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BOUNDED PLATEAU AND WEIERSTRASS MARTINGALES WITH INFINITE VARIATION IN EACH DIRECTION

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1. Introduction

In this note we discuss a dyadic model for real harmonic functions $x_1, \dots, x_n$ defined on the unit disk $\{z = u + iv : |z| < 1\}$ whose partial derivatives satisfy the compatibility condition

$$\sum_{i=1}^{m} x_{i,v}^2 = \sum_{i=1}^{m} x_{i,v}^2 \quad \text{and} \quad \sum_{i=1}^{m} x_{i,u} x_{i,v} = 0.$$

We restrict the discussion to the cases $m = 2, 3, 4$.

The problem we have in mind is the following. Suppose that $|x(z)| \le 1$, does it then follow that there exists a ray R connecting 0 to $e^{i\theta}$ such that

$$\int_R \sum_{i=1}^{m} |x_{i,v}(z)| + |x_{i,u}(z)|\,|dz| < \infty?$$

For the case $m = 2$ J. Bourgain has shown in [**1**] that the answer is "yes". Where as for $m = 4$ (and hence for $m \ge 4$) P. W. Jones has shown in [**2**] that the answer is "no". We now turn to $m = 3$. In [**4**] N. Nadirashvili constructs an example of of three bounded harmonic functions satisfying

$$\sum_{i=1}^{3} x_{i,v}^2 = \sum_{i=1}^{3} x_{i,v}^2 \quad \text{and} \quad \sum_{i=1}^{3} x_{i,u} x_{i,v} = 0.$$

and

$$\int_R \sum_{i=1}^{3} |x_{i,v}(z)| + |x_{i,u}(z)|\,|dz| = \infty$$

for every ray R connecting 0 to the boundary of the unit disk. The construction uses hard analytic estimates and is highly combinatiorial in nature.

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The purpose of this note is to illustrate the method of Nadirashvili by isolating the combinatorial pattern of his proof. This is done by studying martingale models that are well adapted to the problem at hand.

When $m = 2$, $x_1 + ix_2$ is either holomorphic in the unit disk or antiholomorphic. Recall that the discrete model for that is provided by a complex valued martingale $F_n = G_n + iH_n$ such that the increments satisfy

$$\mathbf{E}_{n-1}((G_n - G_{n-1})^2) = \mathbf{E}_{n-1}((H_n - H_{n-1})^2) \quad \text{and}$$
$$\mathbf{E}_{n-1}(G_n - G_{n-1})(H_n - H_{n-1}) = 0.$$

This is equivalent to $\mathbf{E}_{n-1}(F_n^2) = F_{n-1}^2$. Martingales satisfying this condition are called conformal martingales. Many authors have exploited the analogy between conformal martingales and analytic functions to obtain significant results in complex analysis and probability. (See especially [**3**] and [**5**].)

In Section 2 we will introduce discrete analouges for triples of harmonic functions satisfying the compatibility condition:

$$\sum_{i=1}^{3} x_{i,v}^2 = \sum_{i=1}^{3} x_{i,v}^2 \quad \text{and} \quad \sum_{i=1}^{3} x_{i,u} x_{i,v} = 0.$$

These $\mathbf{R}^3$-valued martingales also extend the notion of conformal martingales. We will isolate an example in this class of martingales that is uniformly bounded, yet has infinite variation in each direction. The construction given below controls the oscillations, that give rise to infinite variation, by exploiting orthogonality in $\mathbf{R}^3$. (See Lemma 1 below.)

Harmonic functions x_1, x_2, x_3 in the unit disk which satisfy the above compatibility conditions admit the so called Weierstrass representation. That means that there are analytic function f, g in the unit disk such that

$$x_1(z) = \mathrm{Re} \int_0^z f(\zeta)(1 - g^2(\zeta))\, d\zeta,$$
$$x_2(z) = i\mathrm{Re} \int_0^z f(\zeta)(1 + g^2(\zeta))\, d\zeta,$$
$$x_3(z) = 2\mathrm{Re} \int_0^z f(\zeta) g(\zeta))\, d\zeta.$$

In Section 3 we will motivate and define an $\mathbf{R}^3$ valued martingale model for harmonic function that are given by the Weierstrass representation.

This model provides another extension of the notion of conformal martingales which is different from the one discussed in Section 2. Here too we will find an example which is uniformly bounded and has infinite variation in each direction. Again the construction starts by creating oscillation, (for the lower bounds on the variation of the martingale) and then makes use of the resulting cancellation to obtain pointwise upper estimates on the value of the martingale. At that point the geometry of the Euclidian ball in $\mathbf{R}^3$ (uniform convexity) becomes crucial.

2. Plateau Martingals

Let Ω be a metric space endowed with a probability measure P. Let the sequence of semicontinuous functions $F_n\colon \Omega \to \mathbf{R}^3$ be a martingale with respect to the filtration $\mathcal{F}_n$, $n \in \mathbf{N}$. Then this martingale is called a Plateau martingale provided that the martingale difference sequence $F_{n+1} - F_n$ admits the following representation

$$(F_{n+1} - F_n)(w) = R_1(w)g + R_2(w)h, \quad w \in \Omega, \tag{1}$$

where $g, h \in \mathbf{R}^3$ satisfy

$$|g| - |h| = 0 = \langle g, h\rangle, \tag{2}$$

and where $R_i\colon \Omega \to \mathbf{R}$ are independent random variables, independent of $\mathcal{F}_n$, measurable with respect to $\mathcal{F}_{n+1}$ such that

$$\mathbf{E}(R_i|\mathcal{F}_n) = 0 \quad \text{and} \quad \mathbf{E}(R_i^2|\mathcal{F}_n) = 1, \quad i \in \{1, 2\}. \tag{3}$$

We would like to point out that in the above definition $\langle g, h\rangle$ denotes the scalar product in $\mathbf{R}^3$ of g and h.

Theorem 1. *There exists a Plateau martingale F_n, $n \in \mathbf{N}$, so that for every $w \in \Omega$*

$$\sup_{n\in\mathbf{N}} |F_n(w)| \leq 4, \tag{4}$$

$$\sum_{n=1}^{\infty} |(F_{n+1} - F_n)(w)| = \infty. \tag{5}$$

The construction of this martingale uses the following elementary observation.

Lemma 1. *Let $n \in \mathbf{N}$. For $F \in \mathbf{R}^3$, there exist $g, h \in \mathbf{R}^3$ so that*

$$\begin{aligned} \langle F, g\rangle &= \langle F, h\rangle = 0,\\ |g| - |h| &= 0 = \langle h, g\rangle,\\ |\epsilon_1 g + \epsilon_2 h| &= \frac{1}{n}, \quad \textit{where} \quad \epsilon_1, \epsilon_2 \in \{\pm 1\}. \end{aligned}$$

Proof. Let $E \subseteq \mathbf{R}^3$ be the plane orthogonal to F that contains the origin. Then choose $g, h \in E$ such that $\langle g, h\rangle = 0$ and $|g| = |h| = 1/n\sqrt{2}$. Then clearly $|\epsilon_1 g + \epsilon_2 h| = \sqrt{|g|^2 + |h|^2} = 1/n$. $\square$

Proof of Theorem 1. We will build the Plateau martingal (F_n) on the interval $[0, 1]$ endowed with Lebesque measure. The filtration $\mathcal{F}_n$ will be the σ-algebra generated by the first $2n$ Rademacher functions $r_1, \dots, r_{2n}$. Let $\Delta_n = F_{n+1} - F_n$ denote the martingale difference sequence. Applying Lemma 1 we will construct (F_n) in such a way that for every $w \in [0,1]$

$$\langle F_n(w), \Delta_n(w) \rangle = 0, \tag{6}$$

$$|\Delta_n(w)| = \frac{1}{n}. \tag{7}$$

(Recall that $\langle \cdot, \cdot \rangle$ denotes scalar product in $\mathbf{R}^3$.)

By (6), (7) and induction we obtain for every $w \in [0,1]$,

$$\begin{aligned} |F_{n+1}(w)|^2 &= |(F_n + \Delta_n)(w)|^2 \\ &= |F_n(w)|^2 + |\Delta_n(w)|^2 \\ &= \sum_{m=1}^{n} |\Delta_m(w)|^2 = \sum_{m=1}^{n} \frac{1}{m^2}. \end{aligned}$$

On the other hand, for every $w \in [0,1]$,

$$\sum_{m=1}^{n} |\Delta_m(w)| = \sum_{m=1}^{n} \frac{1}{m}.$$

Hence (4) and (5) hold for martingales satisfying (6) and (7).

Summing up, we have observed so far that it suffices to construct a Plateau martingale $F_n\colon [0,1] \to \mathbf{R}^3$ satisfying (6) and (7). We start the construction by choosing $F_0 \in \mathbf{R}^3$. We let $r_1\colon [0,1] \to \{\pm 1\}$ be the Rademacher function which is 1 on $[0, \frac{1}{2})$ and -1 on $[\frac{1}{2}, 1]$. When continued periodically to $\mathbf{R}$, the first Rademacher function can be used to define the second Rademacher function by setting $r_2(w) = r_1(2w)$, $w \in [0,1]$.

By Lemma 1 there are $g, h \in \mathbf{R}^3$ so that

$$\langle F_0, g \rangle = \langle F_0, h \rangle = 0 \tag{8}$$

$$|g| - |h| = 0 = \langle g, h \rangle \tag{9}$$

$$|r_1(w)g + r_2(w)h| = \frac{1}{n}, \quad \text{for } w \in [0,1]. \tag{10}$$

We define $F_1\colon [0,1] \to \mathbf{R}^3$ by

$$F_1(w) = F_0 + r_1(w)g + r_2(w)h, \quad w \in [0,1].$$

$\mathcal{F}_1$ is the σ-algebra generated by r_1, r_2.

The construction of F_{n+1} follows the same pattern. We are given $F_1, \dots, F_n$ so that (1), (2) and (3) as well as (6) and (7) are satisfied. The σ-algebra $\mathcal{F}_n$

is generated by the Rademacher functions $r_1, \ldots, r_{2n}$. Let the halfopen interval $I \subset [0,1)$ be an atom in $\mathcal{F}_n$. F_n is constant on I, and we put $F_I = F_n(w)$, $w \in I$. The Rademacher functions r_{2n+1}, r_{2n} are independent of $\mathcal{F}_n$ and satisfy

$$\mathbf{E}(r_i|\mathcal{F}_n) = 0 \quad \text{and} \quad \mathbf{E}(r_i^2|\mathcal{F}_n) = 1; \quad i \in \{2n+1, 2n\}. \tag{11}$$

By Lemma 1 there exist $g_I, h_I \in \mathbf{R}^3$, so that

$$\langle F_I, g_I \rangle = \langle F_I, h_I \rangle = 0 \tag{12}$$

$$|g_I| - |h_I| = 0 = \langle g_I, h_I \rangle \tag{13}$$

$$|r_{2n+1}(w) g_I + r_{2n}(w) h_I| = \frac{1}{n}, \quad w \in I. \tag{14}$$

Then on I we define F_{n+1} by

$$F_{n+1}(w) = F_I + r_{2n+1}(w) g_I + r_{2n}(w) h_I, \quad w \in I.$$

As $F_n(w) = F_I$ for $w \in I$, we have for $w \in I$,

$$F_{n+1}(w) = F_n(w) + r_{2n+1}(w) g_I + r_{2n}(w) h_I. \tag{15}$$

We obtain the definition of F_{n+1} on the whole of $[0,1)$ by successively considering the atoms I of $\mathcal{F}_n$ and applying the above construction. By (12) and (14), we have for $w \in [0,1)$,

$$\langle F_n(w), \Delta_n(w) \rangle = 0 \quad \text{and} \quad |\Delta_n(w)| = \frac{1}{n}.$$

Moreover by (11), (13) and (15), the sequence $F_1, \ldots, F_{n+1}$ is a Plateau martingale, when $\mathcal{F}_{n+1}$ is defined as the σ-algebra generated by $r_1, \ldots, r_{2n+2}$. This completes the construction of a bounded Plateau martingale that is of infinite variation in each direction.

3. Weierstrass Martingales

We first motivate our martingale model of harmonic functions given by the Weierstrass representation.

Let $f = a + iHa$, $g = m + iHm$ where a, m are real harmonic, and where H denotes the Hilbert transform. We further abbreviate $c = m^2 - (Hm)^2$. Then $Hc = 2mHm$. With this notation we have

$$\begin{aligned} f(1-g^2) &= a - ac + 2HaHc \\ &\quad + i(Ha - (cHa - aHc), \\ if(1+g^2) &= -Ha - cHa - aHc \\ &\quad + i(a + ac - HaHc), \\ fg &= am - HaHm + i(aHm + mHa). \end{aligned}$$

To display the symmetries in the above expressions, we rewrite, using scalar products

$$f(1-g^2) = a + \left\langle \begin{pmatrix} a \\ Ha \end{pmatrix}, \begin{pmatrix} -c \\ Hc \end{pmatrix} \right\rangle + iHa - i \left\langle \begin{pmatrix} a \\ Ha \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c \\ Hc \end{pmatrix} \right\rangle,$$
$$if(1+g^2) = -Ha - \left\langle \begin{pmatrix} a \\ Ha \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c \\ Hc \end{pmatrix} \right\rangle + ia + i \left\langle \begin{pmatrix} a \\ Ha \end{pmatrix}, \begin{pmatrix} -c \\ Hc \end{pmatrix} \right\rangle,$$
$$fg = \left\langle \begin{pmatrix} a \\ Ha \end{pmatrix}, \begin{pmatrix} m \\ -Hm \end{pmatrix} \right\rangle - i \left\langle \begin{pmatrix} a \\ Ha \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} m \\ -Hm \end{pmatrix} \right\rangle.$$

We conclude from this discussion that **if** we can find harmonic functions a, m (hence c) so that the following integrals are bounded,

$$\int_0^z a\,dr, \int_0^z Ha\,dr;$$
$$\int_0^z \left\langle \begin{pmatrix} a \\ Ha \end{pmatrix}, \begin{pmatrix} -c \\ Hc \end{pmatrix} \right\rangle dr;$$
$$\int_0^z \left\langle \begin{pmatrix} a \\ Ha \end{pmatrix}, \begin{pmatrix} -c \\ Hc \end{pmatrix} \right\rangle dr;$$
$$\int_0^z \left\langle \begin{pmatrix} a \\ Ha \end{pmatrix}, \begin{pmatrix} m \\ -Hm \end{pmatrix} \right\rangle, \int_0^z \left\langle \begin{pmatrix} a \\ Ha \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} m \\ -Hm \end{pmatrix} \right\rangle,$$

then the harmonic functions x_1, x_2, x_3 given by Weierstrass representation are also bounded. Therefore they parametrize a bounded minimal surface. If moreover

$$\int_\gamma \left| \left\langle \begin{pmatrix} a \\ Ha \end{pmatrix}, \begin{pmatrix} -c \\ Hc \end{pmatrix} \right\rangle \right| + \left| \left\langle \begin{pmatrix} a \\ Ha \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c \\ Hc \end{pmatrix} \right\rangle \right| ds = \infty,$$

when γ connects 0 to the boundary of the unit disk, then the resulting minimal surface is complete.

Now we tie these observation to the martingale model given below. We interpret $a + iHa$ as the infinitesimal increment of the analytic function $\int_0^z a + iHa\,dz = \int_0^z f\,dz$ and $c + iHc$ as the infinitesimal increment of the analytic function $\int_0^z g^2\,dz$.

In the model we replace holomorphic functions by $\mathbf{R}^2$ valued conformal martingales and infinitesimal increments by martingale difference. Then the integral expressions above will guide us to the formulation of the $\mathbf{R}^3$-valued martingales in Weierstrass representation. We start by defining conformal martingales with values in $\mathbf{R}^2$.

Let Ω be a metric space endowed with a probability measure P. Let the sequence $F_n : \Omega \to \mathbf{R}^2$ be a martingale with respect to the filtration $\mathcal{F}_n$, $n \in \mathbf{N}$. Then F_n is called a conformal martingale if the martingale differences admit the representation

$$(F_n - F_{n-1})(w) = R_1(w)g + R_2(w)h, \quad w \in \Omega$$

where $g, h \in \mathbf{R}^2$ satisfy
$$|g| - |h| = 0 = \langle g, h\rangle$$
and where $R_i \colon \Omega \to \mathbf{R}$ are independent random variables, independent of $\mathcal{F}_n$, measurable with respect to $\mathcal{F}_{n+1}$ such that
$$\mathbf{E}(R_i|\mathcal{F}_n) = 0 \quad \text{and} \quad E(R_i^2|\mathcal{F}_n) = 1, \quad i \in \{1, 2\}.$$
(Here $\langle g, h\rangle$ denotes scalar product in $\mathbf{R}^2$.)

We say that (F_l, G_l, M_l) is a triple of conformal martingales if for each l there exist independent random variables $R_i \colon \Omega \to \mathbf{R}$ as above, and the martingale differences satisfy
$$\begin{aligned} F_l - F_{l-1} &= R_1 g + R_2 h \\ G_l - G_{l-1} &= R_1 m + R_2 n \\ M_l - M_{l-1} &= R_1 u + R_2 w \end{aligned}$$
where $|g| - |h| = |m| - |n| = |u| - |w| = 0 = \langle g, h\rangle = \langle m, n\rangle = \langle u, w\rangle$.

The significance of the above setting is that one and the same pair R_1, R_2 of independent random variables appears in the representation of $\Delta F_n, \Delta G_n$ and ΔM_n.

Now we are ready to define the notion of Weierstrass martingales.

Let $(X_m, Y_m, Z_m) \colon \Omega \to \mathbf{R}^3$ be a martingale. If there exists a triple (F_m, G_m, M_m) of conformal martingales so that
$$\begin{aligned} X_m &= \sum_{n=1}^{m} \langle \Delta F_n, \Delta G_n\rangle, \\ Y_m &= \sum_{n=1}^{m} \left\langle \Delta F_n, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Delta G_n \right\rangle, \\ Z_m &= \sum \langle \Delta F_n, \Delta M_n\rangle, \end{aligned}$$
where $|\Delta M_n| \le \sqrt{\Delta G_n}$, then we say that (X_m, Y_m, Z_m) is a Weierstrass martingale and (F_m, G_m, M_n) is its Weierstrass representation.

Theorem 2. *There exists an uniformly bounded Weierstrass martingale with infinite variation in each direction.*

In the construction below we use again the Pythagorean theorem. We create oscillation to ensure infinite variation and we use uniform convexity, as expressed by the Pythagorean theorem when we give a pointwise estimate for the Weierstrass martingale. This remark is made precise in the following proposition, which describes our construction level by level. Theorem 2 follows from the next proposition.

Proposition 1. *There exist conformal martingales $F_k, G_k, M_k \colon [0,1] \to \mathbf{R}^2$ so that $|\Delta M_k| \le \sqrt{\Delta G_k}$ and so that the following holds.*

(1) *For each k, $|\Delta F_k| = k^{-2}$, and in $\mathbf{R}^2$ the vector*

$$\left(\langle \Delta F_k, \Delta G_k \rangle, \left\langle \Delta F_k, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Delta G_k \right\rangle \right)$$

has length $\frac{1}{k}$.

(2) *For each $w \in [0,1]$, the vector in $\mathbf{R}^2$ with the components*

$$\langle \Delta F_k(w), \Delta G_k(w) \rangle$$

and

$$\left\langle \Delta F_k(w), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Delta G_k(w) \right\rangle,$$

is orthogonal to $V = (V_1, V_2)$, where

$$\begin{aligned} V_1(w) &= \sum_{l=1}^{k-1} \langle \Delta F_l(w), \Delta G_l(w) \rangle, \\ V_2(w) &= \sum_{l=1}^{k-1} \left\langle \Delta F_l(w), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Delta G_l(w) \right\rangle. \end{aligned}$$

(3) *The partial sums*

$$\begin{aligned} &\sum_{l=1}^{k} \langle \Delta F_l, \Delta G_l \rangle \\ &\sum_{l=1}^{k} \left\langle \Delta F_l, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Delta G_l \right\rangle \\ &\sum_{l=1}^{k} \langle \Delta F_l, \Delta M_l \rangle \end{aligned}$$

form martingales.

Proof. We let $F_0 = G_0 = M_0 = 0$. Now we assume that $F_1, \dots, F_{k-1}$, $G_1, \dots, G_{k-1}$ and $M_1, \dots, M_{k-1}$ have been determinded, and we let $\mathcal{F}_{k-1}$ be the σ-algebra generated by these functions.

We fix an atom I in $\mathcal{F}_{k-1}$. Next we choose independent Rademacher functions r_1, r_2 which are also independent of $\mathcal{F}_{k-1}$. On I the martingales we whish to build are of the following form:

$$F_k - F_{k-1} = g r_1 + h r_2,$$

where $g, h \in \mathbf{R}^2$, $|g| - |h| = 0 = \langle h, g \rangle$;

$$G_k - G_{k-1} = mr_1 + nr_2,$$

where $m, n \in \mathbf{R}^2$, $|m| - |n| = 0 = \langle m, n \rangle$.

We will now determine the vectors g, h, m, n such that the conditions (1)–(3) are satisfied. We first make a rather arbitrary choice to select m and n, then we determine g, h depending on the history of $F_l(w), G_l(w)$, $w \in I$, $l \leq k-1$. Here we will use that $G_l(w) = G_l(I)$ and $F_l(w) = F_l(I)$ when $l \leq k-1$.

Finally we will choose ΔM_k depending on g and h. We start by putting $m = (k, 0)$ and $n = (0, k)$. Now we determine g, h such that the orthogonality condition (2) in Proposition 1 holds and so that the partial sums

$$\sum_{l=1}^{p} \langle \Delta F_l, \Delta G_l \rangle,$$
$$\sum_{l=1}^{p} \left\langle \Delta F_l, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Delta G_l \right\rangle$$

form martingales for $p \leq k$.

By choice of m, n we have

$$\langle \Delta F_k, \Delta G_k \rangle = k(g_1 + (h_1 + g_2) r_1 r_2 + h_2) \tag{16}$$

and

$$\left\langle \Delta F_k, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Delta G_k \right\rangle = k(g_2 + (h_2 - g_1) r_1 r_2 - h_1). \tag{17}$$

Hence, if we choose $g_1 = -h_2$ and $g_2 = h_1$ then the random variables

$$\langle \Delta F_k, \Delta G_k \rangle \quad \text{and} \quad \left\langle \Delta F_k, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Delta G_k \right\rangle$$

are martingale differences. Moreover this choice gives $|g| - |h| = 0 = \langle g, h \rangle$. We denote the quantities apearing in (16) resp. (17) by $\alpha(w)$ resp. $\beta(w)$. By our preliminary choice of g, h, we have for $w \in I$,

$$\begin{pmatrix} \alpha(w) \\ \beta(w) \end{pmatrix} = 2 r_1(w) r_2(w) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \tag{18}$$

Our plan is now to use the remaining degree of freedom to ensure that $V(w) = (V_1(w), V_2(w))$ is orthogonal to $(\alpha(w), \beta(w))$, whenever $w \in I$. By definition, $V(w)$ is measurable with respect to $\mathcal{F}_{k-1}$. Hence $V(w) = V(I)$ for $w \in I$. Now we choose $h = (h_1, h_2)$ to be orthogonal to $V(I)$. Consequently, by (18), the vector $(\alpha(w), \beta(w)) = \pm h$ is orthogonal to $V(I)$, when $w \in I$. Finally we let $|h_1| = |h_2| = 1/k^2$.

Summing up, we have now determined the coefficients of g, h, m, n. We have done this so that F_k and G_k are conformal martingales, and so that the assertions (1) and (2) of Proposition 1 hold. It remains to determine M_k so that, $\langle \Delta F_k, \Delta M_k\rangle$ is a martingale difference, and

$$M_k - M_{k-1} = ur_2 + wr_2,$$

where $u, w \in \mathbf{R}^2$, and $|u| - |w| = \langle u, w\rangle$.

Expanding the scalar product we obtain

$$\langle \Delta F_k, \Delta M_k\rangle = g_1u_1 + h_1w_1 + g_2u_2 + h_2w_2 + r_1r_2(g_1w_1 + h_1v_1 + g_2w_2 + h_2u_2).$$

As $g_1 = -h_2, g_2 = h_1$ we see that the choice $u_1 = -\alpha g_2$, $u_2 = \alpha g_2$, $w_1 = +\alpha g_1$ and $w_2 = +\alpha g_2$, makes $\langle \Delta F_k, \Delta M_k\rangle$ a martingale difference. Moreover we have $|u| - |w| = 0 = \langle u, w\rangle$, hence M_k is a conformal martingale. Finally we let $\alpha = \sqrt{k}/\sqrt{2}$ then $|\Delta M_k| = \sqrt{k} \le \sqrt{\Delta G_k}$. This completes the proof of Proposition 1.$\square$

Proof of Theorem 2. Let F_k, G_k and M_k be conformal martingales satisfying conditions (1), (2) and (3) of Proposition 1 and let (X_k, Y_k, Z_k) be the corresponding Weierstrass martingale. By Proposition 1 (1), for each $w \in [0,1]$ we have

$$\begin{aligned}\sum_{k=1}^{\infty} &|\Delta X_k|(w) + |\Delta Y_k|(w) + |\Delta Z_k|(w)\\ &\ge \sum_{l=1}^{\infty} |\langle \Delta F_l(w), \Delta G_l(w)\rangle| + \left|\left\langle \Delta F_l(w), \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} \Delta G_l(w)\right\rangle\right| = \infty.\end{aligned}$$

Again by condition (1) in Proposition 1, the martingale F_k is uniformly bounded, and for Z_k we have the simple estimate $|Z_k| \le \sum_{l=1}^{k} l^{-3/2}$.

To find bounds for X_k, Y_k, we use the orthogonality condition (2) in Proposition 1. We write

$$(X_k, Y_k) = (X_{k-1}, Y_{k-1}) + (X_k - X_{k-1}, Y_k - Y_{k-1})$$

and by Proposition 1.(2) for each $w \in [0,1]$, the vector $(X_{k-1}(w), Y_{k-1}(w))$ is orthogonal to $((X_k - X_{k-1})(w), (Y_k - Y_{k-1})(w))$ in $\mathbf{R}^2$. Hence by induction and the Pythagorean theorem:

$$|(X_k, Y_k)|^2 = \sum_{l=1}^{k} |(X_l - X_{l-1}, Y_l - Y_{l-1})|^2 = \sum_{l=1}^{k} l^{-2}.$$

This estimate completes the proof of Theorem 2.

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