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THE LAGRANGE THEOREM FOR MULTIDIMENSIONAL DIOPHANTINE APPROXIMATION

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Abstract. In this paper we give a necessary and sufficient condition for z in the floor of the Poincaré half-space to have periodicity in the multidimensional Diophantine approximation by convergents using the Hermite algorithm. We examine in detail the structure of the corresponding sequences and give some examples

1. Introduction

We consider an hyperbolic reflection group W_S generated by the finite set S of the reflections in the faces of a fundamental chamber $\mathscr{C}$ in the Poincaré half-space H^p of $\mathbb{R}^p$. We suppose that $\mathscr{C}$ is of finite volume and that the only vertex at infinity is ∞. For z in the floor $\mathbb{R}^{p-1} = \partial H^p$, a moving point on the vertical line (∞z) from ∞ to z crosses a sequence $w_0(\mathscr{C}), w_1(\mathscr{C}), \dots, w_n(\mathscr{C}), \dots$ of adjacent chambers (with $w_0 = 1$). This algorithm (originally due to Hermite [**5**]) produces parabolic points $w_n(\infty)$ which are the convergents of a multidimensional continued fraction expansion of z. The main purpose of this paper is to prove the Lagrange Theorem for this framework. We assert that z is a loxodromic fixed point if and only if there are two integers N and $k > 0$, and $w \in W_S$ such that $w_{N+nk} = w_N w^n$ for $n \geq 0$. In this case, the asymptotic behavior of the $w_n^{-1}(\infty z)$ is described by a finite graph. In the last section, we give a way to study exact periodicity with some examples.

2. Stabilizer

For $p \geq 2$, we denote by $H^p = \{x \in \mathbb{R}^p \,|\, x_p > 0\}$ the Poincaré upper half-space of the Euclidean space $\mathbb{R}^p$ with an orthonormal basis $(\epsilon_i)_{1 \leq i \leq p}$. The floor is $\mathbb{R}^{p-1} = \{x \in \mathbb{R}^p \,|\, x_p = 0\}$ and we use the notation $\widetilde{\mathbb{R}}^{p-1}$ for $\mathbb{R}^{p-1} \cup \{\infty\}$. The group Möb$(H^p)$ of Möbius transformations acting on $\{x \in \mathbb{R}^p \,|\, x_p \geq 0\} \cup \{\infty\}$ is generated by the inversions in half spheres and half hyperplanes orthogonal to $\mathbb{R}^p$.

A Möbius transformation has at least one fixed point. A transformation with a fixed point in H^p is termed **elliptic** and is conjugate to an Euclidean motion. A

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transformation w with no fixed point in H^p has exactly one or two fixed points in $\widetilde{\mathbb{R}}^{p-1}$ and we say that w is respectively **parabolic** or **loxodromic**. A parabolic transformation is conjugate to an affine Euclidean motion. A loxodromic transformation is conjugate to a similarity and one of its fixed points is **attracting** and the other is **repulsive** (see [**10**]).

In this paper, W_S denote an hyperbolic reflection group generated by the finite set S of reflections in the walls of the fundamental chamber $\mathscr{C}$ which is supposed to have a finite volume. Hence the set of vertices of $\mathscr{C}$ in $\widetilde{\mathbb{R}}^{p-1}$ is finite. We see W_S as an injective representation of an abstract Coxeter group (see [**3**]). The length $l_S(w)$ of $w \in W_S$ is the smallest integer q such that $w = s_1 \cdots s_q$ with $s_i \in S$.

We denote the stabilizer of z in $H^p \cup \widetilde{\mathbb{R}}^{p-1}$ by $\mathrm{Stab}(z)$. If z is in $\cup_{w \in W_S} w(\mathscr{C})$ then $\mathrm{Stab}(z) = W_{S(z)}$ where $S(z)$ is the set of reflections s in W_S (necessarily conjugate to reflections of S) such that $s(z) = z$.

If $z \in \widetilde{\mathbb{R}}^{p-1}$ is equivalent to a vertex $\varpi \in \mathscr{C}$ then $\mathrm{Stab}(z)$ is conjugate to $W_{S(\varpi)}$. We say that z is a **parabolic** fixed point.

Suppose that $z \in \widetilde{\mathbb{R}}^{p-1}$ is not parabolic. In this case there is unique $y \neq z$ in $\widetilde{\mathbb{R}}^{p-1}$ such that $\mathrm{Stab}(y) = \mathrm{Stab}(z)$. The elements in $\mathrm{Stab}(z)$ of infinite order are loxodromic with axis the geodesic (yz) in H^p. We say that z is a **loxodromic** fixed point. The set of elements of finite order in $L(y,z) = \mathrm{Stab}(y) = \mathrm{Stab}(z)$ is the finite normal subgroup

$$L_0(y,z) = W_{S(y)} \cap W_{S(z)} = W_{S(y) \cap S(z)},$$

and $L(y,z)/L_0(y,z)$ is an infinite cyclic group. The group $L(y,z)/L_0(y,z)$ acts on (yz) as a discrete group of dilatations.

We note that for any w in W_S, the inner automorphism $u \mapsto wuw^{-1}$ of W_S induces the isomorphisms $L(y,z) \to L(w(y), w(z))$ and $L_0(y,z) \to L_0(w(y), w(z))$.

3. The Hermite Algorithm

We denote by P_s the mirror of the reflection $s \in S$. It is a half-sphere or a half-hyperplane with added ∞. The corresponding face of $\mathscr{C}$ is $F_s = P_s \cap \mathscr{C}$.

Let y and z be different points in $\widetilde{\mathbb{R}}^{p-1}$. The geodesic (yz) in H^p is an Euclidean half-circle or a half-line orthogonal to $\widetilde{\mathbb{R}}^{p-1}$. We suppose that (yz) is oriented from y to z: $(zy) = -(yz)$. We denote $(yz) \cup \{y,z\}$ by $[y,z]$.

We define $S_{succ}(y,z)$ to be the subset of S formed by the reflections s such that

- (yz) goes out of $\mathscr{C}$ through F_s
- (yz) is not in P_s

A necessary and sufficient condition to have $S_{succ}(y,z) \neq \emptyset$ is $[y,z] \cap \mathscr{C} \neq \emptyset$.

An example is illustrated on figure 1 in the case of $W_{\{s_1,s_2,s_3\}} = PGL_2(\mathbb{Z})$ for the golden number $\phi = \frac{1+\sqrt{5}}{2}$.

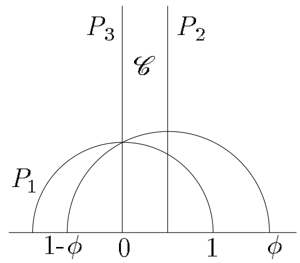


Figure 1. $S_{succ}(1-\phi,\phi) = \{s_2\}$ and $S_{succ}(\phi, 1-\phi) = \{s_1, s_3\}$.

We introduce an oriented graph $\Gamma(y,z)$ whose set of vertices is

$$\mathrm{Arc}(y,z) = \{w[y,z] \,|\, \mathscr{C} \cap w[y,z] \neq \emptyset\}.$$

The edges are (a,s,b) with $b = s(a)$ for $s \in S_{succ}(a)$. This graph is simple and without loops. It is possible to deduce $\Gamma(z,y)$ from $\Gamma(y,z)$ by reversing the orientation of vertices and edges. For w in W_S, we have $\Gamma(w(y), w(z)) = \Gamma(y,z)$.

Each vertex of $\Gamma(y,z)$ has at least one successor and one predecessor. The infinite paths give us a way to cover geodesics by reduced galleries.

Let $\sigma = (\sigma_0, \dots, \sigma_{n-1})$ be a finite path in $\Gamma(y,z)$ with edges $\sigma_i = (a_i, s_{i+1}, a_{i+1})$. We denote by $w_i(\sigma)$ the word $s_1 \cdots s_i$ of W_S with $w_0(\sigma) = 1$. We know that $(s_1, \dots, s_i)$ is a reduced decomposition of $w_i(\sigma)$. This means that the length $l_S(w_i(\sigma))$ of $w_i(\sigma)$ is i. The intersection of $[y,z]$ with $\cup_{0\le i\le n} w_i(\mathscr{C})$ is a connected part of $[y,z]$.

Let a_0 be a vertex in $\Gamma(y,z)$ and $\sigma = (\sigma_n)_{n\in\mathbb{Z}}$ be an infinite path such that a_0 is the beginning of σ_0. Setting $w_{-n}(\sigma) = s_{-1} \cdots s_{-n}$ for $n > 0$, we obtain an infinite reduced gallery $(w_n(\mathscr{C}))_{n\in\mathbb{Z}}$ which covers a_0 minus its extremities (see [**7**]).

4. Finite Graphs

It is possible to characterized the finite graphs. They are closely related to closed geodesics in H^p/W_S.

From the fact that $(w_n(\mathscr{C}))$ covers a_0, we deduce that there is at least one vertex a in $\Gamma(y,z)$ such that $a \cap \mathscr{C}$ is not a point. We say that in this case a is **general**.

Theorem 1. *The graph $\Gamma(y,z)$ is finite if and only if (yz) is the axis of a loxodromic transformation in W_S. In this case, we have:*

(i) *the graph $\Gamma(y,z)$ is a circuit*
(ii) *the general vertices belong to all circuits*
(iii) *let a be a vertex in $\Gamma(y,z)$ and σ be an elementary circuit with extremity a. The element $w(\sigma) \in W_S$ is the word of minimum length which generates*

the cyclic group $L(a)/L_0(a)$ and for which the ending extremity of a is attracting. This length is independent of a.

Proof. Suppose that $\Gamma(y,z)$ is finite. There exists a circuit σ in $\Gamma(y,z)$ because all vertex have a successor. Let a be the extremity of σ. From the fact that σ^n is in $\Gamma(y,z)$ for all $n \geq 0$, we deduce that a is the axis of an element of infinite order. Hence z is a loxodromic fixed point.

Suppose that (yz) is the axis of a loxodromic transformation. Without loss of generality, we may assume that (yz) is a general vertex of $\Gamma(y,z)$.

Let u be a generator of the infinite cyclic group $L(y,z)/L_0(y,z)$. Let a be a vertex of $\Gamma(y,z)$: there is $w \in W_S$ such that $a = w[y,z]$. We can find an integer $n \in \mathbb{Z}$ such that $(yz) \cap w^{-1}(\mathscr{C})$ is between $(yz) \cap u^n(\mathscr{C})$ and $(yz) \cap u^{n+1}(\mathscr{C})$. Hence $u^{-n}w^{-1}(\mathscr{C})$ is a chamber intersecting (yz) between $(yz) \cap \mathscr{C}$ and $(yz) \cap u(\mathscr{C})$. There is only a finite number of possibilities because W_S is discrete. From $a = wu^n[y,z]$, we deduce that $\Gamma(y,z)$ is finite.

(i) Going over $\Gamma(y,z)$ from a, and over $\Gamma(z,y)$ from $-a$, we obtain reduced galleries which intersect $u^n(\mathscr{C})$ and $u^{n+1}(\mathscr{C})$ by composing with w^{-1}. Thus we get a circuit which contains a and $[y,z]$.

(ii) In particular, when σ is a circuit with extremity a, we deduce that $[y,z]$ is a vertex of some σ^n by considering the infinite path $(\sigma^n)_{n\in\mathbb{Z}}$. But the set of vertices of σ and σ^n are the same.

(iii) Without loss of generality, we may suppose that $a = [y,z]$. Let u be a generator of $L(a)/L_0(a)$. Changing u in u^{-1}, we can suppose that z is the attracting point for u and for $w(\sigma)$.

First, we suppose that a is general. In this case, there is an integer $n > 0$ such that $(yz) \cap w(\sigma)(\mathscr{C})$ is in $(yz) \cap u^n(\mathscr{C})$. The corresponding gallery must intersect $(yz) \cap u(\mathscr{C})$. We deduce that $n = 1$ because σ is elementary.

It is easy to see that $w^{-n}(\sigma)u^n \in L_0(a)$ for all n because $w^{-1}(\sigma)u \in L_0(a)$. Let $N = \sup\,\{l_S(w) \,|\, w \in L_0(a)\}$. From the following inequalities:

$$nl_S(w(\sigma)) = l_S(w^n(\sigma)) \leq l_S(u^n) + N \leq nl_S(u) + N,$$

we obtain $l_S(w(\sigma)) \leq l_S(u)$.

We have the unicity because $w(\sigma)$ is $(\emptyset, S(y) \cap S(z))$-reduced (see [**3**]).

If a is not general then we note that σ contains a general vertex b. By circular permutation of the edges of σ, we get an elementary circuit τ with extremity b. The words $w(\sigma)$ and $w(\tau)$ are conjugate.

By conjugation, the length is independent of the choice of a. □

Remark 1. From (iii), we have $w(\sigma) = w(\tau)$ whenever σ and τ are two elementary paths with the same extremities.

5. Pseudo-periodicity

We consider the 2^{p-1}-dimensional associative and unitary algebra Cl_{p-1} generated by $\epsilon_2, \cdots \epsilon_p$ verifying

$$\begin{cases} \epsilon_i^2 & = -1 \\ \epsilon_i \epsilon_j & = -\epsilon_j \epsilon_i, \;\; i \neq j\,. \end{cases}$$

By identifying ϵ_1 with 1, one can see $\mathbb{R}^p = \oplus_{i=1}^p \mathbb{R}\epsilon_i$ as a vector subspace of Cl_{p-1}. The products of non null vectors form the Clifford group Γ_{p-1}. We put an Euclidean norm $|\quad|$ on the vector space Cl_{p-1} which coincides with the Euclidean norm $\|\quad\|$ on $\mathbb{R}^p$. For x and y in Γ_{p-1}, we have $|xy| = |x|\,|y|$.

Let f be a homography in $\text{Möb}(H^p)$. There are a, b, c and d in $\Gamma_{p-1} \cup \{0\}$ such that $f(z) = (az+b)(cz+d)^{-1}$ and $ad^* - bc^* = 1$ where $x \mapsto x'$ is the anti-automorphism of Cl_{p-1} defined by $\epsilon_i^* = \epsilon_i$ (see [**1**]). For an anti-homography f, using the automorphism $x \mapsto x'$ of Cl_{p-1} defined by $\epsilon_i' = -\epsilon_i$, we have the existence of a, b, c and d in $\Gamma_{p-1} \cup \{0\}$ such that $f(z) = (az'+b)(cz'+d)^{-1}$ and $ad^* - bc^* = -1$ (see [**8**]). These transformations are respectively denoted by the Cliffordian matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix}'.$$

In each case, (a,b,c,d) is unique up to multiplication by ± 1.

From now on, we suppose that ∞ is the only parabolic vertex of $\mathscr{C}$. Let z be in $\mathbb{R}^{p-1}$. Let $\sigma = (\sigma_n)_{n \in \mathbb{N}}$ be a path in $\Gamma(\infty, z)$ beginning with $[\infty, z]$. We say that the corresponding sequence (w_n) is a H-sequence associated to z. We know that $\lim_{n\to\infty} w_n(\infty) = z$. Using Cliffordian matrices, it is possible to write $w_n(\infty) = p_n q_n^{-1}$ with p_n and q_n in $\Gamma_{p-1} \cup \{0\}$. The fractions $p_n q_n^{-1}$ have the same properties as those obtained from the usual continued fraction expansions. We say that the $p_n q_n^{-1}$ are convergents. The discrete sequence of $|q_n|$ increases, and $\lim_{n\to\infty} |q_n| < \infty$ if and only if z is parabolic.

Lemma 1. *Let (yz) be the axis of a loxodromic element w in W_S. Let (w_n) be a H-sequence associated to z. For all $\varepsilon > 0$, we can find an integer N such that, for a given $n \geq N$, there is $[\alpha, \beta]$ in the finite set* $\text{Arc}(y,z)$ *verifying $w_n^{-1}(z) = \alpha$ and $|w_n^{-1}(\infty) - \beta| \leq \varepsilon$.*

Proof. It is clear that $w_n^{-1} w w_n$ is loxodromic with axis $w_n^{-1}(yz)$. We have:

$$|w_n^{-1}(y) - w_n^{-1}(\infty)| = \frac{1}{|q_n|^2 |w_n(\infty) - y|}.$$

From the fact that $\lim_{n\to\infty} |q_n| = \infty$ and $\lim_{n\to\infty} w_n(\infty) = z \neq y$, we deduce that $w_n^{-1}[\infty, z]$ and $w_n^{-1}[y,z]$ are as near as we want. Moreover, the number of $w_n^{-1}[y,z]$ is finite because $\{w_n^{-1} w w_n\}$ is finite from Theorem 4 in [**7**]. We have

$w_n^{-1}[\infty, z] \cap \mathscr{C} \neq \emptyset$ for all n. This implies the existence of an integer p such that $w_n^{-1}[y, z] \cap \mathscr{C} \neq \emptyset$ for $n \geq p$. In this case $w_n^{-1}[y, z]$ is a vertex of $\Gamma(y, z)$. □

When (yz) is the axis of a loxodromic transformation, the graph $\Gamma(y, z)$ depends only on z. We shall say that it is the reduced graph for z denoted by $\Gamma_{red}(z)$. The graphs $\Gamma(\infty, z)$ and $\Gamma_{red}(z)$ are related by

Corollary 1. *Let z be a loxodromic fixed point and (w_n) be a H-sequence associated to z. There are an integer N and a path σ in $\Gamma_{red}(z)$ verifying $w_n = w_n(\sigma)$ for all $n \geq N$.*

Proof. From Lemma 1, there is an integer N such that, for $n \geq N$:

$$S_{succ} w_n^{-1}[\infty, z] \subset S_{succ} w_n^{-1}[y, z]. \qquad \square$$

Definition 1. We say that a H-sequence (w_n) is **pseudo-periodic** if there are two integers k and N, and $w \in W_S$ such that $w_{N+nk} = w_N w^n$ for all $n \geq 0$.

The main result of this paper is the Lagrange Theorem for the Hermite Algorithm:

Theorem 2. *Let $z \in \widetilde{\mathbb{R}}^{p-1} \setminus W_S(\infty)$. A H-sequence (w_n) associated to z is pseudo-periodic if and only if z is a loxodromic fixed point.*

Proof. Suppose that the H-sequence (w_n) associated to z is pseudo-periodic. We have $\lim_{n \to \infty} w_N w^n(\infty) = z$ and w is of infinite order. We deduce that w is parabolic or loxodromic. But w is not parabolic because z is not equivalent to ∞.

Let z be a loxodromic fixed point. From Corollary 1, we have an integer N and an elementary circuit σ such that $w_{N+nk} = w_N w^n(\sigma)$ where k is the length of $w(\sigma)$. □

We say that the length in Theorem 1(iii) is the **pseudo-period** of z denoted by $l(z)$.

For $z \in \widetilde{\mathbb{R}}^{p-1} \setminus W_S(\infty)$, we have defined in [**7**] the approximation constant by

$$\gamma(z) = \limsup \left(|q|\,|p - zq|\right)^{-1},$$

the limsup being taken over all $pq^{-1} \in W_S(\infty)$. From Theorem 3 in [**7**], we deduce

Corollary 2. *Let z be a loxodromic fixed point in $\mathbb{R}^{p-1}$. We have*

$$\gamma(z) = \sup \left\{|\alpha - \beta| \mid [\alpha, \beta] \in \mathrm{Arc}_{red}(z)\right\}$$

where $\mathrm{Arc}_{red}(z)$ denotes the finite set of vertices of $\Gamma_{red}(z)$.

6. An Example

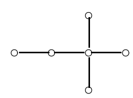
Following an idea of Professor Y. Hellegouarch, we consider the hyperbolic reflection group $W_S \subset \text{Möb}(H^5)$ generated by

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}' : x \mapsto 1/x', \qquad s_2 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}' : x \mapsto -x' + 1,$$

$$s_3 = \begin{pmatrix} -\xi' & 0 \\ 0 & \xi \end{pmatrix}' : x \mapsto -\xi' x' \xi', \qquad s_4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}' : x \mapsto -ix'i,$$

$$s_5 = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}' : x \mapsto -jx'j, \qquad s_6 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}' : x \mapsto -kx'k,$$

where $\xi = \frac{1}{2}(1 + i + j + k)$. Its Coxeter scheme is



Let $y = \frac{1}{2} + \frac{1+\sqrt{5}}{4} i + \frac{1-\sqrt{5}}{4} j$ and $z = \frac{1}{2} + \frac{1-\sqrt{5}}{4} i + \frac{1+\sqrt{5}}{4} j$. The geodesic $(y\, z)$ is an edge of the fundamental chamber and the axis of a loxodromic transformation. The reduced graph of z is illustrated on Figure 2.

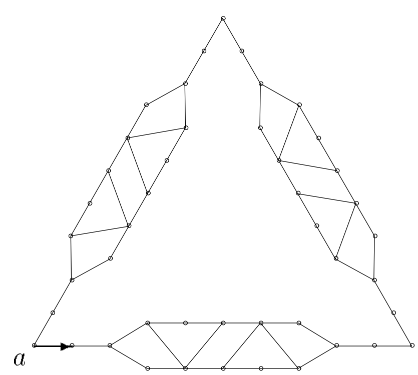


Figure 2. $\Gamma_{red}(\frac{1}{2} + \frac{1-\sqrt{5}}{4} i + \frac{1+\sqrt{5}}{4} j)$.

The pseudo-period is 30. The vertex $a = (y\, z)$ is general and the corresponding word of Theorem 1 is

$$w = s_4 s_3 s_6 s_2 s_3 s_4 s_1 s_2 s_3 s_6 s_5 s_3 s_4 s_2 s_3 s_5 s_1 s_2 s_3 s_4 s_6 s_3 s_5 s_2 s_3 s_6 s_1 s_2 s_3 s_5.$$

With a computer, we obtained $\gamma(z) = \sqrt{5/2}$. This is the Hurwitz constant for the approximation in $\mathbb{R}^4$ respect to the ring of the Hurwitz integers $\mathbb{Z}[i, j, \frac{1}{2}(1 + i + j + ij)]$. One can see [**9**] and for other proofs of this result.

7. Periodicity

A H-sequence (w_n) associated to z in $\widetilde{\mathbb{R}}^{p-1}$ is said to be **periodic** if there are two integers N and $T > 0$ such that $s_{n+T} = s_n$ for $n \geq N$. For a loxodromic fixed point, it is natural to study periodicity. The difficulty increases with the dimension. We did not obtain definitive results, but it is sometimes possible to know if there is a periodic H-sequence associated to a given loxodromic fixed point by working in the Euclidean framework. We present the method.

Let z be a loxodromic fixed point in $\widetilde{\mathbb{R}}^{p-1}$ and (w_n) be a H-sequence associated to z. We consider a general vertex $a = (z_1 z_2)$ in $\Gamma_{red}(z_2)$. Let w be the corresponding loxodromic transformation. From Corollary 1, we know that there is an integer N verifying

(i) $w_N^{-1}(z) = z_2$
(ii) $w_{N+k\,l(w)} = w_N w^k$ for $k \geq 0$.

We get $w_{N+(k+1)l(w)} = w_{N+k\,l(w)} w$ by the elementary circuit $\sigma_i = (s_1^{(i)}, \ldots, s_{l(w)}^{(i)})$ if $w_{N+k\,l(w)}^{-1}[\infty, z]$ and the faces $F_j^{(i)} = s_1^{(i)} \cdots s_j^{(i)}(F_{s_{j+1}^{(i)}})$ are intersecting for $0 \leq j < l(w)$. Let $u \in \text{Möb}(H^p)$ such that $u[z_1, z_2] = [\infty, 0]$. We use σ_i if $uw_{N+k\,l(w)}^{-1}(\infty)$ is in the orthogonal projection $R_j^{(i)}$ of $u(F_j^{(i)})$ on the floor $\mathbb{R}^{p-1}$. We note that

$$R^{(i)} = \bigcap_{0 \leq j \leq l(w)-1} R_j^{(i)}$$

is the convex Euclidean hull of a finite number of points. The attracting point of the similarity uwu^{-1} is $0 \in R^{(i)}$. We consider an Euclidean ball B with center 0 in $\cup_i R^{(i)}$. There is an integer k_1 such that $w_{N+k\,l(w)}^{-1}(\infty) \in B$ for $k \geq k_1$. From the fact that $B^{(i)} = R^{(i)} \cap B$ is a spherical cone, we can replace the similarity $uwu^{-1} \colon x \mapsto \alpha x \alpha^*$ or $x \mapsto -\alpha x' \alpha^*$ by the associated Euclidean isometry $v \colon x \mapsto \alpha x \alpha^* / |\alpha|^2$ or $v \colon x \mapsto -\alpha x' \alpha^* / |\alpha|^2$. Hence a necessary condition for the appearance of the elementary circuit σ_i at the step $k \geq k_1$ is $v^{k-k_1} u w_{N+k_1\,l(w)}^{-1}(\infty) \in B^{(i)}$. This condition is not sufficient when

$$v^{k-k_1} u w_{N+k_1\,l(w)}^{-1}(\infty) \in B^{(i_1)} \cap B^{(i_2)}$$

with $i_1 \neq i_2$. The problem of periodicity is related to the behavior of the orbit of $uw_{N+k_1\,l(w)}^{-1}(\infty)$ under the semi-group $\langle v \rangle$ in the tessellation of B by the $B^{(i)}$. It is possible that the periodicity would depend on $w_{N+k_1\,l(w)}^{-1}(\infty)$. But from the fact that $uW_{S(z_1)\cap S(z_2)}u^{-1}$ is a finite group, only a finite number of cases may occur.

The simplest result that we can deduce from the preceding discussion in higher dimension is the following: if the Euclidean transformation v associated to w is of finite order then there is a periodic H-sequence associated to z.

For approximation in $\mathbb{R}$ (see [**6**]) or $\mathbb{C}$ (see [**4**]), we can state

Theorem 3.

(i) *Let z be a loxodromic fixed point in $\mathbb{R}$. The H-sequences associated to z are periodic. The period is $l(z)$ (when $l(z)$ is even) or $2l(z)$ (when $l(z)$ is odd).*

(ii) *Let z be a loxodromic fixed point in $\mathbb{C}$. If $\Gamma_{red}(z)$ is not a elementary circuit and the Euclidean transformation associated to is of infinite order then there is no periodic H-sequence associated to z.*

Proof. (i) The Euclidean motion is $x \mapsto x$ or $x \mapsto -x$ on $\mathbb{R}$. The first case corresponds to homography and the second to anti-homography.

(ii) The orbit is dense in a circle with center 0. Consequently, it is not possible to encounter the $B^{(i)}$ periodically. □

Example 1. For $q \geq 3$, we consider the group W_q acting on the upper complex half-plane and generated by the reflections

$$\begin{cases} s_1 \colon z \mapsto 1/\bar{z} \\ s_2 \colon z \mapsto 2\cos(\pi/q) - \bar{z} \\ s_3 \colon z \mapsto -\bar{z} \end{cases}$$

in the walls of the fundamental chamber

$$\mathscr{C}_q = \{z \in \mathbb{C} \mid 1 \leq |z|, 0 \leq \Re(z) \leq \cos(\pi/q)\} \cup \{\infty\}.$$

The subgroup W_q^+ formed by the homographies is the Hecke group (see [**2**]). In particular, W_3^+ is isomorphic to the modular group $PSL_2(\mathbb{Z})$.

The transformation $w = s_2 s_1 s_3 \colon z \mapsto 1/\bar{z} + \lambda_q$ is loxodromic with repulsive point $y_q = (\lambda_q - \sqrt{\lambda_q^2 + 4})/2$ and attracting point $z_q = (\lambda_q + \sqrt{\lambda_q^2 + 4})/2$ where $\lambda_q = 2\cos(\pi/q)$. The geodesic $(y_q\, z_q)$ and the interior of $\mathscr{C}_q$ are intersecting. Hence $a = (y_q\, z_q)$ is a general vertex of $\Gamma_{red}(z_q)$. We have two elementary circuits with extremity a (see Figure 3):

$$\begin{cases} \sigma_1 &= (s_2, s_1, s_3) \\ \sigma_2 &= (s_2, s_3, s_1). \end{cases}$$

The H-sequences associated to any real number equivalent to z_q are periodic with period $\sigma_1\sigma_2 = (s_2, s_1, s_3, s_2, s_3, s_1)$.

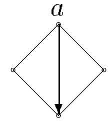


Figure 3. $\Gamma_{red}(z_q)$.

Example 2. We consider the hyperbolic reflection group W acting on the Poincaré half-space of $\mathbb{R}^3$ and generated by the reflections s_i which are defined on $\mathbb{C} \cup \{\infty\}$ by

$$\begin{cases} s_1\colon z \mapsto -1/\bar{z} \\ s_2\colon z \mapsto 1-\bar{z} \\ s_3\colon z \mapsto i\,\bar{z} \\ s_4\colon z \mapsto \bar{z} \end{cases}$$

The subgroup W^+ formed by homographies is isomorphic to $PGL_2(\mathbb{Z}[i])$.

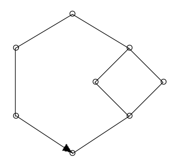


Figure 4. $\Gamma_{red}(\frac{1+i\sqrt{3}}{2})$.

Let $z = (1+i\sqrt{3})/2$. The geodesic $(\bar{z}\,z)$ is an edge of the fundamental chamber and z is a loxodromic fixed point. We obtain $a = (\bar{z}\,z)$ as a general vertex of $\Gamma_{red}(z)$. We have two elementary circuits with extremity a (see Figure 4):

$$\begin{cases} \sigma_1 = (s_3, s_2, s_3, s_1, s_2, s_3, s_4) \\ \sigma_2 = (s_3, s_2, s_1, s_3, s_2, s_3, s_4). \end{cases}$$

The corresponding loxodromic transformation on $\mathbb{C} \cup \{\infty\}$ is

$$w\colon z \mapsto \frac{-\bar{z}+1+i}{i\,\bar{z}+1}.$$

The Euclidean transformation s corresponding to w is a reflection because the pseudo-period is odd. One can verify that if $u \in L(a)$ then $u(\infty)$ lies on the union of the axes of s_1, s_2 and $s_1 s_2 s_1$ which is globally invariant under s because $s_2 = s s_1 s$ (see Figure 5). Moreover, the axis of s separates $B^{(1)}$ and $B^{(2)}$. This proves that H-sequences associated to any complex number equivalent to z are periodic with period $\sigma_1\sigma_2$.

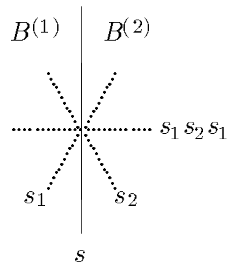


Figure 5.

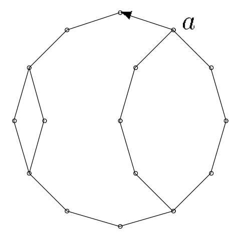


Figure 6. $\Gamma_{red}\left(\sqrt{\frac{1+\sqrt{5}}{2}}+i\sqrt{\frac{-1+\sqrt{5}}{2}}\right)$.

Example 3. We give another example from the previous group. We consider the loxodromic transformation acting on $\mathbb{C}\cup\{\infty\}$ by

$$w\colon z\mapsto \frac{(1+i)z+1+2i}{z+1+i}.$$

The attracting point is $z=\sqrt{\frac{1+\sqrt{5}}{2}}+i\sqrt{\frac{-1+\sqrt{5}}{2}}$. The other extremity of the axis is $y=-z$. One can verify that $a=(y\,z)$ is a general vertex of $\Gamma_{red}(z)$.

We have four elementary circuits with extremities a (see Figure 6):

$$\begin{cases} \sigma_1=(s_2,s_3,s_2,s_1,s_4,s_2,s_3,s_2,s_3,s_4,s_3,s_4)\\ \sigma_2=(s_2,s_3,s_2,s_4,s_1,s_2,s_3,s_2,s_3,s_4,s_3,s_4)\\ \sigma_3=(s_2,s_3,s_2,s_1,s_4,s_2,s_3,s_2,s_4,s_3,s_4,s_3)\\ \sigma_4=(s_2,s_3,s_2,s_4,s_1,s_2,s_3,s_2,s_4,s_3,s_4,s_3)\end{cases}$$

The corresponding Euclidean motion is a rotation r with angle θ verifying $\cos 2\theta = \frac{-105+1456\sqrt{5}}{3929}$. By considering the Chebyshev polynomials over $\mathbb{Q}(\sqrt{5})$, one can verify that the order of r is infinite. Hence H-sequences associated to any complex number equivalent to z are not periodic.

References

1. Ahlfors L. V., *Möbius transformations in $\mathbb{R}^n$ expressed through 2×2 matrices of Clifford numbers*, Complex Variables **5** (1986), 215–224.

2. Beardon A. F., *The Geometry of Discrete Groups*, Graduate texts in Math **91**, Springer-Verlag, New York, 1983.

3. Bourbaki N., *Groupes et algèbres de Lie, Chap. 4-6*, Hermann, Paris, 1968.

4. Ford L. R., *Rational approximations to irrational complex numbers*, T.A.M.S. **19** (1918), 1–42.

5. Hermite C., *Sur la théorie des formes quadratiques*, Crelle's Journal f.d. Math. **47** (1854).

6. Humbert G., *Sur la méthode d'approximation d'Hermite*, Journal de Maths **2** (1916), 79–103.

7. Meignen P., *Galleries in Poincaré half-spaces and Diophantine approximation*, to appear in Contributions to Algebra and Geometry.

8. ———, *Orthochronous Lorentz Group from the Cliffordian point of view*, AACA **6** (2) (1996), 233–248.

9. Schmidt A. L., *Farey simplices in the space of quaternions*, Math. Scand. **24** (1969), 31–65.

10. Tukia P., *On isomorphisms of geometrically finite Möbius groups*, Publ. Math. I.H.E.S. **61** (1985), 171–214.

11. Vinberg E. B., *Geometry II, Encyclopaedia of Math. Sciences* **29**, Springer-Verlag, Berlin Heidelberg, 1993.

12. Vulakh L. Ya., *Diophantine approximation in* $\mathbb{R}^n$, T.A.M.S. **347** (1995), 573–585.

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