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A DARBOUX PROPERTY OF $\mathcal{I}_1$-APPROXIMATE PARTIAL DERIVATIVES

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ABSTRACT. Some Darboux property for functions of two variables is studied. In particular, it is shown that $\mathcal{I}_2$-approximately continuous functions and $\mathcal{I}_1$-approximate partial derivatives of separately $\mathcal{I}_1$-approximately continuous functions are Darboux.

Let $\Re(\Re^2)$ denote the real line (the plane) and $\mathcal{N}$-the set of all positive integers. All topological notations, except for the case where a topology $\mathcal{T}$ is specifically mentioned, are given with respect to the natural topology on $\Re$ or $\Re^2$.

Let $\mathcal{S}_1(\mathcal{S}_2)$ denote the σ-field of sets of $\Re(\Re^2)$ having the Baire property. $\mathcal{I}_1(\mathcal{I}_2)$ will denote the σ-ideal of sets of $\Re(\Re^2)$ of the first category.

Recall that 0 is an $\mathcal{I}_1$-density point of a set $A \in \mathcal{S}_1$ if and only if, for each increasing sequence of positive integers $\{n_m\}_{m\in\mathcal{N}}$, there is a subsequence $\{n_{m_p}\}_{p\in\mathcal{N}}$ such that

$$\{x : \chi_{n_{m_p}\cdot A\cap[-1,1]}(x) \not\to 1\} \in \mathcal{I}_1$$

where $n \cdot A = \{nx : x \in A\}$ (see [**8**] and, for two variables, [**2**]).

A point $x_0 \in \Re$ is said to be an $\mathcal{I}_1$-density point of $a \in \mathcal{S}_1$ if and only if 0 is an $\mathcal{I}_1$-density point of the set $\{x - x_0 : x \in A\}$.

A point $x_0 \in \Re$ is said to be an $\mathcal{I}_1$-dispersion point of $A \in \mathcal{S}_1$ if and only if x_0 is an $\mathcal{I}_1$-density point of $\Re \setminus A$.

For each $A \in \mathcal{S}_1$, we denote

$$\Phi_1(A) = \{x \in \Re : x \text{ is an } \mathcal{I}_1\text{-density point of } A\},$$
$$\Psi_1(A) = \{x \in \Re : x \text{ is an } \mathcal{I}_1\text{-dispersion point of } A\}.$$

In [**8**] it was proved that $\mathcal{T}_{\mathcal{I}_1} = \{A \in \mathcal{S}_1 \colon A \subset \Phi_1(A)\}$ is a topology on the real line. Every function which is continuous with respect to the $\mathcal{T}_{\mathcal{I}_1}$-topology is called an $\mathcal{I}_1$-approximately continuous function.

We say that x_0 is a deep $\mathcal{I}_1$-density point of a set A if and only if there exists a closed set $F \subset A \cup \{x_0\}$ such that $x_0 \in \Phi_1(F)$. In [**9**] it was proved that if f is an $\mathcal{I}_1$-approximately continuous function then, for every open set U, if $x_0 \in f^{-1}(U)$, then x_0 is a deep $\mathcal{I}_1$-density point of the set $f^{-1}(U)$.

The following result will be useful (see [**5**]).

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Lemma 1. *Let G be an open subset of the real line; then 0 is an $\mathcal{I}_1$-dispersion point of G if and only if, for each $n \in \mathcal{N}$, there exist $k \in \mathcal{N}$ and a real $\delta > 0$ such that, for any $h \in (0, \delta)$ and $i \in \{1, \dots, n\}$, there exist two numbers $j, j' \in \{1, \dots, k\}$ such that*

$$G \cap \Big(\Big(\frac{i-1}{n} + \frac{j-1}{nk}\Big) \cdot h, \Big(\frac{i-1}{n} + \frac{j}{nk}\Big) \cdot h\Big) = \emptyset$$

and

$$G \cap \Big(\Big(-\frac{i-1}{n} + \frac{j'}{nk}\Big) \cdot h, -\Big(\frac{i-1}{n} + \frac{j'-1}{nk}\Big) \cdot h\Big) = \emptyset.$$

In [**2**], the definition of an $\mathcal{I}_2$-density point of a set $A \in \mathcal{S}_2$ was introduced. The authors obtained analogous results as in [**8**], on the plane. They defined the topology on the plane in the following way: $\mathcal{T}_{\mathcal{I}_2} = \{A \in \mathcal{S}_2 : A \subset \Phi_2(A)\}$ where

$$\Phi_2(A) = \{(x, y) \in \Re^2 : (x, y) \text{ is an } \mathcal{I}_2\text{-density point of } A\}.$$

We shall denote by $\Phi_2^{++}(A)$, for each $A \in \mathcal{S}_2$, the set of $\mathcal{I}_2$-density points of the set A with respect to the first quarter on the plane. For the remaining quarters, we use the symbols $\Phi_2^{-+}(A)$, $\Phi_2^{+-}(A)$ and $\Phi_2^{--}(A)$. By $\Psi_2^{++}(A)$, $\Psi_2^{-+}(A)$, $\Psi_2^{+-}(A)$ and $\Psi_2^{--}(A)$ we denote sets of $\mathcal{I}_2$-dispersion points of the set A with respect to each quarter on the plane, respectively [**2**]. Functions which are continuous with respect to the $\mathcal{T}_{\mathcal{I}_2}$-topology will be called $\mathcal{I}_2$-approximately continuous.

In a similar way as Lemma 1 we may prove the following

Lemma 2. *Let G be an open set on the plane; then $(0, 0) \in \Psi_2^{++}(G)$ if and only if, for each $n \in \mathcal{N}$, there exist $k \in \mathcal{N}$ and a real number $\delta > 0$ such that, for any $h \in (0, \delta)$ and $i, i' \in \{1, \dots, n\}$, there exist two numbers $j, j' \in \{1, \dots, k\}$ such that*

$$\begin{aligned} G \cap \Big(\Big(\frac{i-1}{n} + \frac{j-1}{nk}\Big) \cdot h, \Big(\frac{i-1}{n} + \frac{j}{nk}\Big) \cdot h\Big) \\ \times \Big(\Big(\frac{i'-1}{n} + \frac{j'-1}{nk}\Big) \cdot h, \Big(\frac{i'-1}{n} + \frac{j'}{nk}\Big) \cdot h\Big) = \emptyset. \end{aligned}$$

The definition of a separately $\mathcal{I}_1$-approximately continuous function was introduced in the obvious manner in [**10**] and was considered in [**10**] and [**1**].

In [**6**], the definition of the $\mathcal{I}_1$-approximative derivative of a function f of one variable was introduced. Many properties of $\mathcal{I}_1$-approximate derivatives and $\mathcal{I}_1$-differentiable functions were considered there.

Definition 3 ([**6**]). *Let $f \colon \Re \to \Re$ have the Baire property in a neighbourhood of x_0. The upper $\mathcal{I}_1$-approximate limit of f at x_0 ($\mathcal{I}_1\text{-}\limsup_{x \to x_0} f(x)$) is the greatest lower bound of the set $\{y : \{x : f(x) > y\}$ has x_0 as an $\mathcal{I}_1$-dispersion point$\}$. The lower $\mathcal{I}_1$-approximate limit, the right-hand and left-hand upper and lower $\mathcal{I}_1$-approximate limits are defined similarly. If $\mathcal{I}_1\text{-}\limsup_{x \to x_0} f(x) = \mathcal{I}_1\text{-}\liminf_{x \to x_0} f(x)$, their common value will be called the $\mathcal{I}_1$-approximate limit of f at x_0 and denoted by $\mathcal{I}_1\text{-}\limsup_{x \to x_0} f(x)$.*

Let $f \colon \Re^2 \to \Re$ and $(x_0, y_0) \in \Re^2$. Put

$$U_{(x_0, y_0)}(x) = \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} \quad \text{for } x \in \Re, x \neq x_0.$$

Definition 4 ([**6**]). *Let $f\colon \Re^2 \to \Re$ be any function defined in some neighbourhood of $(x_0, y_0) \in \Re^2$ and having there the Baire property in the direction of the ox axis. We define the upper right $\mathcal{I}_1$-approximate partial derivative of f at (x_0, y_0) in the direction of ox as the corresponding extreme limit of $U_{(x_0,y_0)}(x)$ as x tends to x_0 from the right. The other extreme $\mathcal{I}_1$-approximate partial derivatives in the direction of ox are defined similarly. If all these derivatives are equal and finite, we call their common value the $\mathcal{I}_1$-approximate partial derivative of f at (x_0, y_0) and denote it by $f_{\mathcal{I}_1,x}(x_0, y_0)$.*

In a similar way we can define the partial $\mathcal{I}_1$-approximate derivate in the direction of the oy axis.

The partial $\mathcal{I}_1$-approximate derivatives are considered in [**3**] and [**4**].

Definition 5. *Let $f\colon \Re^2 \to \Re$. We shall say that f has the Darboux property if and only if, for each open interval $J \subset \Re^2$, $f(J)$ is a connected set.*

Definition 6 ([**7**]). *A set $D \subset \Re^2$ is Darboux if and only if*

- *for each $x \in D$, there exists a closed interval I such that $x \in I$ and $\operatorname{int}(I) \subset D$,*
- *for two points $x, y \in D$, there are $k \in \mathcal{N}$ and $Q_1, Q_2, \ldots, Q_k$ such that, for each $i \in \{1, \ldots, k\}$, $\operatorname{int}(\operatorname{cl}(Q_i)) \subset Q_i \subset D$, $\operatorname{cl}(Q_i)$ is a closed interval, $x \in Q_1$, $y \in Q_k$ and $Q_i \cap Q_{i+1} \neq \emptyset$ for $i = 1, \ldots, k-1$.*

Definition 7. *Let $f\colon \Re^2 \to \Re$. We shall say that f is Darboux if and only if, for every Darboux set Q, $f(Q)$ is a connected set.*

Definition 8. *Let $f\colon \Re^2 \to \Re$. We shall say that f is a connected function if and only if, for every connected set A, $f(A)$ is connected.*

By [**2**], we have the following theorem.

Theorem 9. *Let $f\colon \Re^2 \to \Re$ be an $\mathcal{I}_2$-approximately continuous function. Then f has the Darboux property.*

Corollary 10. *Every open interval is a connected set with respect to the $\mathcal{T}_{\mathcal{I}_2}$-topology.*

Proposition 11. *Every Darboux set is connected with respect to the $\mathcal{T}_{\mathcal{I}_2}$-topology.*

Proof. It is enough to prove that each set $Q \subset \Re^2$, such that $\operatorname{cl}(Q)$ is a closed interval and $\operatorname{int}(\operatorname{cl}(Q)) \subset Q$, is connected with respect to $\mathcal{T}_{\mathcal{I}_2}$. We put $A = \operatorname{int}(\operatorname{cl}(Q))$ and assume that $Q \setminus A \neq \emptyset$. We observe that if $(x, y) \in Q \setminus A$, then $(x, y) \in \Phi_2^{++}(A)$ or $(x, y) \in \Phi_2^{-+}(A)$ or $(x, y) \in \Phi_2^{+-}(A)$. Therefore, for each $U \in \mathcal{T}_{\mathcal{I}_2}$ such that $(x, y) \in U$, $U \cap A \neq \emptyset$.

We suppose that there exist two sets U_1, $U_2 \in \mathcal{T}_{\mathcal{I}_2}$ such that $Q \cap U_1 \neq \emptyset$, $Q \cap U_2 \neq \emptyset$, $Q \cap U_1 \cap U_2 = \emptyset$ and $Q \cap (U_1 \cup U_2) = \emptyset$. Since A is $\mathcal{T}_{\mathcal{I}_2}$-connected, therefore $A \subset U_1$ or $A \subset U_2$. We assume that $A \subset U_1$. Thus $\emptyset \neq U_2 \cap A \subset U_2 \cap U_1 \cap Q$, a contradiction. Hence every Darboux set is $\mathcal{T}_{\mathcal{I}_2}$-connected. $\square$

Theorem 12. *Let $f\colon \Re^2 \to \Re^2$ be an $\mathcal{I}_2$-approximately continuous function. Then f is a Darboux function.*

Proposition 13. *There exists a set $A \subset \Re^2$ such that A is connected with respect to the natural topology and A is not connected with respect to the $\mathcal{T}_{\mathcal{I}_2}$-topology.*

Proof. It is enough to show that there exist two disjoint nonempty sets A_1 and A_2 such that $A_1 \in \mathcal{T}_{\mathcal{I}_2}$, $A_2 \in \mathcal{T}_{\mathcal{I}_2}$, and $A_1 \cup A_2$ is a connected set with respect to the natural topology.

Let

$$A_1 = \left\{(x, y) \in \Re^2 : -\frac{1}{2}x^2 < y < \frac{1}{2}x^2\right\}$$

and

$$A_2 = (\Re^2 \setminus \{(x, y) \in \Re^2 : -x^2 \le y \le x^2\}) \cup \{(0, 0)\}.$$

Then $A_1 \in \mathcal{T}_{\mathcal{I}_2}$ and $A_1 \cup A_2$ is a connected set with respect to the natural topology. We shall show that $A_2 \in \mathcal{T}_{\mathcal{I}_2}$. Since $A_2 \setminus \{(0, 0)\}$ is an open set we only prove that $(0, 0) \in \Phi_2(A_2)$. It is obvious that $(0, 0) \in \Phi_2^{-+}(A_2)$ and $(0, 0) \in \Phi_2^{--}(A_2)$.

Let $n \in \mathcal{N}$. We put $k = 2$ and $\delta = \frac{1}{2n}$. Let $0 < h < \delta$, $(i_1, i_2) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$ and

$$(x_0, y_0) \in \left(\frac{i_1 - 1}{n}h, \frac{2i_1 - 1}{2n}h\right) \times \left(\frac{2i_2 - 1}{2n}h, \frac{i_2}{n}h\right).$$

Then $y_0 > \frac{2i_2-1}{2n}h > (2i_2 - 1)h^2 \ge h^2$ and $0 < x_0 < h$. Thus $y_0 > x_0^2$ and $(x_0, y_0) \in A_2$. Therefore there exists $(j_1, j_2) = (1, 2) \in \{1, 2\} \times \{1, 2\}$ such that

$$\Big(\Big(\frac{i_1 - 1}{n} + \frac{j_1 - 1}{nk}\Big) \cdot h, \Big(\frac{i_1 - 1}{n} + \frac{j_1}{nk}\Big) \cdot h\Big) \\ \times \Big(\Big(\frac{i_2 - 1}{n} + \frac{j_2 - 1}{nk}\Big) \cdot h, \Big(\frac{i_2 - 1}{n} + \frac{j_2}{nk}\Big) \cdot h\Big) \subset A_2.$$

Hence, by Lemma 2, $(0, 0) \in \Phi^{++}(A_2)$. In a similar way we can prove that $(0, 0) \in \Phi^{+-}(A_2)$ and the proof of the proposition is completed. □

Proposition 14. *There exists a function $f\colon \Re^2 \to \Re$ such that f is $\mathcal{I}_2$-approximately continuous and is not a connected function.*

Proof. Let A_1, A_2 be defined in the same way as in Proposition 13. Let $f\colon \Re^2 \to \Re$ be a continuous function at each $(x, y) \in \Re^2 \setminus \{(0, 0)\}$ such that $f(A_1) = \{1\}$ and $f(A_2) = \{0\}$. Since $(0, 0) \in \Phi_2(A_2)$ we have that f is $\mathcal{I}_2$-approximately continuous on $\Re^2$. By $f(A_1 \cup A_2) = \{0, 1\}$, we know that f is not connected. □

Lemma 15. *Let $[a, b] \subset \Re$ and let A_1, A_2 be two nonempty sets having the Baire property such that $[a, b] = A_1 \cup A_2$. Then $A_1 \cap ((a, b) \setminus \Psi_1(A_2)) \ne \emptyset$ or $A_2 \cap ((a, b) \setminus \Psi_1(A_1)) \ne \emptyset$.*

Proof. First we assume that $A_1 \cap A_2 \notin \mathcal{I}_1$. Then, by [**8**], $(a, b) \cap A_1 \cap A_2 \cap \Phi_1(A_1 \cap A_2) \ne \emptyset$ and we choose $x_0 \in (a, b) \cap A_1 \cap A_2 \cap \Phi_1(A_1 \cap A_2)$. Then $x_0 \in A_1 \cap ((a, b) \setminus \Psi_1(A_2))$.

Now, let $A_1 \cap A_2 \in \mathcal{I}_1$. We put $B_1 = (A_1 \setminus (A_1 \cap A_2)) \cap (a,b)$ and $B_2 = A_2 \cap (a,b)$. Then, by [**8**], $\Psi_1(B_1) = \Psi_2(A_1)$ and $\Psi_1(B_2) = \Psi_1(A_2)$. We suppose that $B_1 \subset \Psi_1(B_2)$ and $B_2 \subset \Psi_1(B_1)$. Then $B_1 \subset \Phi(B_1)$ and $B_2 \subset \Phi(B_2)$. Hence B_1, B_2 are open sets with respect to the $\mathcal{T}_{\mathcal{I}_1}$-topology, $B_1 \cup B_2 = (a,b)$ and $B_1 \cap B_2 = \emptyset$. This is impossible since (a,b) is a connected set the with respect to the $\mathcal{T}_{\mathcal{I}_1}$-topology [**8**]. Thus $B_1 \cap ((a,b) \setminus \Psi_1(B_2)) \neq \emptyset$ or $B_2 \cap ((a,b) \setminus \Psi_1(B_1)) \neq \emptyset$, and $A_1 \cap ((a,b) \setminus \Psi(A_2) \neq \emptyset$ or $A_2 \cap ((a,b) \setminus \Psi(A_1)) \neq \emptyset$. □

Lemma 16. *Let $f, g\colon \Re \to \Re$ be $\mathcal{I}_1$-approximately continuous functions. If 0 is not an $\mathcal{I}_1$-dispersion point of a set $A \in \mathcal{S}_1$ then there exists a sequence $\{y_n\}_{n\in\mathcal{N}} \subset A$ such that $\lim_{n\to\infty} y_n = 0$, $\lim_{n\to\infty} f(y_n) = f(0)$ and $\lim_{n\to\infty} g(y_n) = g(0)$.*

Proof. We may assume that 0 is not a right-side $\mathcal{I}_1$-dispersion point of the set $A \in \mathcal{S}_1$. By Lemma 1, there exists $n \in \mathcal{N}$ such that, for any $k \in \mathcal{N}$ and a real $\delta > 0$, there exist $h = h(k,\delta) \in (0,\delta)$ and $i = i(h) \in \{1,\dots,n\}$ such that, for each $j \in \{1,\dots,k\}$,

$$\Big(\frac{(i-1)k+j-1}{nk}h, \frac{(i-1)k+j}{nk}h\Big) \cap A \notin \mathcal{I}.$$

Let $p \in \mathcal{N}$. We put $C_p = \big\{y : |f(y)-f(0)| < \frac{1}{p}\big\}$ and $B_p = \big\{y : |g(y)-g(0)| < \frac{1}{p}\big\}$. Since f and g are $\mathcal{I}_1$-approximately continuous, 0 is a deep $\mathcal{I}_1$-density point of $C_p \cap B_p$. Therefore, by Lemma 1, there exist $k_1 \in \mathcal{N}$ and $\delta_1 > 0$ such that, for any $i \in \{1,\dots,n\}$ and $h \in (0,\delta_1)$, there exists $j = j(i,h) \in \{1,\dots,k_1\}$ such that

$$\Big(\frac{(i-1)k_1+j-1}{nk_1}h, \frac{(i-1)k_1+j}{nk_1}h\Big) \subset C_p \cap B_p.$$

Let $\delta_0 = \min\big(\frac{1}{p},\delta_1\big)$. We put $h = h(k_1,\delta_0)$, $i = i(h)$ and $j = j(i,h)$. Then we may choose

$$y_p \in \Big(\frac{(i-1)k_1+j-1}{nk_1}h, \frac{(i-1)k_1+j}{nk_1}h\Big) \cap A \subset C_p \cap B_p.$$

Thus $0 < y_p < \frac{1}{p}$, $|f(y_p) - f(0)| < \frac{1}{p}$ and $|g(y_p) - g(0)| < \frac{1}{p}$.

Hence $\lim_{p\to\infty} y_p = 0$, $\lim_{p\to\infty} f(y_p) = f(0)$ and $\lim_{p\to\infty} g(y_p) = g(0)$. □

Theorem 17. *Let $f\colon \Re^2 \to \Re$ be a separately $\mathcal{I}_1$-approximately continuous function. If f is $\mathcal{I}_1$-approximately differentiable with respect to x at every point, then $f_{\mathcal{I}_1,x}$ is a Darboux function.*

Proof. By the assumption and by the result of [**10**], we have that f has the Baire property. Therefore, by [**3**], $f_{\mathcal{I},x}$ has the Baire property, too.

First, we show that if $I = [a,b] \times [c,d]$, then $f_{\mathcal{I}_1,x}(I)$ is a connected set. If it is not true, there exists $x_0 \in \Re$ and two nonempty sets A and B having the Baire property, such that $I = A \cup B$ and $f_{\mathcal{I}_1,x}(A) \subset (-\infty, x_0)$ and $f_{\mathcal{I}_1,x}(B) \subset (x_0, +\infty)$.

For $y \in [c,d]$, let $H_y = \{(x,y) : x \in [a,b]\}$. Since $f_{\mathcal{I}_1,x}(x,y)$, as a function of x, has Darboux property, [**6**], we have that $f_{\mathcal{I}_1,x}(H_y)$ is a connected set. Then $H_y \subset A$ or $H_y \subset B$. Hence there exist A_1, A_2 such that $A = [a,b] \times A_1$ and $B = [a,b] \times A_2$. By Lemma 15, we may assume that there exists a point $y_0 \in A_1$ which is not an $\mathcal{I}_1$-dispersion point of A_2. Thus, by the above and the $\mathcal{I}_1$-approximate continuity of the functions $f(a,y)$ and $f(b,y)$ as functions of y, we may choose a sequence $\{y_n\}_{n\in\mathcal{N}} \subset A_2$ such that $\lim_{n\to\infty} y_n = y_0$, $\lim_{n\to\infty} f(b,y_n) = f(b,y_0)$ and $\lim_{n\to\infty} f(a,y_n) = f(a,y_0)$ (see Lemma 16). Since, for each $n \in \mathcal{N}$, $f(x,y_n)$ is $\mathcal{I}_1$-approximately differentiable as a function of x, by the mean-value property [**6**], we have that there exists $z_n \in (a,b)$ such that

$$\frac{f(b,y_n) - f(a,y_n)}{b-a} = f_{\mathcal{I}_1,x}(z_n,y_n).$$

Hence

$$\lim_{n\to\infty} f_{\mathcal{I}_1,x}(z_n,y_n) = \frac{f(b,y_0) - f(a,y_0)}{b-a}.$$

Applying the mean-value property to the function $f(x,y_0)$, we can find $z_0 \in (a,b)$ such that

$$\frac{f(b,y_0) - f(a,y_0)}{b-a} = f_{\mathcal{I}_1,x}(z_0,y_0).$$

Hence

$$\lim_{n\to\infty} f_{\mathcal{I}_1,x}(z_n,y_n) = f_{\mathcal{I}_1,x}(z_0,y_0).$$

Since $\{(z_n,y_n)\}_{n\in\mathcal{N}} \subset B$, we have that $\{f_{\mathcal{I}_1,x}(z_n,y_n)\}_{n\in\mathcal{N}} \subset f_{\mathcal{I}_1,x}(B) \subset (x_0,\infty)$ and $f_{\mathcal{I}_1,x}(z_0,y_0) \geq x_0$. This contradicts the fact that $f_{\mathcal{I}_1,x}(z_0,y_0) \in f(A) \subset (-\infty,x_0)$.

To complete the proof, it suffices to show that, for each set Q such that $\operatorname{int}(\operatorname{cl}(Q)) \subset Q$ and $\operatorname{cl}(Q)$ is a closed interval, $f_{\mathcal{I}_1,x}(Q)$ is a connected set. If Q is an open interval then $Q = \cup_{n\in\mathcal{N}}[a_n,b_n] \times [c_n,d_n]$ where, for each $n \in \mathcal{N}$, $[a_n,b_n] \times [c_n,d_n] \subset [a_{n+1},b_{n+1}] \times [c_{n+1},d_{n+1}]$. Since $f_{\mathcal{I}_1,x}([a_n,b_n] \times [c_n,d_n])$ is a connected set for each $n \in \mathcal{N}$, therefore $f(\mathcal{I}_1,x)(Q)$ is a connected set, too. If Q is not an open interval, we may assume that there exists $p_0 \in Q \setminus \operatorname{int}(Q)$. Let $I = [a,b] \times [c,d]$ be an interval included in $\operatorname{cl}(Q)$, having p_0 as a vertex. Say, $p_0 = (a,d)$. We want to show that $f_{\mathcal{I}_1,x}(\operatorname{int}(I) \cup \{p_0\})$ is connected. Since $\operatorname{int}(I)$ is an open interval, $f_{\mathcal{I}_1,x}(\operatorname{int}(I))$ is connected. Thus the proof will be completed if we show that $f_{\mathcal{I}_1,x}(p_0)$ is a limit of a sequence of points of $f_{\mathcal{I}_1,x}(\operatorname{int}(I))$. Since $f_{\mathcal{I}_1,x}(x,d)$ has the Darboux property, there exists a sequence $\{x_n\}_{n\in\mathcal{N}} \subset (a,b)$ such that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} f_{\mathcal{I}_1,x}(x_n,d) = f_{\mathcal{I}_1,x}(a,d)$.

Let $n \in \mathcal{N}$. Then, by our assumption, there exists $z_n \in (a,b) \setminus \{x_n\}$ such that

$$\left|\frac{f(z_n,d) - f(x_n,d)}{z_n - x_n} - f_{\mathcal{I}_1,x}(x_n,d)\right| < \frac{1}{3n}.$$

We assume that $z_n > x_n$. On the other hand, by the $\mathcal{I}_1$-approximate continuity of $f(z_n,y)$ and $f(x_n,y)$ as functions of y, there exists $y_n \in (c,d)$ such that

$$|f(x_n,d) - f(x_n,y_n)| < \frac{1}{3n}|x_n - z_n|$$

and

$$|f(z_n,d) - f(z_n,y_n)| < \frac{1}{3n}|x_n - z_n|.$$

Then we have

$$\left|\frac{f(x_n,y_n) - f(z_n,y_n)}{x_n - z_n} - f_{\mathcal{I}_1,x}(x_n,d)\right| < \frac{1}{n}\,.$$

By the mean-value theorem for $\mathcal{I}_1$-approximate derivatives (see [**6**]), we can choose a point $t_n \in (x_n, z_n)$ such that $f(x_n, y_n) - f(z_n, y_n) = f_{\mathcal{I}_1,x}(t_n, y_n)(x_n - z_n)$. Then we have

$$|f_{\mathcal{I}_1,x}(t_n, y_n) - f_{\mathcal{I}_1,x}(x_n, d)| < \frac{1}{n}\,.$$

Hence we have the sequence $\{(t_n, y_n)\}_{n\in\mathcal{N}} \subset \operatorname{int}(I)$ satisfying for each $n \in \mathcal{N}$,

$$|f_{\mathcal{I}_1,x}(t_n, y_n) - f_{\mathcal{I}_1,x}(x_n, d)| < \frac{1}{n}\,.$$

Therefore $\lim_{n\to\infty} f_{\mathcal{I}_1,x}(t_n, y_n) = f_{\mathcal{I}_1,x}(a, d)$. □

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