in H_{\kappa_\tau} : y < p\}$, and $R = H_{\kappa_\tau} \setminus I$. Now for all ν with $\tau < \nu < \mathrm{cf}(\kappa)$ there exists a $c \in H_{\kappa_\nu}$ such that $I < c < R$. Therefore as $\{y \in H_{\kappa_\xi} : y < p\} \subset I$, we have $\sup_{\mu<\rho} z_\mu < c < p$ which is a contradiction. Consequently $p \notin {}^{<\kappa}2_*$. □

For the remainder of this paper, let $Q = \bigcup_{\xi<\mathrm{cf}(\kappa)} P_\xi$. Clearly $Q \subseteq Q_\kappa$. It also follows that Q is a set of first $\mathrm{cf}(\kappa)$-category. Lastly, $|P_\xi| = |H_{\kappa_\xi}| = 2^{\kappa_\xi}$. Thus $|Q| = |{}^{<\kappa}2_*| = \sum_{\xi<\mathrm{cf}(\kappa)} 2^{\kappa_\xi}$.

A well known result in analysis is that every perfect set in **R** is of cardinality $2^{\aleph_0}$. Reminder, ${}^{<\kappa}2_*$ is a generalization of the set of rational numbers and ${}^{\kappa}2_*$ its Dedekind completion, so it is important to realize what results can be extended to ${}^{\kappa}2_*$. Since $P_\xi \subset Q$, and since $|Q| = |{}^{<\kappa}2_*|$, we find that a perfect set in ${}^{\kappa}2_*$ need not necessarily be of the same power as ${}^{\kappa}2_*$.

Another well known result in analysis is that every residual set contains a perfect set. Folley in [**1**], erroneously established that every ω_α-residual set contains a perfect set. In [**1**], Folley was working with a complete set C which has a dense η_κ-set of cardinality κ, hence under Folley's assumption, C is isomorphic to ${}^{\kappa}2_*$. (Note that Folley's assumption implied $\mathrm{cf}(\kappa) = \kappa$.) We have already shown that Q_κ is a set of first $\mathrm{cf}(\kappa)$-category. Now every perfect set is infinite, thus it must contain a sequence of type ω or ω^*, and so it must contain a point of character (ω, κ) or (κ, ω). Also $\mathrm{cf}(\kappa) = \kappa > \omega$ (if $\kappa = \omega$ we would be discussing the set of real numbers $\mathbf{R}$ and Baire category). Consequently every perfect set intersects Q_κ. It will be shown in the following section that ${}^{\kappa}2_*$ is a $\mathrm{cf}(\kappa)$-Baire set. Thus $\widetilde{Q_\kappa}$ is a $\mathrm{cf}(\kappa)$-residual set, and so we find that there does exist a $\mathrm{cf}(\kappa)$-residual set which does not contain a perfect set.

Theorem 5.5. *For κ satisfying $\mathrm{cf}(\kappa) = \kappa > \omega$:*

(i) *there exists a perfect set $P \subset {}^{\kappa}2_*$ such that P is of cardinality $|{}^{<\kappa}2_*|$;*
(ii) *there exists a set X of first κ-category such that every perfect set in ${}^{\kappa}2_*$ intersects X.*

The proof follows from the above remarks.

6. ${}^{\kappa}2_*$ is a $\mathrm{cf}(\kappa)$-Baire set

Theorem 6.1. *${}^{\kappa}2_*$ is a $\mathrm{cf}(\kappa)$-Baire set.*

Proof. We will show that is a $\mathrm{cf}(\kappa)$-*good* set. By Theorem 4.1 we see that ${}^{\kappa}2_*$ is a $\mathrm{cf}(\kappa)$-Baire set. For all $\xi < \mathrm{cf}(\kappa)$, $P_\xi = \overline{H_{\kappa_\xi}} \setminus H_{\kappa_\xi}$. Let $P_\xi^- = \{f \in P_\xi : f$ *is a left hand limit of* $H_{\kappa_\xi}\}$, and let $P_\xi^+ = P_\xi \setminus P_\xi^-$. The set P_ξ has an initial element denoted by $m_\xi \in P_\xi^+$, and a terminal element denoted by $n_\xi \in P_\xi^-$.

Let p be any element such that $p \in P_\xi^- \setminus \{n_\xi\}$, there exists a $q_p \in P_\xi^+$ such that q_p is the immediate successor of p in P_ξ. This result follows from the fact P_ξ is a β-free set for some $\beta < \kappa$, and that ${}^{<\kappa}2_*$ is everywhere dense in ${}^{\kappa}2_*$.

Now consider any t such that $t \in {}^{<\kappa}2_* \setminus Q$, and any open interval (a, b) containing t. Then there exists a $\xi_0 < \mathrm{cf}(\kappa)$ such that both $(a, t) \cap H_{\kappa_{\xi_0}} \neq \emptyset$ and $(t, b) \cap H_{\kappa_{\xi_0}} \neq \emptyset$. This can be seen by the following argument: Since ${}^{<\kappa}2_*$ is dense in ${}^{\kappa}2_*$, there exist ξ, τ such that $(a, t) \cap H_{\kappa_\xi} \neq \emptyset$, and $(t, b) \cap H_{\kappa_\tau} \neq \emptyset$. Let $\xi_0 = max(\xi, \tau)$.

For each $\xi < \mathrm{cf}(\kappa)$, let

$$\mathcal{U}_\xi = \{O : O = (p, q_p), p \in P_\xi^-, \, p \neq n_\xi, \, q_p \in P_\xi^+\}.$$

Set $\mathcal{U} = \bigcup_{\xi < \mathrm{cf}(\kappa)} \mathcal{U}_\xi$. For all $O_1, O_2 \in \mathcal{U}_\xi$ if $O_1 \neq O_2$ then $O_1 \cap O_2 = \emptyset$.

For all $t \in {}^{\kappa}2_* \setminus Q$, the collection $\{O \in \mathcal{U}_{{}^{\kappa}2_*} : t \in O\}$ forms a basis for the open sets which contain t. This follows from the following argument. Let U be any open set containing t. Then there exists an open interval (a, b) such that $t \in (a, b) \subset U$. By the above argument, there exists a $\xi < \mathrm{cf}(\kappa)$ such that both (a, t), (t, b) intersect $H_{\kappa_{\xi_0}}$. We will assume that both (a, t) and (t, b) contain at least two elements of $H_{\kappa_{\xi_0}}$. It follows then that both (a, t) and (t, b) intersect P_{ξ_0}.

Let $p = \sup_{\substack{y \in P_{\xi_0} \\ y<t}} y$. As P_{ξ_0} is closed, $p \in P_{\xi_0}$, hence $p \in P^-_{\xi_0}$, thus there exists $q_p \in P^+_{\xi_0}$. It follows then that $t < q_p$. Hence $p < t < q_p$, and so there exists an $(p, q_p) \in \mathcal{U}_{\xi_0}$ such that $t \in (p, q_p) \subset O$.

Now Q is a set of first cf(κ)-category. Further ${}^{\kappa}2_*$ contains a dense $\eta_{\mathrm{cf}(\kappa)}$-set. Consequently ${}^{\kappa}2_*$ is cf(κ)-*good*, and so ${}^{\kappa}2_*$ is a cf(κ)-Baire set. □

Let

$$\mathcal{U}^{(2)} = \bigcup_{\mu<\mathrm{cf}(\kappa)} \left(\bigcup_{\xi<\mathrm{cf}(\kappa)} \{O_1 \times O_2 : O_1 \in \mathcal{U}_\mu, O_2 \in \mathcal{U}_\xi\} \right).$$

Then $\mathcal{U}^{(2)}$ can be written as the union of cf(κ) many collections, i.e. $\mathcal{U}^{(2)} = \bigcup_{\tau<\mathrm{cf}(\kappa)} \mathcal{U}^{(2)}_\tau$ where for each τ there exists $\xi, \mu < \mathrm{cf}(\kappa)$ such that $\mathcal{U}^{(2)}_\tau = \{O_1 \times O_2 : O_1 \in \mathcal{U}_{{}^{\kappa}2_*,\mu}, O_2 \in \mathcal{U}_{{}^{\kappa}2_*,\xi}\}$. Now for all $U_1, U_2 \in \mathcal{U}_{({}^{\kappa}2_*)^2}$ if $U_1 \neq U_2$ then $U_1 \cap U_2 = \emptyset$. For all $\zeta \in ({}^{\kappa}2_*)^2 \setminus [({}^{\kappa}2_* \times Q) \cup (Q \times {}^{\kappa}2_*)]$, the collection $\{O \in \mathcal{U}_{({}^{\kappa}2_*)^2} : \zeta \in O\}$ forms a basis for the open sets which contain ζ.
It follows then that $({}^{\kappa}2_*)^2$ is also a cf(κ)-Baire set.

7. Examples

Example 7.1. Denote

$$C = \{(x, y) \in \mathbf{R}^2 : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ (x, y) \neq (0, 0), (1, 1)\}$$

and give C the lexicographic order. Then C is a continuous set without the first and the last element, where every element of C has character (ω, ω). The smallest cardinal for which there exists a basis is $2^{\aleph_0}$ (i.e. the weight of C is $2^{\aleph_0}$). However, $Q = \{(x, 0) : (x, 0) \in C\}$ is nowhere dense in C. Hence it is a set of first ω-category.

Let $\mathcal{B} = \{(a_n, b_n)\}_{n<\omega}$ denote the open intervals of (0,1) with rational end points. We use $\langle \ldots \rangle$ to denote an open interval in C to avoid confusion between a point and an interval. Now let $\mathcal{U}_C = \bigcup_{n<\omega} \{\langle (x, a_n),\ (x, b_n) \rangle : x \in [0, 1]\}$. Then C is ω_0-*good* set. Hence C is an ω_0-Baire space (i.e. a Baire space).

Example 7.2. Let $\kappa > \omega$ be an initial ordinal number, and $f, g \in {}^{\kappa}2_*$ where $[f, g] \subset {}^{\kappa}2_*$. Now let $C = \{(x, y) : x \in [f, g], y \in [0, 1],\ (x, y) \neq (f, 0), (g, 1)\}$, and give C the lexicographic order. Then C is a continuous set without the first and the last element. Every element of C has character (λ, τ) where both $\lambda, \tau \geq \omega$. Then the set $Q = \{(x, 0) : (x, 0) \in C\}$ is nowhere dense in C. Let $\{(a_n, b_n)\}_{n<\omega}$ denote the open intervals in (0,1) with rational end points, and set

$$\mathcal{U}_c = \bigcup_{n<\omega} \{\langle (x, a_n),\ (x, b_n) \rangle : x \in [f, g]\}.$$

Then C is an ω_0-*good* set. Hence C is an ω_0-Baire space.

This example illustrates that using the weight of the space, as in $*$-category, to distinguish between"large" (i.e. an s-set) and "small" (i.e. an f-set) inefficiently classifies sets. This follows from the fact that the weight of C exceeds ω_0.

Observe that both of the above examples have the property that C can be represented as $C = A \times B$, where B is an ω_0-*good* set, and that C is given the lexicographic ordering. The consequence is that C is ω_0-*good* set.

Example 7.3. Let κ be an regular ordinal number $> \omega$ (i.e. $\mathrm{cf}(\kappa) = \kappa$), and $[f, g] \subset {}^{\kappa}2_*$. Thus ${}^{\kappa}2_*$ is cofinal and coinitial with κ. Let

$$C = \{(x, y) : x \in [0, 1],\, y \in [f, g],\, (x, y) \neq (0, f), (1, g)\}$$

and give C the lexicographic order. Then C is a continuous set without the first and the last element. Every element of C has character (λ, τ) where at least one of λ or τ equals κ (because κ is regular). It would be best to use κ-category to classify subsets of C as "large" or "small". For there exists a dense set of first κ-category in C, the set $\{(x, y) \in C : y \in {}^{<\kappa}2_*\}$. Let $Q = \{(x, y) \in C : y \in Q_\kappa\}$, and $\mathcal{V}_\xi$ be the intervals of $\mathcal{U}_\xi$ which are subsets of $[f, g]$. Now let $\mathcal{U} = \bigcup_{\xi<\kappa} \{\langle (x, t_1),\, (x, t_2)\rangle : x \in [0, 1],\, (t_1, t_2) \in \mathcal{V}_\xi\}$. As Q is a set of first κ-category in C, we find that C is a κ-*good* set. Consequently C is a κ-Baire set.

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