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# WEAK CONGRUENCE SEMIDISTRIBUTIVITY LAWS AND THEIR CONJUGATES

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*Dedicated to the memory of Viktor Aleksandrovich Gorbunov*

Abstract. Lattice Horn sentences including Geyer's $SD(n, 2)$ and their conjugates $C(n, 2)$ are considered. $SD(2, 2)$ is the meet semidistributivity law $SD_\wedge$. Both $SD(n, 2)$ and $C(n, 2)$ become strictly weaker when $n$ grows. For varieties $\mathcal{V}$ the satisfaction of $SD(n, 2)$ in $\{\mathrm{Con}(A) : A \in \mathcal{V}\}$ is characterized by a Mal'cev condition. Using this Mal'cev condition it is shown that $C(n, 2) \models_{\mathrm{con}} SD(n, 2)$, which means that, for every variety $\mathcal{V}$, whenever $C(n, 2)$ holds in $\{\mathrm{Con}(A) : A \in \mathcal{V}\}$ then so does $SD(n, 2)$. In particular, $C(2, 2) \models_{\mathrm{con}} SD(2, 2)$, which is a stronger statement than $SD_\vee \models_{\mathrm{con}} SD_\wedge$, the only previously known $\models_{\mathrm{con}}$ result between lattice Horn sentences "not below congruence modularity". Some other $\models_{\mathrm{con}}$ statements are also presented.

## I. Introduction and the Main Results

This paper is primarily concerned with Mal'cev conditions and the consequence relation $\models_{\mathrm{con}}$ between lattice Horn sentences in congruence (quasi)varieties.

Given a variety $\mathcal{V}$ of algebras, the class of congruence lattices of members of $\mathcal{V}$ will be denoted by

$$\mathrm{Con}(\mathcal{V}) = \{\mathrm{Con}(A) : A \in \mathcal{V}\}.$$

By a (universal lattice) Horn sentence we mean a first order sentence

$$(1) \qquad (\forall x_0, \dots , x_{t-1}) \big((p_1 = q_1 \;\&\; \dots \;\&\; p_k = q_k) \Longrightarrow p = q\big)$$

where $p_1, \dots , p_k, q_1, \dots , q_k, p$ and $q$ are lattice terms of the variables $x_0, \dots , x_{t-1}$. Notice that using "$\leq$" instead of "$=$" in (1) would give the same notion modulo lattice theory. Lattice identities are special Horn sentences with $k = 0$ (or with $p_i = x_0$ and $q_i = x_0$ for all $i$). For convenience, lattice operations will be denoted by $+$

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(join) and $\cdot$ (meet); $\bigwedge$ and & will denote conjunctions. The join semidistributivity law

$$SD_\vee : \qquad x + y = x + z \Longrightarrow x + y = x + yz$$

and the meet semidistributivity law

$$SD_\wedge : \qquad xy = xz \Longrightarrow xy = x(y + z)$$

are the most known Horn sentences that are not equivalent to lattice identities.

For a lattice $H$ resp. class $H$ of lattices and a Horn sentence $\lambda$ let $H \models \lambda$ denote the fact that $\lambda$ holds in $H$ resp. in all members of $H$. The same symbol is used for the standard consequence relation between Horn sentences $\lambda$ and $\mu$: $\lambda \models \mu$ means that for every lattice $L$ if $L \models \lambda$ then $L \models \mu$. If $\mathrm{Con}(\mathcal{V}) \models \lambda$ implies $\mathrm{Con}(V) \models \mu$ for every variety $\mathcal{V}$ then the notation

$$\lambda \models_{\mathrm{con}} \mu$$

is used. The statement $\lambda \models_{\mathrm{con}} \mu$ is said to be nontrivial if $\lambda \not\models \mu$. This fact, i.e. the conjunction of $\lambda \models_{\mathrm{con}} \mu$ and $\lambda \not\models \mu$, will be denoted by $\lambda \models^{\mathrm{nt}}_{\mathrm{con}} \mu$. Starting with Nation [**22**], there are many results of the form $\lambda \models^{\mathrm{nt}}_{\mathrm{con}} \mu$, cf., e.g., Day [**6**], [**7**], Day and Freese [**8**], Freese, Herrmann and [**11**], Jónsson [**17**], [**18**], Mederly [**21**], and [**2**], with various lattice identities. (As a related deep result, Freese [**10**] is also worth mentionig here.) These results are "below congruence modularity" in the sense that modularity $\models_{\mathrm{con}} \mu$. The only known $\lambda \models^{\mathrm{nt}}_{\mathrm{con}} \mu$ type result not below congruence modularity is

$$SD_\vee \models^{\mathrm{nt}}_{\mathrm{con}} SD_\wedge \tag{2}$$

from Hobby and McKenzie [**14**, p. 112]. One of our goals is to strengthen (2) and, by generalizing (2), to present infinitely many $\lambda \models^{\mathrm{nt}}_{\mathrm{con}} \mu$ results not below modularity.

Given a lattice identity $\lambda$, the class of varieties $\{\mathcal{V} : \mathrm{Con}(\mathcal{V}) \models \lambda\}$ is a weak Mal'cev class by Wille [**26**] and Pixley [**24**]. In other words, (the satisfaction of) $\lambda$ (in congruence varieties) can be characterized by a weak Mal'cev condition. In many cases, all being covered by Chapter XIII in Freese and McKenzie [**12**], $\{\mathcal{V} : \mathrm{Con}(\mathcal{V}) \models \lambda\}$ is known to be a Mal'cev class. E.g., the distributivity resp. modularity are characterized by the famous Mal'cev conditions given by Jónsson [**16**] resp. Day [**5**].

Now let $\lambda$ be a Horn sentence. Then $\{\mathcal{V} : \mathrm{Con}(\mathcal{V}) \models \lambda\}$ is known to be a weak Mal'cev class only in certain cases described in [**3**]; these cases include $SD_\wedge$ and $SD_\vee$. Using commutator theory, Lipparini [**20**] and Kearnes and Szendrei [**19**] have recently proved that $\{\mathcal{V} : \mathrm{Con}(\mathcal{V}) \models SD_\wedge\}$ is a Mal'cev class. For a direct approach (and also for an important application of the corresponding Mal'cev condition) cf. Willard [**25**], and cf. also Hobby and McKenzie [**14**] for the locally finite case. Using ideas from [**1**], [**3**] and [**25**] we present Mal'cev conditions for infinitely many Horn sentences. These Mal'cev conditions provide the key to our $\lambda \models^{\mathrm{nt}}_{\mathrm{con}} \mu$ type achievements.

For $n \geq 2$ put $\mathbf{n} = \{0, 1, \ldots, n-1\}$ and let $P_2(\mathbf{n})$ denote $\{S : S \subseteq \mathbf{n}$ and $|S| \geq 2\}$. For $\emptyset \neq H \subseteq P_2(\mathbf{n})$ we define the generalized meet semidistributivity law $SD(n, H)$ as follows:

$$\alpha\beta_0 = \alpha\beta_1 = \ldots = \alpha\beta_{n-1} \Longrightarrow \alpha \prod_{I \in H} \sum_{i \in I} \beta_i \leq \beta_0.$$

Equivalently, $SD(n, H)$ is

$$\alpha\beta_0 = \alpha\beta_1 = \ldots = \alpha\beta_{n-1} \Longrightarrow \alpha\beta_0 = \alpha \prod_{I \in H} \sum_{i \in I} \beta_i.$$

When $H = \{S \in P_2(\mathbf{n}) : |S| = 2\}$, $SD(n, H)$ will be denoted by $SD(n, 2)$. Notice that

$$SD(n,2): \qquad \alpha\beta_0 = \alpha\beta_1 = \ldots = \alpha\beta_{n-1} \Longrightarrow \alpha \prod_{0 \leq i < j < n} (\beta_i + \beta_j) \leq \beta_0$$

has been studied by Geyer [**13**], and $SD(2, 2)$ is exactly $SD_\wedge$.

Now with $SD(n, H)$ we associate its conjugate Horn sentence $C(n, H)$ as follows. Let $\alpha$ and $\beta_{i,I}$ $(i \in I \in H)$ be the variables of $C(n, H)$. Denoting $\{I \in H : j \in I\}$ by $H_j$, $C(n, H)$ is

$$\bigwedge_{I \in H} \Big( (\alpha \leq \sum_{i \in I} \beta_{i,I}) \;\&\; \bigwedge_{i \in I} (\beta_{i,I} \leq \alpha + \sum_{j \in I \setminus \{i\}} \beta_{j,I}) \Big) \Longrightarrow$$
$$\alpha \leq \sum_{I \in H_0} \beta_{0,I} + \alpha\big(\sum_{I \in H_1} \beta_{1,I} + \alpha\big(\sum_{I \in H_2} \beta_{2,I} + \alpha\big(\ldots + \alpha \sum_{I \in H_{n-1}} \beta_{n-1,I}\big)\ldots\big).$$

The conjugate of $SD(n, 2)$ is denoted by $C(n, 2)$; it is the following Horn sentence:

$$\Big( \bigwedge_{i<j}^{0,n-1} (\alpha \leq \beta_{ij} + \beta_{ji}) \;\&\; \bigwedge_{i \neq j}^{0,n-1} (\beta_{ij} \leq \alpha + \beta_{ji}) \Big) \Longrightarrow$$
$$\alpha \leq \sum_{j \neq 0}^{0,n-1} \beta_{0j} + \alpha\big(\sum_{j \neq 1}^{0,n-1} \beta_{1j} + \alpha\big(\sum_{j \neq 2}^{0,n-1} \beta_{2j} + \alpha\big(\ldots \alpha \sum_{j \neq n-1}^{0,n-1} \beta_{n-1,j}\big)\ldots\big).$$

For example, $C(2, 2)$, the conjugate of $SD_\wedge$, is (clearly equivalent to):

(3) $$C(2,2): \qquad x + y = x + z = y + z \Longrightarrow x + y = x + yz.$$

Our main results are as follows; the proofs will be given in the next chapter.

**Theorem 1.** *For every* $n \geq 2$ *and* $\emptyset \neq H \subseteq P_2(\mathbf{n})$, $\{\mathcal{V} : \mathcal{V}$ *is a variety and* $\mathrm{Con}(\mathcal{V}) \models SD(n, H)\}$ *is a Mal'cev class.*

A concrete Mal'cev condition will be given in Theorem 9.

**Theorem 2.** *For every* $n \geq 2$ *and* $\emptyset \neq H \subseteq P_2(\mathbf{n})$, $C(n, H) \models_{\mathrm{con}} SD(n, H)$.

**Theorem 3.** *For every* $n \geq 2$ *and* $\emptyset \neq H \subseteq P_2(\mathbf{n})$, *(*$SD(n, H)$ *and modularity)* $\models_{\mathrm{con}}$ *distributivity.*

To justify the notation used in Theorem 3 let us mention that the conjunction of two Horn sentences is equivalent to a single Horn sentence modulo lattice theory. While ($SD_\wedge$ and modularity) $\models$ distributivity, the five element nonmodular lattice $M_3$ witnesses that ($SD(n, 2)$ and modularity) $\not\models$ distributivity for $n > 2$. Hence $\models_{\mathrm{con}}$ in Theorem 3 is nontrivial in many cases. The same is true for Theorem 2, as it is pointed out by the following

**Corollary 4.** *For every* $n \geq 2$, $C(n, 2) \models^{\mathrm{nt}}_{\mathrm{con}} SD(n, 2)$.

Notice that $C(2, 2)$ is a weaker Horn sentence than $SD_\vee$. Indeed, $SD_\vee \models C(2, 2)$ is trivial, and $C(2, 2) \not\models SD_\vee$ is witnessed by

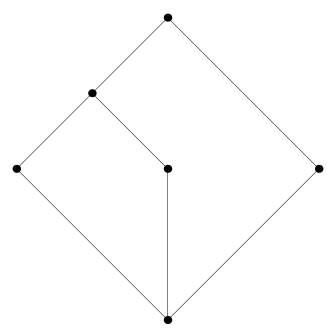

**Figure 1.**

Hence Corollary 4 for $n = 2$ is a stronger result than (2), and it is worth separate formulating.

**Corollary 5.** $C(2, 2) \models^{\mathrm{nt}}_{\mathrm{con}} SD_\wedge$.

Now we formulate a statement on the relations among the Horn sentences $C(n, H)$ and $SD(n, H)$.

**Proposition 6.** *Let* $k > 2$, $m \geq 2$, $n \geq 2$, $\emptyset \neq H \subseteq P_2(\mathbf{n})$ *and* $\emptyset \neq K \subseteq P_2(\mathbf{m})$. *Then*

(a) $SD(k, 2)$ *is strictly weakening in* $k$, *i.e.,* $SD(k - 1, 2) \models SD(k, 2)$ *but* $SD(k, 2) \not\models SD(k - 1, 2)$*;*

(b) *$C(k,2)$ is strictly weakening in $k$, i.e., $C(k-1,2) \models C(k,2)$ but $C(k,2) \not\models C(k-1,2)$;*
(c) *$SD(2,2) \models SD(n,H)$;*
(d) *$SD(m,K) \not\models C(n,H)$;*
(e) *$C(m,2) \not\models SD(n,H)$ and, moreover, $SD_\vee \not\models SD(n,H)$.*

Since Proposition 6 does not answer all questions, the remarks concluding the paper will add some further information. Part (d) of Proposition 6 can be strengthened to

**Theorem 7.** *For any $m,n \geq 2$, $\emptyset \neq K \subseteq P_2(\mathbf{m})$ and $\emptyset \neq H \subseteq P_2(\mathbf{n})$ we have $SD(m,K) \not\models_{\mathrm{con}} C(n,H)$.*

The Mal'cev conditions we are going to present in the following chapter are far from being simple. However, they are useful to prove Theorems 2 and 3. Interestingly enough, for all known $\lambda \models^{\mathrm{nt}}_{\mathrm{con}} \mu$ statement $\{\mathcal{V} : \mathrm{Con}(\mathcal{V}) \models \mu\}$ is known to be a Mal'cev class (even if $\lambda \models_{\mathrm{con}} \mu$ was proved or can be proved without Mal'cev conditions). The proof of Theorem 7 is also based on our Mal'cev condition, and resorting to Theorem 7 is, at present, the only way to prove (d) of Proposition 6. On the other hand, we could not solve the naturally arising problem if $SD(n,2) \models_{\mathrm{con}} SD(n-1,2)$ is true or not.

## II. Proofs and Technical Statements

Like in some previous papers, e.g. in [**1**] and [**3**], our Mal'cev conditions will be given by certain graphs. This is not just an economic way to establish the appropriate Mal'cev conditions, it is also a possible way to work with them. For any lattice term $p(\alpha_0, \dots, \alpha_{n-1})$ and integer $k \geq 2$ we define a graph $G_k(p)$ associated with $p$. The edges of $G_k(p)$ will be coloured by the variables $\alpha_0, \dots, \alpha_{n-1}$, and two distinguished vertices, the so-called left and right **endpoints**, will have special roles. In figures, the endpoints will always be placed on the left-hand side and on the right-hand side, respectively. By $E(G_k(p))$ we denote the edge set of

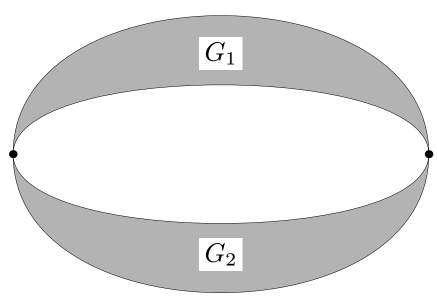


**Figure 2.**

$G_k(p)$. An $\alpha$-coloured edge connecting the vertices $x$ and $y$ will often be denoted by $(x, \alpha, y)$. Before defining $G_k(p)$ we introduce two kinds of operations for graphs. We obtain the **parallel connection** of graphs $G_1$ and $G_2$ by taking disjoint copies of $G_1$ and $G_2$ and identifying their left (right, resp.) endpoints, cf. Figure 2.

By taking disjoint graphs $H_1, \dots, H_k$ ($k \geq 2$) such that $H_i \cong G_1$ for $i$ odd and $H_i \cong G_2$ for $i$ even, and identifying the right endpoint of $H_i$ and the left endpoint of $H_{i+1}$ for $i = 1, 2 \dots, k-1$ we obtain the **serial connection** of length $k$ of $G_1$ and $G_2$. (The left endpoint of $H_1$ and the right one of $H_k$ are the endpoints of the serial connection, cf. Figure 3.)

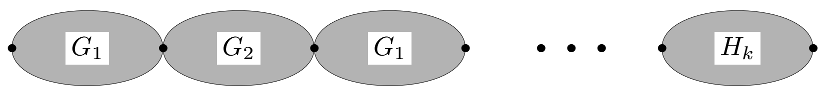


**Figure 3.**

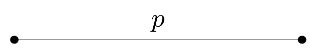


**Figure 4.**

Now, if $p$ is a variable then, for any $k \geq 2$, let $G_k(p)$ be the graph depicted in Figure 4, which consists of a single edge coloured by $p$. Let $G_k(p_1 + p_2)$ resp. $G_k(p_1 p_2)$ be the serial connection of length $k$ resp. the parallel connection of graphs $G_k(p_1)$ and $G_k(p_2)$. Now we have defined $G_k(p)$ for lattice terms $p$ with **binary** operations. However, $p$ is often given by means of $\sum$ and $\prod$ as well. Then we always assume a fixed binary representation of $p$. Although each fixed binary form makes the rest of the paper work and the corresponding $G_2(p)$ does not depend too much on this form, we note that $G_k(p)$ ($k \geq 3$) heavily depends on the binary representation chosen. E.g., $G_3\big((\beta_0 + \beta_1) + \beta_2\big)$ has eight vertices while $G_3\big(\beta_1 + (\beta_2 + \beta_0)\big)$ has only six.

For an algebra $A$, a lattice term $p = p(\alpha_0, \dots, \alpha_{n-1})$, congruences $\hat{\alpha}_0, \dots, \hat{\alpha}_{n-1} \in \mathrm{Con}(A)$, $a_0, a_1 \in A$ and $k \geq 2$ we say that $a_0$ and $a_1$ **can be connected by** $G_k(p)$ **in the algebra** $A$ if there is a map $\varphi$ (referred to as the connecting map) from the vertex set of $G_k(p)$ into $A$ such that $a_0$ and $a_1$ are the images of the left and right endpoints, respectively, and for every edge $(x, \alpha_i, y) \in E(G_k(p))$ we have $\big(\varphi(x), \varphi(y)\big) \in \hat{\alpha}_i$. If it is necessary, we can emphasize that the colour $\alpha_i$ is represented by the congruence $\hat{\alpha}_i$. The following statement from [**3**] was proved by an easy induction.

**Lemma 8.** *With the above notations,* $(a_0, a_1) \in p(\hat{\alpha}_0, \dots, \hat{\alpha}_{n-1})$ *iff* $a_0$ *and* $a_1$ *can be connected by* $G_k(p)$ *in* $A$ *for some* $k \geq 2$ *iff there is a* $k_0 \geq 2$ *such that* $a_0$ *and* $a_1$ *can be connected by* $G_k(p)$ *in* $A$ *for all* $k \geq k_0$.

Now with any pair of (finite coloured) graphs $G'$ and $G''$ we associate a strong Mal'cev condition $U(G' \leq G'')$ in the following way, cf. [**3**]. Let $\alpha_0, \dots, \alpha_{n-1}$ be the colours occurring on edges of $G'$ and $G''$, and let $X = \{x_0, x_1, \dots, x_{t-1}\}$ and $F = \{f_0, f_1, \dots\}$ be the vertex sets of $G'$ and $G''$, respectively, with $x_0, x_1, f_0, f_1$ being the endpoints. For $0 \leq j \leq t-1$ and $0 \leq i \leq n-1$ let $\alpha_i(j)$ be the smallest $s$ such that there is an $\alpha_i$-coloured path in $G'$ connecting $x_j$ and $x_s$. (By convention, the empty path connecting $x_j$ with itself is $\alpha_i$-coloured.) Now $U(G' \leq G'')$ is defined to be the following (strong Mal'cev) condition:

> "There exist $t$-ary terms $f(x_0, \dots, x_{t-1})$ $(f \in F)$ which satisfy (1) the **endpoint identities** $f_0(x_0, \dots, x_{t-1}) = x_0$ and $f_1(x_0, \dots, x_{t-1}) = x_1$, and (2) for every edge $(f, \alpha_i, g) \in E(G'')$ the corresponding identity $f(x_{\alpha_i(0)}, x_{\alpha_i(1)}, \dots, x_{\alpha_i(t-1)}) = g(x_{\alpha_i(0)}, x_{\alpha_i(1)}, \dots, x_{\alpha_i(t-1)})$."

The identity associated with the edge $(f, \alpha_i, g)$ above will often be denoted by $I(f, \alpha_i, g)$.

Now let $n \geq 2$ be fixed, and define lattice terms $\beta_i^{(k)} = \beta_i^{(k)}(\alpha, \beta_0, \dots, \beta_{n-1})$, $0 \leq i < n$, $0 \leq k$, via induction as follows. Let $\beta_i^{(0)} = \beta_i$, and let $\beta_i^{(j+1)} = \beta_i + \alpha\beta_{i+1}^{(j)}$. Here the subscript $i+1$ is understood modulo $n$, and the same convention applies for subscripts of $\beta$ in the sequel. Theorem 1 is an easy consequence of the following theorem.

**Theorem 9.** *Let $n \geq 2$ and $\emptyset \neq H \subseteq P_2(\mathbf{n})$. Then, for an arbitrary variety $\mathcal{V}$, the following three conditions are equivalent.*

*(i)* $\mathrm{Con}(\mathcal{V}) \models SD(n, H)$.

*(ii) The Mal'cev condition*

$$\text{“there is a } k \geq 2 \text{ such that } U_k := U\big(G_2(\alpha \prod_{I \in H} \sum_{i \in I} \beta_i) \leq G_k(\beta_0^{(k)})\big)\text{”}$$

*holds in $\mathcal{V}$.*

*(iii)* $(x_0, x_1) \in \beta_0^{(k)}(\hat\alpha, \hat\beta_0, \dots, \hat\beta_{n-1})$ *for some $k$ where $X$ is the vertex set of $G_2 = G_2(\alpha \prod_{I \in H} \sum_{i \in I} \beta_i)$, $x_0$ and $x_1$ are the endpoints, $\hat\alpha$ resp. $\hat\beta_i$ denote the congruence generated by $\{(x, y) \in X^2 : (x, \alpha, y) \in E(G_2)\}$ resp. $\{(x, y) \in X^2 : (x, \beta_i, y) \in E(G_2)\}$ in the free algebra $F_{\mathcal{V}}(X)$.*

*Proof.* (i) $\Longrightarrow$ (iii): Let $A = F_{\mathcal{V}}(X)$. With the notation $\hat\beta_i^{(k)} = \beta_i^{(k)}(\hat\alpha, \hat\beta_0, \dots, \hat\beta_{n-1})$, an evident induction gives $\hat\beta_i^{(0)} \subseteq \hat\beta_i^{(1)} \subseteq \hat\beta_i^{(2)} \subseteq \dots$ for $0 \leq i < n$. Hence $\hat\beta_i^{(\omega)} := \bigcup_{k=0}^{\infty} \hat\beta_i^{(k)} \in \mathrm{Con}(A)$. Suppose $(a, b) \in \hat\alpha \cap \hat\beta_i^{(\omega)}$. Then $(a, b) \in \hat\alpha \cap \hat\beta_i^{(k)}$ for some $k$, which gives $(a, b) \in \hat\alpha \cap \hat\beta_{i-1}^{(k+1)} \subseteq \hat\alpha \cap \hat\beta_{i-1}^{(\omega)}$ for all $i$, i.e.,

$$\hat\alpha \cap \hat\beta_0^{(\omega)} \supseteq \hat\alpha \cap \hat\beta_1^{(\omega)} \supseteq \dots \supseteq \hat\alpha \cap \hat\beta_{n-1}^{(\omega)} \supseteq \hat\alpha \cap \hat\beta_0^{(\omega)}.$$

Hence all the $\hat\alpha \cap \hat\beta_i^{(\omega)}$ are equal, and (i) gives $\hat\alpha \prod_{I\in H} \sum_{i\in I} \hat\beta_i^{(\omega)} \le \hat\beta_0^{(\omega)}$. Using Lemma 8 we conclude

$$(x_0, x_1) \in \hat\alpha \prod_{I\in H} \sum_{i\in I} \hat\beta_i \subseteq \hat\alpha \prod_{I\in H} \sum_{i\in I} \hat\beta_i^{(\omega)} \subseteq \hat\beta_0^{(\omega)}.$$

Hence $(x_0, x_1) \in \hat\beta_0^{(k)} = \beta_0^{(k)}(\hat\alpha,\ \hat\beta_0, \dots, \hat\beta_{n-1})$ for some $k$, i.e., (iii) holds.

(iii) $\Longrightarrow$ (ii): Suppose (iii). By Lemma 8, $x_0$ and $x_1$ can be connected by $G_t(\beta_0^{(k)})$ in $F_{\mathcal V}(X)$ for some $t \ge 2$. Since $\beta_0^{(k)} \le \beta_0^{(k+1)}$ in all lattices, it is not hard to see that both $k$ and $t$ can be enlarged, and therefore $t = k$ can be assumed$^\dagger$. Now the routine technique of deriving strong Mal'cev conditions, cf. e.g. Wille [**26**], Pixley [**24**] and [**3**], yields that $U_k$ holds in $\mathcal V$.

(ii) $\Longrightarrow$ (i): Suppose $k \ge 2$, $U_k$ holds in $\mathcal V$, $A \in \mathcal V$, $\hat\alpha, \hat\beta_0, \dots, \hat\beta_{n-1} \in \mathrm{Con}(A)$ and $\hat\alpha\hat\beta_0 = \dots = \hat\alpha\hat\beta_{n-1}$. Let $(a_0, a_1)$ belong to $\hat\alpha \prod_{I\in H} \sum_{i\in I} \hat\beta_i$; we have to show that $(a_0, a_1) \in \hat\beta_0$. By Lemma 8, there is an $s \ge 2$ such that $a_0$ and $a_1$ can be connected by $G_s(\alpha \prod_{I\in H} \sum_{i\in I} \beta_i)$ in $A$. Hence there are finitely many elements $c_{I,0} = a_0$, $c_{I,1}, \dots, c_{I,m_I} = a_1$ for each $I \in H$ such that $(c_{I,j}, c_{I,j+1}) \in \bigcup_{i\in I} \hat\beta_i$ for $0 \le j < m_I$.

Now $G_2(\alpha \prod_{I\in H} \sum_{i\in I} \beta_i)$ is depicted in Figure 5 where $I, J \ldots \in H$, $I = \{i_1 < i_2 < i_3 < \dots\}$ and $J = \{j_1 < j_2 < j_3 < \dots\}$. The inner (i.e., not endpoint) vertices of this graph are denoted by $y_{I,1}$, $y_{I,2}, \dots$ $(I \in H)$; the corresponding variables in the Mal'cev condition $U_k$ are called **inner variables**.

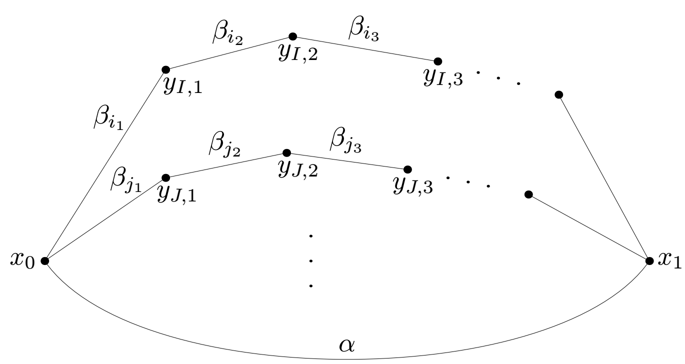


**Figure 5.**

Now we define some subgraphs, referred to as **permitted subgraphs**, of $G_k(\beta_0^{(k)})$. The only permitted subgraph of height $k$ is $G_k(\beta_0^{(k)})$ itself. By definition, $G_k(\beta_0^{(k)})$ is a serial connection of length $k$ of $G_k(\alpha\beta_1^{(k-1)})$ and the single edge graph $G_k(\beta_0)$; the copies of $G_k(\alpha\beta_1^{(k-1)})$ in the serial connection are the permitted subgraphs of height $k-1$. Each copy of $G_k(\beta_1^{(k-1)})$, i.e. each permitted

$^\dagger$Essentially by the same reason, $U_k \models U_{k+1}$, i.e., "$(\exists k)(U_k)$" is a Mal'cev condition, indeed.

subgraph of height $k-1$ without its $\alpha$-edge connecting its endpoints, is a serial connection of length $k$ of $G_k(\beta_1)$ and $G_k(\alpha\beta_2^{(k-2)})$; the copies of $G_k(\alpha\beta_2^{(k-2)})$ are the permitted subgraphs of height $k-2$. And so on, for $0 \le j < k$, the permitted subgraphs of height $j$ are isomorphic to $G_k(\alpha\beta_{k-j}^{(j)})$, and each of them is a subgraph of a permitted subgraph of height $j+1$. (Of course, according to our general agreement, the subscript $k-j$ is understood modulo $n$.) In particular, the permitted subgraphs of height 0 are isomorphic to $G_k(\alpha\beta_k^{(0)}) = G_2(\alpha\beta_k)$. For $k = 4$ the situation is outlined in Figure 6. The expression "permitted subgraph" will mean a permitted subgraph of $G_k(\beta_0^{(k)})$ of height $j$ for some $0 \le j \le k$.

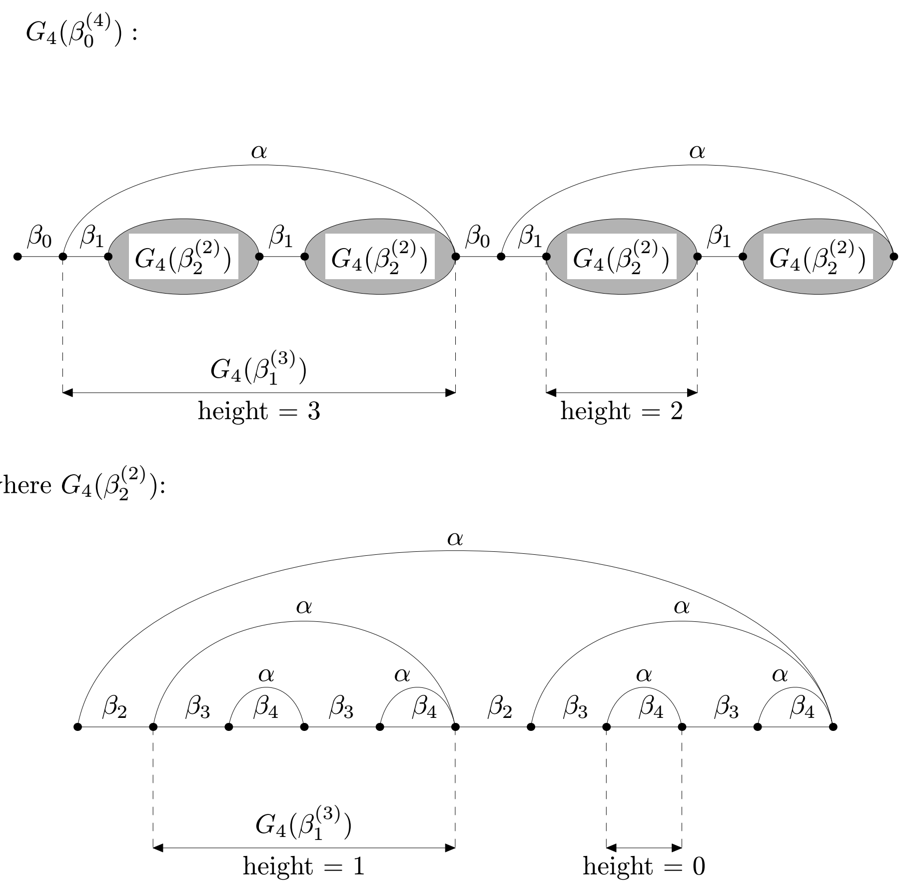


**Figure 6.**

The term symbols in the strong Mal'cev condition $U_k$ are vertices in $G_k(\beta_0^{(k)})$, so they are endpoints of permitted subgraphs; this fact will be utilized in the sequel. Let $m = 2 + \sum_{I\in H}(|I|-1)$, the number of vertices in $G_2(\alpha\prod_{I\in H}\sum_{i\in I}\beta_i)$.

**Claim 10.** *Let $f$ and $g$ be the endpoints of a permitted subgraph and let*

$$\vec{u} = (a_0, a_1, d_2, \dots , d_{m-1}) \in \{a_0\} \times \{a_1\} \times A^{m-2}$$

*be arbitrary. Then $f(\vec{u})\ \hat{\alpha}\ g(\vec{u})$.*

Since $(f, \alpha, g)$ is an edge of the permitted subgraph in question, using the identity $I(f, \alpha, g)$ associated with this edge we obtain

$$f(\vec{u})\ \hat{\alpha}\ f(a_0, a_0, d_2, \dots , d_{m-1}) = g(a_0, a_0, d_2, \dots , d_{m-1})\ \hat{\alpha}\ g(\vec{u}),$$

proving Claim 10.

**Claim 11.** *Let $f$ and $g$ be the endpoints of a permitted subgraph. If there exists a $\vec{u} \in \{a_0\}\times\{a_1\}\times\{a_0, a_1\}^{m-2}$ with $f(\vec{u})\ \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{n-1}\ g(\vec{u})$ then $f(\vec{v})\ \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{n-1}$ $g(\vec{v})$ holds for all $\vec{v} \in \{a_0\} \times \{a_1\} \times \{a_0, a_1\}^{m-2}$.*

It suffices to show that if $2 \le i < m$ and the $i$-th component of $\vec{u} = (a_0,\, a_1,\, u_2,\, \dots,\, u_{m-1})$ is $u_i = a_0$ then $f(\vec{v})\ \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{n-1}\ g(\vec{v})$ holds for $\vec{v} = (a_0,\, a_1,\, u_2,\, \dots,\, u_{i-1},\, a_1,\, u_{i+1},\, \dots,\, u_{m-1})$. Fix an $I \in H$ and consider the $m$-tuples $\vec{w}^{(j)} = (a_0,\, a_1,\, u_2,\, \dots,\, u_{i-1},\, c_{I,j},\, u_{i+1},\, \dots,\, u_{m-1})$, $j = 0, 1, \dots , m_I$. Then $\vec{w}^{(0)} = \vec{u}$ and $\vec{w}^{(m_I)} = \vec{v}$, so it suffices to show via induction that for all $j \le m_I$

$$f(\vec{w}^{(j)})\ \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{n-1}\ g(\vec{w}^{(j)}). \tag{4}$$

When $j = 0$, (4) states what we have assumed. Now suppose (4) for some $j < m_I$. Since $(c_{I,j}, c_{I,j+1}) \in \bigcup_{\ell \in I} \hat{\beta}_\ell$, there is an $\ell \in I$ with $(c_{I,j}, c_{I,j+1}) \in \hat{\beta}_\ell$, and we have $f(\vec{w}^{(j)})\ \hat{\beta}_\ell\ f(\vec{w}^{(j+1)})$ and $g(\vec{w}^{(j)})\ \hat{\beta}_\ell\ g(\vec{w}^{(j+1)})$. Using (4) for $j$ and transitivity we infer $f(\vec{w}^{(j+1)})\ \hat{\beta}_\ell\ g(\vec{w}^{(j+1)})$. By Claim 10, $f(\vec{w}^{(j+1)})\ \hat{\alpha}\ g(\vec{w}^{(j+1)})$. Since $\hat{\alpha}\hat{\beta}_0 = \dots = \hat{\alpha}\hat{\beta}_{m-1}$, we conclude (4) for $j + 1$. We have shown that $a_0$ can be changed to $a_1$ at the $i$th component; the transition from $a_1$ to $a_0$ follows similarly. This proves Claim 11.

**Claim 12.** *Let $f$ and $g$ be the endpoints of a permitted subgraph $S$. Then for all $\vec{u} \in \{a_0\} \times \{a_1\} \times \{a_0, a_1\}^{m-2}$ we have $f(\vec{u})\ \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{n-1}\ g(\vec{u})$.*

We prove this claim via induction on the height of $S$. Suppose $S$ is of height 0, i.e., $S = G_k(\alpha\beta_k)$. We define $\vec{u} = (u_0, \dots , u_{m-1}) \in \{a_0\} \times \{a_1\} \times \{a_0, a_1\}^{m-2}$ as follows. Let $u_0 = a_0$, and for all edge $(x_0, \beta_k, y_{I,1}) \in E\big(G_2(\alpha \prod_{I \in H} \sum_{i \in I} \beta_i)\big)$, cf. Figure 5, let the component of $\vec{u}$ corresponding to $y_{I,1}$ be $a_0$. Let the rest of the components be defined as $a_1$. Since $2 \le |I|$ for all $I \in H$, for each $\beta_k$-coloured edge of $G_2(\alpha \prod_{I \in H} \sum_{i \in I} \beta_i)$ the components of $\vec{u}$ corresponding to the endpoints of this edge are equal. Hence the identity $I(f, \beta_k, g)$ applies and we obtain $f(\vec{u}) = g(\vec{u})$. This gives $f(\vec{u})\ \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{n-1}\ g(\vec{u})$ for one $\vec{u}$, whence it holds for all $\vec{u}$ in virtue of Claim 11.

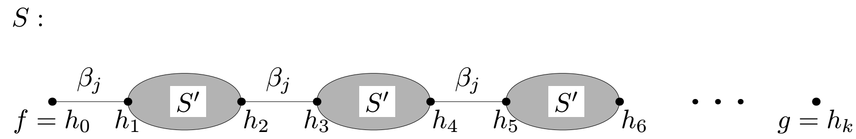


**Figure 7.**

Now let $S$ be of height $k-j$, $0 \le j < k$. Then $S$ is a serial connection of length $k$ of graphs $G_k(\beta_j)$ and $S' = G_k(\alpha\beta_{j+1}^{(k-j-1)})$. Let $h_0 = f$, $h_1$, ..., $h_k = g$ be the endpoints of copies of $G_k(\beta_j)$ and $S'$ in this serial connection, cf. Figure 7.

As previously, we can choose a $\vec{u} \in \{a_0\}\times\{a_1\}\times\{a_0,a_1\}^{m-2}$ such that, applying the identity associated with $(h_t, \beta_j, h_{t+1}) \in E(S)$, we obtain $h_t(\vec{u}) = h_{t+1}(\vec{u})$ for $t$ even, $0 \le t < k$. Since each copy of $S'$ in Figure (7) is a permitted subgraph of height $k-j-1$, the induction hypothesis yields $h_t(\vec{u})\ \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{m-1}\ h_{t+1}(\vec{u})$ for $0 < t < k$, $t$ odd. By transitivity, $\big(f(\vec{u}), g(\vec{u})\big) = \big(h_0(\vec{u}), h_k(\vec{u})\big) \in \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{n-1}$. This holds for one carefully chosen $\vec{u}$, whence for all $\vec{u} \in \{a_0\}\times\{a_1\}\times\{a_0,a_1\}^{m-2}$ in virtue of Claim 11. Claim 12 has been shown.

Now let us apply Claim 12 for the whole graph $G_k(\hat{\beta}_0^{(k)})$ with endpoints $f_0$ and $f_1$; we obtain $(a_0, a_1) = \big(f_0(\vec{u}), f_1(\vec{u})\big) \in \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{m-1} \subseteq \hat{\beta}_0$ for arbitrary $\vec{u} \in \{a_0\} \times \{a_1\} \times \{a_0, a_1\}^{m-2}$. This proves (ii) $\Longrightarrow$ (i) and Theorem 9. □

*Proof of Theorem* 2. Let $\mathcal{V}$ be a variety with $\mathrm{Con}(\mathcal{V}) \models C(n,H)$, and let us consider the graph $G_2(\alpha \prod_{I\in H} \sum_{i\in I} \beta_i)$, cf. Figure 5. The vertex set of this graph is denoted by $X$. For $i \in I \in H$, the path $x_0$, $y_{I,1}$, $y_{I,2}$, ..., $x_1$ contains a unique $\beta_i$-coloured edge; let $\hat{\beta}_{i,I}$ be the smallest congruence of the free algebra $F_{\mathcal{V}}(X)$ that collapses the endpoints of this edge. The congruence generated by $(x_0, x_1)$ is denoted by $\hat{\alpha}$. Clearly, $\hat{\alpha}$ and the $\hat{\beta}_{i,I}$ ($i \in I$, $I \in H$) satisfy the premise of $C(n,H)$. Since $C(n,H)$ holds in $\mathrm{Con}(F_{\mathcal{V}}(X))$,

$$(x_0, x_1) \in \hat{\alpha} \le \hat{\beta}_0 + \hat{\alpha}(\hat{\beta}_1 + \hat{\alpha}(\hat{\beta}_2 + \dots + \hat{\alpha}\hat{\beta}_{n-1})\dots) \tag{5}$$

where $\hat{\beta}_i := \sum_{I\in H_i} \hat{\beta}_{i,I}$ ($0 \le i < n$, $H_i = \{I \in H : i \in I\}$). Notice that the right-hand side of (5) is just $\beta_0^{(n-1)}(\hat{\alpha}, \hat{\beta}_0, \dots, \hat{\beta}_{n-1})$, and $\hat{\alpha}$, $\hat{\beta}_0$, ..., $\hat{\beta}_{n-1}$ are exactly the congruences occurring in (iii) of Theorem 9. Hence $\mathrm{Con}(\mathcal{V}) \models SD(n,H)$ by Theorem 9. The proof is complete. □

*Proof of Theorem* 3. Suppose, to obtain a contradiction, that $\mathcal{V}$ is a congruence modular but not congruence distributive variety such that $\mathrm{Con}(\mathcal{V}) \models SD(n,H)$. Let $k := 1 + \sum_{I\in H} |I-1| = |X| - 1$ where $X$ is the vertex set of $G_2 := G_2(\alpha \prod_{I\in H} \sum_{i\in I} \beta_i)$, cf. Figure 5. Since $|I| \ge 2$ for $I \in H$, $k \ge 2$. If $k = 2$ then $SD(n,H)$ is equivalent to $SD_\wedge$ modulo lattice theory, and the theorem follows from ($SD_\wedge$ and modularity) $\models$ distributivity. Thus we can assume that $k \ge 3$.

Now, recalling Huhn's lattice identity

$$\text{dist}_k: \qquad x\sum_{i=0}^{k} y_i = \sum_{j=0}^{k}\big(x\sum_{i\neq j}^{0,\,k} y_i\big),$$

it is known that $\text{dist}_k \models_{\text{con}}$ distributivity, cf. Nation [**22**]. Therefore $\text{Con}(\mathcal{V}) \not\models \text{dist}_k$, so we can take an algebra $A \in \mathcal{V}$ with $\text{Con}(A) \not\models \text{dist}_k$. We conclude from Huhn [**15**, Thm. 1.1(C)] that there is a prime field $K$ such that $L(PG_k(K))$, the subspace lattice of the $k$-dimensional projective geometry over $K$, is a sublattice of $\text{Con}(A)$. Let $M$ be the vector space over $K$ freely generated by $X$. Then $L(PG_k(K))$ is isomorphic to $L(M)$, the subspace lattice of $M$, so we conclude that $SD(n, H)$ holds in $L(M)$.

Now the desired contradiction proving Theorem 3 is supplied by the following statement.

**Claim 13.** *$SD(n, H)$ fails in the subspace lattice $L(M)$ defined above.*

Indeed, for $0 \le i < n$, let $\hat\beta_i \in L(M)$ be the subspace spanned by $\{u - v : (u, \beta_i, v) \in E(G_2)\}$, and let $\hat\alpha := K(x_1 - x_0)$, the (cyclic) subspace spanned by $\{u - v : (u, \alpha, v) \in E(G_2)\} = \{x_0 - x_1\}$. Since for each edge $(u, \beta_i, v)$ either $u$ or $v$ is an endpoint of no other $\beta_i$-coloured edge, and $\{u, v\} \neq \{x_0, x_1\}$, it is easy to conclude that $x_1 - x_0 \notin \hat\beta_i$. Hence $\hat\alpha\hat\beta_0 = \ldots = \hat\alpha\hat\beta_{n-1} = 0$. By the construction, $x_1 - x_0 \in \hat\alpha \prod_{I\in H}\sum_{i\in I}\hat\beta_i$ but $x_1 - x_0 \notin \hat\beta_0$. So $SD(n, H)$ fails in $L(M)$. This proves Claim 13 and Theorem 3. □

*Proof of Proposition* 6. (a) $SD(k-1, 2) \models SD(k, 2)$ is evident. It is easy to see that $SD(k, 2)$ holds for any $k+1$ elements in a lattice that do not form an antichain. Let $M_k$ denote the $k+2$ element lattice with a $k$ element antichain, then $SD(k, 2)$ holds but $SD(k-1, 2)$ fails in $M_k$. Hence $SD(k, 2) \not\models SD(k-1, 2)$.

(b) $C(k-1, 2) \models C(k, 2)$ is easy, so we do not detail it. For $t > 1$ let $L_t$ be the lattice depicted in Figure 8.

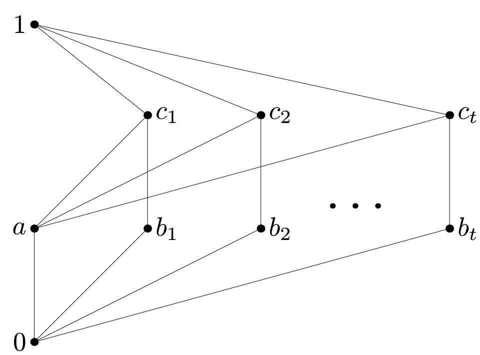


**Figure 8.**

The substitution $\alpha = b_k$, $\beta_{ij} = b_{i+1}$ ($i \neq j$, $0 \leq i < k-1$, $0 \leq j < k-1$) shows that $C(k-1,2)$ fails in $L_k$. Now we show that $C(k,2)$ holds in $L_k$. Suppose the contrary and fix $\alpha, \beta_{ij} \in L_k$ ($i < k$, $j < k$, $i \neq j$) satisfying the premise of $C(k,2)$ such that, with the notation $\beta_i := \sum_{j \neq i} \beta_{ij}$,

$$\alpha \not\leq \beta_0 + \alpha(\beta_1 + \alpha(\beta_2 + \ldots + \alpha\beta_{k-1}) \ldots). \tag{6}$$

Then $\alpha \not\leq \beta_{ij}$, for otherwise $\alpha \leq \beta_i$ would contradict (6). Hence $\beta_{ij} \neq 0$, for otherwise $\alpha \leq \beta_{ij} + \beta_{ji} = \beta_{ji}$, which we have already excluded.

**Case 1**: $\alpha = 1$. Then $\beta_{ij} = a$ would lead to $1 = \alpha = a + \beta_{ji} \Longrightarrow \beta_{ji} = 1 \geq \alpha$, a contradiction. Hence $\{\beta_{ij} : i \neq j\} \subseteq \{b_1, \ldots, b_k, c_1, \ldots, c_k\}$. For a given $i$, the $\beta_{ij}$ must belong to the same $\{b_{\varphi(i)}, c_{\varphi(i)}\}$, for otherwise $\beta_i = 1 \geq \alpha$. Since $\beta_{ij} + \beta_{ji} \geq \alpha = 1$, $\varphi : \{0, \ldots, k-1\} \to \{1, \ldots, k\}$ is injective, and therefore surjective. Hence the right-hand side of (6) is $\sum_{i \neq j} \beta_{ij} \geq b_1 + \ldots + b_k = 1$, a contradiction.

**Case 2**: $\alpha$ is a coatom, say $\alpha = c_1$. If we had $\beta_{ij} \in \{a, b_1\}$ for some pair $(i,j)$, $i \neq j$, then $\alpha \not\leq \beta_{ji} \leq \alpha + \beta_{ij} = \alpha$ and $\alpha \leq \beta_{ij} + \beta_{ji}$ would yield $\{\beta_{ij}, \beta_{ji}\} = \{a, b_1\}$, say $(\beta_{ij}, \beta_{ji}) = (a, b_1)$, and $\beta_i \geq a$ and $\beta_j \geq b_1$ would easily contradict (6). Hence $\{\beta_{ij} : i \neq j\} \subseteq \{b_2, \ldots, b_k, c_2, \ldots, c_k\}$, whence the previous $\varphi$ cannot be injective, a contradiction.

**Case 3**: $\alpha = a$. Then $\{\beta_{ij} : i \neq j\} \subseteq \{b_1, \ldots, b_k\}$, $\varphi$ is a bijection, and $\alpha + \beta_{ij} = \alpha + b_{\varphi(i)} = c_{\varphi(i)} \not\geq b_{\varphi(j)} = \beta_{ji}$ is a contradiction.

**Case 4**: $\alpha$ is another atom, say $\alpha = b_1$. Then $\{\beta_{ij} : i \neq j\} \subseteq \{a, b_2, \ldots, b_k, c_2, \ldots, c_k\}$. If $\beta_{ij} \neq a$ for all $i \neq j$ then $\varphi$ cannot be a bijection. Hence $\beta_{ij} = a$ for some $i \neq j$, and $b_1 = \alpha \leq \beta_{ij} + \beta_{ji} = a + \beta_{ji}$ implies $\alpha \leq \beta_{ji}$, a contradiction. We have seen that $L_k \models C(k,2)$. Hence $C(k,2) \not\models C(k-1,2)$, proving (b).

(c) To show $SD(2,2) \models SD(n,H)$, firstly we assume that $|H| = 1$, say $H = \{\{0, 1, \ldots, t-1\}\}$. Then the statement follows via induction; indeed, after deriving $\alpha(\beta_1 + \ldots + \beta_{t-1}) = \alpha\beta_1 = \alpha\beta_0$ from the induction hypothesis, we can apply $SD_\wedge$ for the elements $\alpha$, $\beta_0$ and $\beta_1 + \ldots + \beta_{t-1}$. From the $|H| = 1$ case the general case is evident.

(d) is a consequence of Theorem 7.

In order to show (e), let $L$ be the set of convex polytopes in the $(n-1)$-dimensional Euclidean space $E_{n-1}$. By a polytope we mean the convex hull of finitely many points. Since polytopes can also be defined as bounded intersections of finitely many half spaces, cf., e.g., Ziegler [**27**], $L$ is a lattice with intersection as meet and convex hull of union as join. First we show that $L \models SD_\vee$. Let $P, Q_1, Q_2 \in L$ such that $P + Q_1 = P + Q_2$. Let $R = P + Q_1 + Q_2 = P + Q_1 = P + Q_2$, and denote by $V$ the vertex set of $R$. Then $\mathrm{conv}(V)$, the convex hull of $V$, is $R$ but $\mathrm{conv}(R \setminus \{v\}) \neq R$ for all $v \in V$. We claim that

$$V \subseteq P \cup (Q_1 \cap Q_2). \tag{7}$$

Suppose $a \in V \setminus \big(P \cup (Q_1 \cap Q_2)\big) = \big(V \setminus (P \cup Q_1)\big) \cup \big(V \setminus (P \cup Q_2)\big)$, then $P+Q_i \subseteq \mathrm{conv}(R\setminus\{a\}) \subset R = P+Q_i$ for $i=1$ or $i=2$, a contradiction. This shows (7). Armed with (7) we conclude $P+Q_1 = R = \mathrm{conv}(V) \subseteq \mathrm{conv}\big(P\cup(Q_1\cap Q_2)\big) = \mathrm{conv}(P) + \mathrm{conv}(Q_1 \cap Q_2) = P + Q_1Q_2$. Hence $L \models SD_\vee$; therefore $L \models C(2,2)$ and, by (b), $L \models C(m,2)$.

Now let $b_0, b_1, \dots, b_{n-1} \in E_{n-1}$ be points in general position, i.e., they do not belong to a hyperplane. Then $S = \mathrm{conv}(\{b_0, \dots, b_{n-1}\})$ is a symplex. For $i = 0, \dots, n-1$ let $\beta_i := \mathrm{conv}(\{b_0, \dots, b_{i-1}, b_{i+1}, \dots, b_{n-1}\})$, a facet of the symplex. Choose an inner point $a$ of the symplex, i.e., $a \in S \setminus \{\beta_0 \cup \beta_1 \cup \ldots \cup \beta_{n-1}\}$. Set $\alpha = \{a\}$. Since $\alpha\beta_i = \{a\} \cap \beta_i = \emptyset$, the polytopes $\alpha, \beta_0, \dots, \beta_{n-1}$ easily witness that $SD(n,H)$ fails in $L$. This yields (e). Proposition 6 is proved. □

*Proof of Theorem* 7. Let $\mathcal{V}$ be the variety of (meet) semilattices. By Papert [**23**] $\mathrm{Con}(\mathcal{V}) \models SD(2,2)$, so $\mathrm{Con}(\mathcal{V}) \models SD(m,K)$ by Proposition 6(c). We intend to show that $\mathrm{Con}(\mathcal{V}) \not\models C(n,H)$; suppose the contrary. The graph $G_2(\alpha \prod_{I\in H} \sum_{i\in I} \beta_i)$ will be denoted by $G_2$. With the notations of the proof of Theorem 2 we have

$$(x_0, x_1) \in \beta_0^{(n-1)}(\hat\alpha, \hat\beta_0, \dots, \hat\beta_{n-1}). \tag{8}$$

For semilattice terms $g_0$ and $g_1$ over the vertex set $X = \{x_0, x_1, \dots\}$ of $G_2$ and for a permitted subgraph $S$ (cf. the proof of Theorem 9) of $G_k(\beta_0^{(n-1)})$ with vertex set $F_S$ and endpoints $f_{0S}$ and $f_{1S}$ we define the following condition:

> "there exist semilattice terms $h(x_0, x_1, \dots)$, $h \in F_S$, which satisfy the identities $f_{0S}(x_0, x_1, \dots) = g_0(x_0, x_1, \dots)$, $f_{1S}(x_0, x_1, \dots) = g_1(x_0, x_1, \dots)$ and for each $(h_1, \gamma, h_2) \in E(S)$ the identity $I(h_1, \gamma, h_2)$."

This condition will be denoted by $U^*(G_2 \le S; f_{0S} = g_0, f_{1S} = g_1)$. For example, $U^*(G_2 \le S; f_{0S} = x_0, f_{1S} = x_1)$ is the same as "$U(G_2 \le S)$ holds in $\mathcal{V}$".

From (8) we obtain $(x_1, x_0) \in \beta_0^{(n-1)}(\hat\alpha, \hat\beta_0, \dots, \hat\beta_{n-1})$, whence, similarly to the proof of (iii) $\Longrightarrow$ (ii) in Theorem 9, we conclude that there is a $k \ge 2$ such that

$$U^*(G_2 \le G_k(\beta_0^{(n-1)}); f_{0S} = x_1, f_{1S} = x_0) \quad \text{holds.} \tag{9}$$

(Interchanging $x_0$ and $x_1$ serves technical purposes.) We will use the fact that each semilattice term is, modulo semilattice theory, the meet of all variables occurring in it.

Multiplying (i.e., meeting) all terms by $x_1$, we infer from (9) that

$$U^*(G_2 \le G_k(\beta_0^{(n-1)}); f_0 = x_1, f_1 = x_0x_1) \quad \text{holds.} \tag{10}$$

We intend to show that for all permitted subgraphs $S$ of $G_k(\beta_0^{(n-1)})$

$$U^*(G_2 \le S; f_{0S} = x_1, f_{1S} = x_0x_1) \quad \text{holds.} \tag{11}$$

This will be done via a downward induction on the height of $S$. If $S$ is of height $n-1$ then (11) coincides with (10).

Now suppose that $S$ is of height $n-1-t>0$, i.e., $S=G_k(\beta_t^{(n-1-t)})$, and $U^*(G_2 \le S; f_{0S}=x_1, f_{1S}=x_0x_1)$ holds. We want to show the same for $T=G_k(\beta_{t+1}^{(n-2-t)})$. Let $g_0=f_{0S}$, $g_1$, $g_2$, $\dots$, $g_k=f_{1S}$ be the endpoints needed to form $S$ from $G_k(\beta_t)$ and $T$ via serial connection, cf. Figure 9, and suppose that all

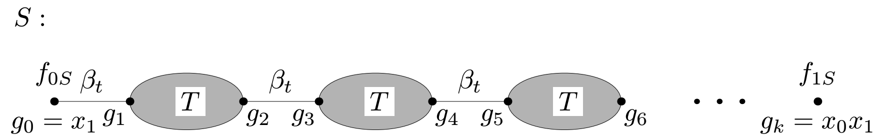


**Figure 9.**

terms are chosen in $\mathcal{V}$ such that they witness $U^*(G_2 \le S; f_{0S}=x_1, f_{1S}=x_0)$. Let $A_t := \{u \in X : (u, \beta_t, x_1) \in E(G_2)\}$. Our argument uses the general convention that the colours on the arcs of $G_2$ (cf. Figure 5) occur from left to right order. This means that if $(x_0, \beta_{i_1}, y_{I,1})$, $(y_{I,1}, \beta_{i_2}, y_{I,2})$, $(y_{I,2}, \beta_{i_3}, y_{I,3})$, $\dots$, $(y_{I,\ell-1}, \beta_{i_\ell}, x_1)$ are adjacent consecutive edges from the left to the right then $i_1 < i_2 < i_3 \dots < i_\ell$. Let $\breve{\beta}_i$ denote the smallest equivalence on $X$ that includes $\{(u,v) \in X^2 : (u, \beta_i, v) \in E(G_2)\}$. It follows from the above-mentioned convention that

$$\text{for } u \in A_t \text{ and } j > t, \quad |[u]\breve{\beta}_j| = 1, \tag{12}$$

i.e., the $\breve{\beta}_j$-class of $u$ is a singleton.

Suppose first that one of the $g_i$ $(0<i<k)$ contains some $u \in A_t$. Let $d$ be the smallest integer such that $g_d$ contains $u$, and let $m$ be the largest integer such that $g_d$, $g_{d+1}$, $\dots$, $g_m$ all contain $u$. Since any two vertices of $T$ are connected by a path containing the colours $\beta_{t+1}$, $\beta_{t+2}$, $\dots$, $\beta_{n-1}$ only, we conclude from (12) that if one of the endpoints of (a copy of) $T$ contains $u$ then all vertices (inner and endpoint vertices) of $T$ contain $u$. Therefore $d$ is odd and $m$ is even, for otherwise $g_{d-1}$ and $g_d$ or $g_m$ and $g_{m+1}$ would be the endpoints of a copy of $T$.

Now we can change $u$ to $x_1$ in all terms (vertices) between $g_d$ and $g_m$ (including $g_d$, $g_m$, and the inner vertices of the corresponding copies of $T$). We claim that the new terms obtained this way still witness that $U^*(G_2 \le S; f_{0S}=x_1, f_{1S}=x_0x_1)$ holds. Since (12) and $|[u]\breve{\alpha}|=1$, $u$ "was not used" within $T$, whence for every copy of $T$ between $g_d$ and $g_m$ the identities associated with the edges of $T$ hold. Since $(u, x_1) \in \breve{\beta}_t$, the identities $I(g_i, \beta_t, g_{i+1})$ remain valid for $d<i<m$, $i$ even, and also for $i=d-1$ and $i=m$. Hence the new terms do the job.

We have seen how to reduce the occurrences of elements of $A_t$. After doing this reduction in a finite number of steps we can get rid of all elements of $A_t$. Hence we can assume that

$$\text{no } u \in A_t \text{ occurs in our terms.} \tag{13}$$

From now on let $m$ be the smallest number such that $x_0$ occurs in $g_m$. We claim that

$$g_j = x_1 \text{ for } 0 \le j < m. \tag{14}$$

This is true for $g_0 = f_{0S}$. If $g_{j-1} = x_1$, $j < m$ and $j-1$ is even then (13) and $I(g_{j-1}, \beta_t, g_j)$ yield $g_j = x_1$. If $g_{j-1} = x_1$, $j < m$ and $j-1$ is odd then the identity $I(g_{j-1}, \alpha, g_j)$ associated with $(g_{j-1}, \alpha, g_j) \in E(T)$ and the lack of $x_0$ in $g_j$ give $g_j = x_1$. This induction shows (14).

If $m-1$ is even then $I(g_{m-1}, \beta_t, g_m)$ cannot hold, for $g_{m-1} = x_1$, $(x_0, x_1) \notin \breve{\beta}_t$ but $x_0$ occurs in $g_m$. Consequently, $m-1$ is odd and $(g_{m-1}, \alpha, g_m) \in E(S)$. Since $g_{m-1} = x_1$ and $x_0$ occurs in $g_m$, the identity $I(g_{m-1}, \alpha, g_m)$ can hold only if $g_m = x_0$ or $g_m = x_0 x_1$. Hence either

$$U^*(G_2 \le T; f_{0T} = x_1, f_{1T} = x_0) \tag{15}$$

or

$$U^*(G_2 \le T; f_{0T} = x_1, f_{1T} = x_0 x_1) \tag{16}$$

holds. Notice that (15) implies (16), for all terms $h$ occurring in (15) can be replaced by $hx_1$. This completes the induction proving (11).

Applying (11) to the subgraphs of height 0, it follows that $U^*(G_2 \le G_k(\alpha\beta_{n-1});$ $f_0 = x_1, f_1 = x_0 x_1)$ holds, which contradicts $(x_0, x_1) \notin \breve{\beta}_{n-1}$. This proves Theorem 7. □

We conclude the paper with some remarks on Proposition 6. The five element nonmodular lattice $N_5$ witnesses that $SD_\vee \not\models C(3, \{\{0,1,2\}\})$ and so $C(2,2) \not\models C(3, \{\{0,1,2\}\})$. This explains why Proposition 6 does not include a "conjugate" counterpart of (c).

We do not know if (e) holds with $C(m, K)$ instead of $C(2,2)$ but the present proof of (e) is not appropriate to decide this. Indeed, if $K$ is the center and $B_0, \dots, B_4$ are consecutive vertices of a (planar) regular pentagon then $\alpha = \operatorname{conv}(\{B_0, B_1, K\})$, $\beta_0 = \operatorname{conv}(\{B_1, B_2\})$, $\beta_1 = \operatorname{conv}(\{B_0, B_3, B_4\})$ and $\beta_2 = \operatorname{conv}(\{B_2, B_3, B_4\})$ witness that $C(3, \{\{0,1,2\}\})$ fails in $L$.

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**Added on June 19, 1998.** As an affirmative answer to the problem raised at the end of the first section, an anonymous referee has proved that $SD(n, H) \models_{\text{con}} SD_\wedge$ for every $n \ge 2$ and $\emptyset \ne H \subseteq P_2(\mathbf{n})$. The proof is based on Kearnes and Szendrei [**19**], Lipparini [**20**], and Theorem 3. Now Theorem 1 becomes a consequence of Proposition 6(c) and Willard [**25**], and the referee's method together with [**3**] gives a shorter proof of Theorem 2. However, the present approach to Theorems 1 and 2 can still be justified. Not only by its role in **finding** the results but also in the proofs of Theorem 7 and (the purely lattice theoretic) Proposition 6(d).

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