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# UNIVALENT HARMONIC MAPPINGS CONVEX IN ONE DIRECTION

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ABSTRACT. In this work some distortion theorems and relations between the coefficients of normalized univalent harmonic mappings from the unit disc onto domains on the direction of imaginary axis are obtained.

## 1. Introduction

J. Clunie and T. Sheil-Small studied the class $S_H$ of all harmonic, complex-valued, sense-preserving, univalent mappings defined on the unit disc $U$, which are normalized by $f(0) = f_z(0) - 1 = 0$. Such functions $f$ can be written in the form $f = h + \bar{g}$ where $h(z) = z + a_2z^2 + \dots$ and $g(z) = b_1z + b_2z^2 + \dots$ are analytic in $U$ and $|g'(z)| < |h'(z)|$ for $z$ in $U$. It follows that $|b_1| < 1$ and hence $f - \overline{b_1 f}$ also belongs to $S_H$. Thus we often restrict ourselves to the subclass $S_H^0$ of $S_H$ consisting of those functions in $S_H$ with $f_{\bar{z}}(0) = 0$. It is proven that $S_H^0$ is a compact and normal family and many other fundamental properties of $S_H^0$ and some of its subclasses are obtained [**2**].

But the general coefficient problems for the functions in the classes $S_H$ and $S_H^0$ are not yet solved. For this reason many mathematicians have tried to solve coefficient problems in the subclasses of $S_H$ [**1**], [**2**], [**3**], [**5**].

This paper is concerned with the subclass $K_H^0(\theta)$ of $S_H^0$ with the images $f(U)$ convex in the direction of $\theta$, $(0 \le \theta < \pi)$. In this subclass we shall obtain distortion theorems and coefficient estimates.

**Lemma 1.1.** [**5**, Theorem 5.7] *First we give two important results that will be used during our work,* [**4**], [**5**]. *A function $f = h + \bar{g}$ in $S_H$ maps $U$ onto a convex domain if and only if the analytic function $h - e^{2i\theta}g$ is univalent and maps $U$ onto a domain convex in direction for all $\theta$, $0 \le \theta < \pi$.*

**Lemma 1.2.** [**4**, Theorem 1] *Let $\varphi(z)$ be a non-constant function regular in $U$. The function $\varphi(z)$ maps univalently onto a domain convex in direction of imaginary axis if and only if there are numbers $\mu$ and $\nu$, $0 \le \mu < 2\pi$ and $0 \le \nu \le \pi$,*

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*such that*

(1) $$\operatorname{Re}\{-ie^{i\mu}(1-2\,z\,e^{-i\mu}\cos\nu+z^2e^{-2i\mu})\varphi'(z)\}\geq 0,\ \ z\in U.$$

## 2. Univalent Harmonic Mappings Convex in the Direction of the Imaginary Axis

Instead of studying a class of functions in the direction of any $\theta$, $0\leq\theta<\pi$, it is enough to study the class of harmonic univalent functions convex in the direction of the imaginary axis. That is because, if the harmonic univalent function $f=h+\bar{g}$ is convex in the direction of some $\theta$, there is a real $\alpha$ so that $F(z)=e^{i\alpha}f(e^{-i\alpha}z)$ is convex in the direction of the imaginary axis.

Let $K_H(i)$ and $K_H^0(i)$ denote the subclasses of $S_H$ and $S_H^0$, respectively, which are convex on the direction of the imaginary axis.

**Remark 2.1.** *A harmonic function $f=h+\bar{g}$ maps $U$ univalently onto a domain convex in the direction of the imaginary axis if and only if the analytic function $h+g$ is univalent and maps $U$ onto a domain convex in the direction of the imaginary axis.*

We obtain the following result from Lemma 1 and Remark 1:

**Remark 2.2.** *A harmonic function $f=h+\bar{g}$ in $K_H(i)$ if and only if there numbers $\mu$, $(0\leq\mu<2\pi)$ and $\nu$, $(0\leq\nu\leq\pi)$, such that*

(2) $$\operatorname{Re}\{-i\,e^{i\mu}(1-2\,z\,e^{-i\mu}\cos\nu+z^2e^{-2i\mu})[h'(z)+g'(z)]\}\geq 0,\quad z\in U.$$

For the functions

$$h(z)=z+\sum_{n=2}^{\infty}a_nz^n\quad\text{ and }\quad g(z)=\sum_{n=2}^{\infty}b_nz^n$$

analytic in $U$, let $f=h+\bar{g}$ in $K_H^0(i)$. If we take

(3) $$q(z)=-i\,e^{i\mu}(1-2\,z\,e^{-i\mu}\cos\nu+z^2e^{-2i\mu})[h'(z)+g'(z)]$$

and

(4) $$p(z)=\frac{q(z)+i\cos\mu}{\sin\mu};$$

then $\operatorname{Re}p(z)>0$ and $p(0)=1$. Therefore the function $p(z)$ belongs the class $P$ of the analytic functions with positive real part. Furthermore, since $\sin\mu\geq 0$ for $\mu\in[0,\pi]$, $\operatorname{Re}q(z)\geq 0$. From (3) and (4)

(5) $$\phi'(z)=h'(z)+g'(z)=\frac{\cos\mu+i\sin\nu\,p(z)}{1-2\,z\,e^{-i\mu}\cos\nu+z^2e^{-2i\mu}}$$

can be obtained.

**Theorem 2.1.** *A harmonic function $f$ in $K_H^0(i)$ if and only if there is analytic function $p_1 \in P$ and two constant $\mu, \nu \in [0, \pi]$ such that*

$$f(z) = \operatorname{Re} \phi(z) + i \operatorname{Im} \int_0^z \phi'(\varsigma)\, p_1(\varsigma)\, d\varsigma. \tag{6}$$

*Proof.* Let $f = h + \bar{g}$ is in $K_H^0(i)$ then we can write

$$f(z) = \operatorname{Re}(h + g) + i \operatorname{Im}(h - g) \tag{7}$$

and

$$h' - g' = (h' + g')\frac{h' - g'}{h' + g'} = \phi' \frac{h' - g'}{h' + g'}. \tag{8}$$

We set $w = -g'/h'$ then the function $w$ is analytic in $U$, $w(0) = 0$ and $|w(z)| < 1$. If we take

$$p_1(z) = \frac{h'(z) - g'(z)}{h'(z) + g'(z)} = \frac{1 + w(z)}{1 - w(z)}$$

then $p_1$ is analytic in $U$ and $p_1(0) = 1$, $\operatorname{Re} p_1 > 0$ and $p_1 \in P$. If we consider (5), (7) and (8) altogether then we obtain (6). □

**Theorem 2.2.** *If $f = h + \bar{g}$ in $K_H^0(i)$ and*

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n\,, \quad z \in U,$$

*then*

$$|a_n| \le \frac{(n+1)(2n+1)}{6}, \quad |b_n| \le \frac{(n-1)(2n-1)}{6} \tag{9}$$

*and*

$$||a_n| - |b_n|| \le n.$$

*Equality occurs for the harmonic Koebe function $k_0 = h + \bar{g}$, where*

$$h(z) = \frac{6z - 3z^2 + z^3}{6(1-z)^3} \quad \text{and} \quad g(z) = \frac{72z^2 + z^3}{6(1-z)^3}.$$

*Proof.* From $h' + g' = \phi'$ and $h' - g' = \phi'\, p_1$ we get

$$\begin{aligned} h'(z) &= e^{-i\mu}[\cos\mu + i\, p(z) \sin\mu] \frac{1}{1 - 2\, z\, e^{-i\mu} \cos\nu + z^2 e^{-2i\mu}} \frac{1 + p_1(z)}{2} \\ &\ll \frac{1+z}{1-z} \frac{1}{(1-z)^2} \frac{1}{1-z}. \end{aligned}$$

Here $\ll$ means that the moduli of the function on the left are bounded by the corresponding coefficients of the function on the right. Thus,

$$h'(z) \ll \sum_{n=0}^{\infty} \frac{(n+1)(n+2)(2n+3)}{6}\, z^n$$

i.e.

$$|na_n| \le \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad |a_n| \le \frac{(n+1)(2n+1)}{6}.$$

Similarly,

$$g'(z) = \phi'(z)\frac{1+p_1(z)}{2} \ll \frac{1+z}{1-z}\frac{1}{(1-z)^2}\frac{-z}{1-z} = \sum_{n=0}^{\infty} -\frac{n(n+1)(2n+1)}{6} z^n$$

i.e.

$$|nb_n| \le \frac{(n-1)\,n\,(2n-1)}{6} \quad \text{and} \quad |b_n| \le \frac{(n-1)(2n-1)}{6}.$$

From (9), we get

$$||a_n| - |b_n|| \le |a_n + b_n| \le n.$$

□

**Theorem 2.3.** *If $f = h + \bar{g}$ in $K_H^0(i)$, then for $|z| = r < 1$, and $b = |\cos\nu|$, $0 \le \nu \le \pi$,*

(10)

$$\frac{1-r}{(1+r)^2(1+2br+r^2)} \le |h'(z)| \le \begin{cases} \dfrac{1+r}{(1-r)^2(1-2br+r^2)} & ; r < \dfrac{1-\sin\nu}{b} \\ \dfrac{1}{(1-r)^3\sin\nu} & ; \dfrac{1-\sin\nu}{b} \le r < 1 \end{cases}$$

*and*

(11)

$$\frac{r(1-r)}{(1+r)^2(1+2br+r^2)} \le |g'(z)| \le \begin{cases} \dfrac{r(1+r)}{(1-r)^2(1-2br+r^2)} & ; r < \dfrac{1-\sin\nu}{b} \\ \dfrac{1}{(1-r)^3\sin\nu} & ; \dfrac{1-\sin\nu}{b} \le r < 1 \end{cases}$$

*Both inequalities are sharp.*

*Proof.* Since $f$ is sense-preserving, the Jacobian of $f$ $J_{f(z)} = |h'(z)|^2 - |g'(z)|^2 > 0$ or $|g'(z)| < |h'(z)|$, $z \in U$. If we define $a(z) = g'(z)/h'(z)$, $a(z)$ satisfies the conditions of Schwarz Lemma. Then by (5)

$$zh'(z)[1 + a(z)] = [\cos\mu + ip(z)\sin\mu]k_\nu(z) \tag{12}$$

where

$$k_\nu(z) = \frac{z}{1 - 2z\cos\nu + z^2}. \tag{13}$$

Since $p \in P$, by [**4**, Lemma 2]

$$\frac{1-r}{1+r} \le |\cos\mu + i\,p(z)\sin\mu| \le \frac{1+r}{1-r}$$

for $|z| = r < 1$, and equality occurs for $\mu = \pi/2$ and for the function $p(z) = (1+z)/(1-z)$. Furthermore by [**4**, Lemma 2]

$$\frac{r}{1+2br+r^2} \le |k_\nu'(z)| \le \begin{cases} \dfrac{r}{1-2br+r^2} & ; r < \dfrac{1-\sin\nu}{b} \\ \dfrac{1}{(1-r^2)\sin\nu} & ; \dfrac{1-\sin\nu}{b} \le r < 1 \end{cases} \tag{14}$$

(12), (13) and (14) together gives (10). The inequality $|g'(z)| \le |z||h'(z)|$ together with (10) gives (11). □

Theorem 1, for $\nu = 0$ or $\nu = \pi$ the top one of the inequalities (10) and (11), and for $\nu = \pi/2$, the bottom one is valid for every $r,\ 0 \le r < 1$.

The following is a result of Theorem 1:

**Remark 2.3.** *If $f = h + \bar{g}$ in $K_H^0(i)$ and $\nu = 0, \pi$, then for $|z| = r < 1$*

$$\frac{1-r}{(1+r)^3} \le |h'(z)| \le \frac{1+r}{(1-r)^4}$$

*and*

$$\frac{r(1-r)}{(1+r)^3} \le |g'(z)| \le \frac{r(1+r)}{(1-r)^4}.$$

*If $\nu = \pi/2$, then*

$$\frac{1-r}{(1+r)(1+r^2)} \le |h'(z)| \le \frac{1}{(1-r)^3}$$

*and*

$$\frac{r(1-r)}{(1+r)(1+r^2)} \le |g'(z)| \le \frac{1}{(1-r)^3}.$$

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