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ON LARGE RANDOM ALMOST EUCLIDEAN BASES

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Abstract. A new class of random proportional embeddings of l_2^n into certain Banach spaces is found. Let $(\xi_i)_{i=1}^n$ be i.i.d. mean zero Cramèr random variables. Suppose $(x_i)_{i=1}^n$ is a sequence in the unit ball of a Banach space with $\mathbf{E}\|\sum_i \varepsilon_i x_i\| \geq \delta n$. Then the system of $\lceil cn \rceil$ independent random vectors distributed as $\sum_i \xi_i x_i$ is well equivalent to the euclidean basis with high probability (c depends on ξ_1 and δ). A connection with combinatorial discrepancy theory is presented.

1. Sign Embeddings And Short Films

G. Schechtman proved that in l_1^n a certain random choise of cn vectors is well equivalent to the euclidean basis ([**Sch1**], see also [**M-S**, 7.15]). More precisely, by ε_i we denote the Rademacher random variables, i.e. independent random variables taking values -1 and 1 with probability 1/2, by e_i the canonical vectors in $\mathbf{R}^n$, and by $c_1, c_2, \dots$ absolute constants. A system $(z_i)_{i=1}^k$ of vectors in a Banach space is said to be **c-equivalent to the euclidean basis** if there is a linear operator $T\colon \operatorname{span}(z_i) \to l_2^k$ sending each z_i to e_i, with $\|T\|\|T^{-1}\| \leq c$. Then Schechtman's theorem says the following. Every system of $\lceil c_1 n \rceil$ independent random vectors in l_1^n distributed as $\sum_{j=1}^n \varepsilon_j e_j$ is c_2-equivalent to the euclidean basis with probability $\geq 1 - \exp(-c_3 n)$.

This result is generalized here in two directions. Instead of the canonical vector basis of l_1^n, we work with arbitrary sequence $(x_j)_{j=1}^n$ of vectors in the unit ball $B(X)$ of a Banach space X satisfying

$$\mathbf{E}\Big\|\sum_{j=1}^n \varepsilon_j x_j\Big\| \geq \delta n \tag{1}$$

for some $\delta > 0$. This estimate is known as the **random δ-sign embedding from l_1^n** condition [**F-J-S**]. In [**Sch1**] it was considered in spaces with a good cotype; our proof does not require cotype restrictions.

Moreover, instead of the Bernoullian distribution of each coordinate, we consider arbitrary distribution of a mean zero r.v. ξ having moment generating function, that is $\mathbf{E}e^{\alpha|\xi|} < \infty$ for some $\alpha > 0$. This is called the **Cramèr condition**,

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and is equivalent to the following: there are constants a, $\alpha > 0$ so that

$$\mathbf{P}\{|\xi| > t\} \le a e^{-\alpha t} \quad \text{for all } t \tag{2}$$

(see [**P**, Lemma III.5]).

Theorem 1.1. *Let* $(\xi_j)_{j=1}^n$ *be independent copies of a mean zero r.v.* ξ *satisfying* (2)*; set* $\alpha_1 = \mathbf{E}|\xi|$*. Suppose* $(x_j)_{j=1}^n$ *is a sequence in* $B(X)$ *satisfying* (1)*, and set* $s = \sqrt{a}/\alpha_1\alpha\delta$*. There is a* $c = c(s) > 0$ *so that the system of* $\lceil cn \rceil$ *independent random vectors distributed as* $\sum_{j=1}^n \xi_j x_j$ *is* $(c_1 s)$*-equivalent to the euclidean basis with probability* $\ge 1 - 2\exp(-c_2 s^{-2} n)$*.*

Remarks. 1. One can set $c(s) = c_2/s^2 \log(c_1 s)$. We see that Theorem 1.1 is controlled by the only parameter s.

2. Actually, we prove that the operator T realizing the equivalence satisfies $\|T\| \le c_3(\alpha_1\delta n)^{-1}$, and $\|T^{-1}\| \le c_4\sqrt{a}\alpha^{-1}n$.

J. Elton [**E**] proved that (1) yields the existence of a subset $A \subset \{1, \dots, n\}$, $|A| \ge c(\delta)n$, such that the sequence $(x_j)_{j\in A}$ is $c'(\delta)$-equivalent to the canonical vector basis of $l_1^{|A|}$. If combined with Schechtman's theorem, this gives another form of proportional euclidean sections of X (however, with a worse dependence on δ: $c(\delta) \sim \delta^2/\log^2(4/\delta)$, $c'(\delta) \sim \delta^{-3}$).

For convenience, we restricted ourselves to identically distributed random variables ξ, but the main result can easily be modified to handle the case when ξ have different distributions.

Theorem 1.1 admits an immediate application to random matrices. The next corolary says that the unit cube in $\mathbf{R}^n$ under the action of a random $k \times n$ matrix (with k proportional to n) is close to the euclidean ball B_2^k. We denote the unit euclidean ball in $\mathbf{R}^k$ by B_2^k.

Corollary 1.2. *Suppose* ξ *is a random variable satisfying* (1)*, then there exist* $c, \mu, \nu > 0$ *such that we have the following. Let* A *be the* $k \times n$ *matrix whose entries are independent random variables distributed as* ξ*. If* $k \le cn$ *then with probability* $\ge 1 - 2\exp(-cn)$

$$\mu B_2^k \subset n^{-1}A([-1,1]^n) \subset \nu B_2^k.$$

Proof. Pass to the dual setting and apply Theorem 1.1 together with Remark 2. □

Now we discuss a relation between almost euclidean bases in l_1^n and combinatorial discrepancies. Given a two-coloring χ, say White and Black, of a finite set Ω, the **discrepancy** $\operatorname{disc}(A, \chi)$ of a set $A \subset \Omega$ is the number of White points in A minus the number of Black points in A (cf. [**A-S**], [**B-S**]). A family $\overline{\chi} = (\chi_j)_{j=1}^n$ of two-colorings on Ω is called a **film** of length n. We define the **film discrepancy** $\operatorname{fdisc}(A, \overline{\chi})$ of a set $A \subset \Omega$ as the average $\frac{1}{n}\sum_{j=1}^n |\operatorname{disc}(A, \chi_j)|$.

The problem is to make a short **homogeneous** film, so that the film discrepancies of any two sets $A, B \subset \Omega$ of equal size be nearly the same: $\operatorname{fdisc}(A, \overline{\chi}) \approx \operatorname{fdisc}(B, \overline{\chi})$ (the relation $x \approx y$ means $c_1 x \le y \le c_2 x$ for some absolute constants $c_1, c_2 > 0$). Since nobody wants to watch a monochromatic film, we require it to be **balanced**, that is the density of each shot be nontrivial: $|\operatorname{disc}(\Omega, \chi_j)| \le (1-c_3)|\Omega|$ for all j and some absolute constant $c_3 > 0$.

One might think that balanced homogeneous films must be fairly long comparing with $|\Omega|$, but this is unjustified.

Theorem 1.3. *Given a finite set Ω, there is a balanced homogeneous film on Ω of length $c_1|\Omega|$.*

Proof. We begin with a geometrical interpretation of the problem, as in [**Sp**]. Let $k = |\Omega|$. A coloring χ on Ω is regarded as a sequence $(\varepsilon_i) \in \{-1, 1\}^k$, assigning 1 to White and -1 to Black. A set $A \subset \Omega$ is identified with its incidence vector $(a_i) \in \{0, 1\}^k$. Then $\operatorname{disc}(A, \chi) = \sum_{i=1}^k \varepsilon_i a_i$.

Now we clarify a relation to Schechtman's result, that is Theorem 1.1 with $\xi_j = \varepsilon_j$ and $(x_j) =$ the canonical vectors in $X = l_1^n$. Let n be the minimal integer such that $\lceil cn \rceil \ge k$. In this case we get with probability $\ge 1 - 2\exp(-c_2 n)$

$$\text{(3)} \qquad \frac{1}{n}\sum_{j=1}^n \Big| \sum_{i=1}^k a_i \varepsilon_{ij} \Big| \approx \Big(\sum_{i=1}^k |a_i|^2 \Big)^{1/2} \quad \text{for all scalars } (a_i),$$

where ε_{ij} are Rademacher random variables (see Remark 2 following Theorem 1.1). Let $\overline{\chi}$ be a random film of length n, so that $\chi_j = (\varepsilon_{ij})_{i=1}^k$. Then (3) yields that, with probability $\ge 1 - 2\exp(-c_2 n)$, every set $A \subset \Omega$ satisfies $\operatorname{fdisc}(A, \overline{\chi}) \approx \sqrt{|A|}$. Hence most films are homogeneous.

It suffices to show that most films are also balanced. Consider a random coloring $\chi = (\varepsilon_i)$ on Ω. Using a subgaussian tail estimate for Rademacher sums (see [**L**] or apply Theorem), we have

$$\mathbf{P}\Big\{ |\operatorname{disc}(\Omega, \chi)| \le \frac{1}{2}|\Omega| \Big\} = \mathbf{P}\Big\{ \Big| \sum_{i=1}^k \varepsilon_i \Big| \le k/2 \Big\} \ge 1 - 2\exp(-k/8).$$

Then the probability that $|\operatorname{disc}(\Omega, \chi_j)| \le \frac{1}{2}|\Omega|$ for all $j = 1, \dots, n$ is at least $1 - 2n\exp(-k/8)$. Since $n \le c_1 k$, this probability tends to 1 as $k \to \infty$. This completes the proof. □

I do not know whether there are asymptotically shorter balanced homogeneous films.

To prove the main result, we will apply a deviation inequality for sums of independent Banach space valued random variables.

Theorem 1.4. *Let $X_1, \dots, X_n$ be independent Banach space valued random variables with $\mathbf{P}\{\|X_i\| > t\} \le ae^{-\alpha_i t}$ for all t and i. Let $d \ge \max_{i\le n} \alpha_i^{-1}$ and $b \ge a\sum_{i=1}^n \alpha_i^{-2}$. Then setting $S_n = \sum_{i=1}^n X_i$ we have*

$$\mathbf{P}\big\{\big|\|S_n\| - \mathbf{E}\|S_n\|\big| > t\big\} \le \begin{cases} 2\exp(-t^2/32b) & \text{for } 0 \le t \le 4b/d \\ 2\exp(-t/8d) & \text{for } t \ge 4b/d. \end{cases}$$

This result can be derived by truncation from known deviation inequalities for sums of bounded random variables (see e.g. [**Le-Ta**, Section 6.2]). However, it is more convenient and more instructive to give a direct proof based on martingales, as in [**Yu**, Sec. 3.3]. A rather short instructive proof is given in §2.

§3 consists of the proof of Theorem 1.1.

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2. Deviations of Sums

In this section we prove Theorem 1.4.

First we recall that problems about Banach space valued independent random variables can often be reduced to a **real valued martingale** case, see [**Le-Ta**, Ch. 6.3].

Let $\mathcal{A}_i$ be the σ-algebra generated by the random variables $X_1, \dots, X_i$, $i \le n$, and $\mathcal{A}_0$ be the trivial σ-algebra. The conditional expectation with respect to $\mathcal{A}_i$ is denoted by $\mathbf{E}^{\mathcal{A}_i}$. Set, for each i, $d_i = \mathbf{E}^{\mathcal{A}_i}\|S_n\| - \mathbf{E}^{\mathcal{A}_{i-1}}\|S_n\|$. Then $(d_i)_{i=1}^n$ forms a real valued martingale difference sequence, and $\sum_{i=1}^n d_i = \|S_n\| - \mathbf{E}\|S_n\|$.

Lemma 2.1. *For every i and every $p \ge 1$*

$$\mathbf{E}^{\mathcal{A}_{i-1}}|d_i|^p \le 2^p\mathbf{E}\|X_i\|^p$$

almost surely.

Proof. Yurinskii's inequality states that $|d_i| \le \|X_i\| + \mathbf{E}\|X_i\|$ almost surely (see [**Le-Ta**, Lemma 6.16]). Then $|d_i|^p \le 2^{p-1}\big(\|X_i\|^p + (\mathbf{E}\|X_i\|)^p\big)$. Hence

$$\begin{aligned}\mathbf{E}^{\mathcal{A}_{i-1}}|d_i|^p &\le 2^{p-1}\big(\mathbf{E}^{\mathcal{A}_{i-1}}\|X_i\|^p + (\mathbf{E}\|X_i\|)^p\big) \\ &= 2^{p-1}\big(\mathbf{E}\|X_i\|^p + (\mathbf{E}\|X_i\|)^p\big) \le 2^p\mathbf{E}\|X_i\|^p,\end{aligned}$$

and we are done. □

Proof of Theorem 1.4. Apply Chebyshev's inequality. For every $\lambda \ge 0$

$$P := \mathbf{P}\Big\{\|S_n\| - \mathbf{E}\|S_n\| > t\Big\} = \mathbf{P}\Big\{\sum_{i=1}^n d_i > t\Big\} \le e^{-\lambda t}\mathbf{E}\exp\Big(\lambda\sum_{i=1}^n d_i\Big). \tag{4}$$

But

$$\begin{aligned}\mathbf{E}\exp(\lambda\sum_{i=1}^n d_i) &= \mathbf{E}\Big(\mathbf{E}^{\mathcal{A}_{n-1}}\exp(\lambda\sum_{i=1}^n d_i)\Big) = \mathbf{E}\Big(\exp(\lambda\sum_{i=1}^{n-1} d_i)\mathbf{E}^{\mathcal{A}_{n-1}}\exp(\lambda d_n)\Big)\\ &\le \|\mathbf{E}^{\mathcal{A}_{n-1}}\exp(\lambda d_n)\|_\infty \mathbf{E}\exp(\lambda\sum_{i=1}^{n-1} d_i) = \cdots \qquad (5)\\ &= \prod_{i=1}^n \|\mathbf{E}^{\mathcal{A}_{i-1}}\exp(\lambda d_i)\|_\infty.\end{aligned}$$

So we are to evaluate

$$\begin{aligned}\mathbf{E}^{\mathcal{A}_{i-1}}\exp(\lambda d_i) &= 1 + \sum_{p=2}^\infty \frac{\lambda^p \mathbf{E}^{\mathcal{A}_{i-1}} d_i^p}{p!} \qquad (\text{since } \mathbf{E}^{\mathcal{A}_{i-1}} d_i = 0)\\ &\le 1 + \sum_{p=2}^\infty \frac{\lambda^p 2^p \mathbf{E}\|X_i\|^p}{p!} \qquad (\text{by Lemma 2.1}).\end{aligned}$$

Note that

$$\mathbf{E}\|X_i\|^p = \int_0^\infty \mathbf{P}\{\|X_i\| > t\}\,dt^p \le \int_0^\infty a e^{-\alpha_i t}\,dt^p = a\alpha_i^{-p} p! \qquad (6)$$

Then for $0 \le \lambda \le \alpha_i/4$

$$\mathbf{E}^{\mathcal{A}_{i-1}}\exp(\lambda d_i) \le 1 + a(2\lambda/\alpha_i)^2 \sum_{p=2}^\infty (2\lambda/\alpha_i)^{p-2} \le 1 + a(2\lambda/\alpha_i)^2 2 \le \exp(8\lambda^2 a\alpha_i^{-2}).$$

Combining this estimate, (5), and (4), we obtain for $0 \le \lambda \le 1/4d$

$$P \le e^{-\lambda t}\prod_{i=1}^n \exp(8\lambda^2 a\alpha_i^{-2}) \le \exp(-\lambda t + 8\lambda^2 b).$$

The minimum here is attained for $\lambda = t/16b$. If $t \le 4b/d$, then the condition $\lambda \le 1/4d$ is satisfied, and $P \le \exp(-t^2/32b)$. If $t \ge 4b/d$, then we take $\lambda := 1/4d$, and get $P \le \exp(-t/8d)$.

Similarly, one obtains the same estimates on $\mathbf{P}\{\|S_n\| - \mathbf{E}\|S_n\| < -t\}$. □

3. Random Euclidean Embeddings

In this section Theorem 1.1 is proved.

We will use a simple symmetrization lemma, see [**Le-Ta**, Lemma 6.3].

Lemma 3.1. *For every finite sequence (X_i) of Banach space valued mean zero random variables*

$$\frac{1}{2}\mathbf{E}\Big\|\sum_i \varepsilon_i X_i\Big\| \le \mathbf{E}\Big\|\sum_i X_i\Big\| \le 2\mathbf{E}\Big\|\sum_i \varepsilon_i X_i\Big\|.$$

Next, we need a known generalization of the Khinchine inequality.

Proposition 3.2. *Let (ξ_i) be a sequence of real valued i.i.d. mean zero random variables. Then for every finite sequence of numbers (a_i)*

$$\frac{1}{2}A_p\|\xi_1\|_{\min(2,p)}\Big(\sum_i |a_i|^2\Big)^{1/2} \le \Big\|\sum_i a_i\xi_i\Big\|_p \le 2B_p\|\xi_1\|_{\max(2,p)}\Big(\sum_i |a_i|^2\Big)^{1/2},$$

where A_p and B_p are the constants from the classical Khinchine inequality.

Actually, we will use the following particular case of the inequality, and give a proof only for this case:

$$\frac{1}{2\sqrt{2}}\|\xi_1\|_1\Big(\sum_i |a_i|^2\Big)^{1/2} \le \Big\|\sum_i a_i\xi_i\Big\|_1 \le \|\xi_1\|_2\Big(\sum_i |a_i|^2\Big)^{1/2}. \tag{7}$$

Proof (sketch). To prove the left-hand side observe that, by Lemma 3.1, $\mathbf{E}|\sum_i a_i\xi_i|$ is nearly the same as $\mathbf{E}|\sum_i \varepsilon_i a_i\xi_i|$. Now it is enough to apply partial integration and use the classical Khinchine inequality (note that $A_1 = 1/\sqrt{2}$ [**Sz**]). Since $\|\sum_i a_i\xi_i\|_1 \le \|\sum_i a_i\xi_i\|_2$, the right-hand side of (7) follows from the orthogonality of (ξ_i) in $L_2(\Omega)$, due to the independentness. □

Another simple consequence of the symmetrization is this.

Lemma 3.3. *Let (η_i) be a finite sequence of real valued i.i.d. mean zero random variables. Then, for any sequence (x_i) in a Banach space,*

$$\mathbf{E}\Big\|\sum_i \eta_i x_i\Big\| \ge \frac{1}{2}\|\eta_1\|_1\mathbf{E}\Big\|\sum_i \varepsilon_i x_i\Big\|.$$

Proof. By the symmetry, $\varepsilon_i|\eta_i|$ has the same distribution as $\varepsilon_i\eta_i$. Using partial integration and the triangle inequality, we have

$$\mathbf{E}\Big\|\sum_i \varepsilon_i\eta_i x_i\Big\| = \mathbf{E}\Big\|\sum_i \varepsilon_i|\eta_i|x_i\Big\| \ge \mathbf{E}\Big\|\sum_i \varepsilon_i\|\eta_i\|_1 x_i\Big\|.$$

Now it is enough to apply Lemma 3.1 with $X_i = \eta_i x_i$. □

Finally, recall a standard approximation lemma (see [**M-S**, 4.1]).

Lemma 3.4. *Let X be a Banach space, and $F\colon X \to \mathbf{R}$ be a non-negative convex homogeneous function. Suppose for some θ-net $\mathcal{N}$ of $S(X)$ one has $a \leq F(x) \leq b$ for every $x \in \mathcal{N}$. Then*

$$a - \frac{\theta}{1-\theta} b \leq F(x) \leq \frac{1}{1-\theta} b$$

for every $x \in S(X)$.

In particular, if $\theta \leq a/3b$, then $\frac{1}{2}a \leq F(x) \leq \frac{3}{2}b$ for every $x \in S(X)$.

Proof of Theorem 1.1. Let (ξ_{ij}) be independent copies of ξ. Let $k \leq cn$. We are to show that the random vectors $y_i = n^{-1}\sum_{j=1}^{n}\xi_{ij}x_j$, $i = 1, \dots, k$, are well equivalent to the euclidean basis.

Fix $\overline{a} = (a_i)_{i=1}^{k}$ in the unit sphere $S(l_2^k)$. Consider a sequence of independent random variables

$$X_{ij} = n^{-1} a_i \xi_{ij} x_j, \quad i = 1, \dots, k, \quad j = 1, \dots, n,$$

and their sum $S(\overline{a}) = \sum_{i=1}^{k}\sum_{j=1}^{n} X_{ij}$. We will prove that, with high probability, $\|S(\overline{a})\|$ is bounded from above and below for every $\overline{a}$.

Theorem 1.4 applied to the sum of X_{ij} helps here. Note that $\mathbf{P}\{\|X_{ij}\| > t\} = \mathbf{P}\{n^{-1}|a_i||\xi| > t\} \leq a\exp(-\alpha n|a_i|^{-1}t)$, thus we take

$$d = \alpha^{-1}n^{-1} \quad \text{and} \quad b = a\sum_{i=1}^{k}\sum_{j=1}^{n}\alpha^{-2}n^{-2}|a_i|^2 = a\alpha^{-2}n^{-1}.$$

Furthermore,

$$\begin{aligned}\mathbf{E}\|S(\overline{a})\| &= n^{-1}\mathbf{E}\Big\|\sum_{j=1}^{n}\Big(\sum_{i=1}^{k} a_i\xi_{ij}\Big)x_j\Big\| \\ &\geq n^{-1}\frac{1}{2}\Big\|\sum_{i=1}^{k} a_i\xi_{ij}\Big\|_1 \mathbf{E}\Big\|\sum_{j=1}^{n}\varepsilon_j x_j\Big\| \quad \text{(by Lemma 3.3)} \\ &\geq \frac{1}{4\sqrt{2}}\alpha_1\delta \quad \text{(by (7) and the condition on } (x_j)).\end{aligned}$$

Conversely, let $\alpha_2 = \|\xi\|_2$. Note that $\alpha_2 \leq \sqrt{2}\sqrt{a}\alpha^{-1}$, as in (6). Then by the triangle inequality and (7)

$$\mathbf{E}\|S(\overline{a})\| \leq n^{-1}\sum_{j=1}^{n}\mathbf{E}\Big|\sum_{i=1}^{k} a_i\xi_{ij}\Big| \leq \alpha_2 \leq \sqrt{2}\sqrt{a}\alpha^{-1}.$$

Now set $t := \frac{1}{8\sqrt{2}}\alpha_1\delta \leq \frac{1}{2}\mathbf{E}\|S(\overline{a})\|$ and apply Theorem 1.4. Clearly, $t \leq 4b/d$ is the case, because $\delta \leq 1$ and $\alpha_1 \leq a\alpha^{-1}$ as in (6). Thus

$$\mathbf{P}\{d_1 \leq \|S(\overline{a})\| \leq d_2\} \geq 1 - 2\exp(-d_3 n),$$

where $d_1 = \frac{1}{8\sqrt{2}}\delta\alpha_1$, $d_2 = \frac{3}{2}\sqrt{2}\sqrt{a}\alpha^{-1}$, and $d_3 = c_4(\alpha\alpha_1\delta/\sqrt{a})^2 = c_4 t^{-2}$.

The preceding observations hold for **fixed** $\overline{a}$. Now let $\overline{a}$ run over a θ-net $\mathcal{N}$ in $S(l_2^k)$, $|\mathcal{N}| \leq \exp(k\log 3/\theta)$, where $\theta = d_1/3d_2$ (there is such a net, cf. [**M-S**, 2.6]). Then

$$\mathbf{P}\{\forall \overline{a} \in \mathcal{N}, \quad d_1 \leq \|S(\overline{a})\| \leq d_2\} \geq 1 - 2\exp(k\log 3/\theta - d_3 n).$$

We conclude by Lemma 3.4,

$$\mathbf{P}\{\forall \overline{a} \in S(l_2^k), \quad d_1/2 \leq \|S(\overline{a})\| \leq 3d_2/2\} \geq 1 - 2\exp(k\log 3/\theta - d_3 n).$$

If c was chosen small enough, then $k\log 3/\theta \leq \frac{1}{2}d_3 n$.

It remains to note that $d_2/d_1 = c_5 s$, and the required $(c_1 s)$-equivalence to the euclidean basis follows. □

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