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FINITE ELEMENTS IN MODELING OF FLOW IN POROUS MEDIA: HOW TO DESCRIBE WELLS

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Abstract. We consider a steady state flow in a porous medium caused by extraction wells and governed by Darcy's law. Point sources and the wells with positive diameters are considered. The conductivity of the soil matrix is not necessarily continuous. Several models and numerical schemes for modeling of wells are presented.

1. Introduction

Transport in porous media is usually multi-component (soil, air, water and volatile contaminant capable of crossing phase boundaries) and multi-phase (solid, liquid, gas and contaminant).

The macroscopic mass balance for component i in phase α can be written as

$$\partial_t(\rho^\alpha\varepsilon^\alpha\omega_i^\alpha) + \nabla\cdot(\rho^\alpha\mathbf{q}^\alpha\omega_i^\alpha) - \nabla\cdot\mathbf{J}_i^\alpha = \rho^\alpha\varepsilon^\alpha\left[f_i^\alpha + e_i^\alpha\right] \tag{1}$$

where ρ^α $\left[\frac{kg}{m^3}\right]$ is the mass density of the phase α; ε^α [1] is the volume fraction occupied by the phase α; $\mathbf{q}^\alpha$ $\left[\frac{m}{s}\right]$ is Darcy's velocity of the phase α; ω_i^α [1] is the mass fraction of component i in the α phase; $\mathbf{J}_i^\alpha$ $\left[\frac{kg}{m^2 s}\right]$ is the flux vector representing the diffusive flux of component i in the phase α; f_i^α $\left[\frac{1}{s}\right]$ is the source of component i in the phase α; e_i^α $\left[\frac{1}{s}\right]$ is the gain of mass of component i due to phase change.

Equation (1) is written under the following constraints

$$\sum_i \omega_i^\alpha = 1, \quad \sum_\alpha \varepsilon^\alpha = 1, \quad \sum_\alpha \rho^\alpha\varepsilon^\alpha e_i^\alpha = 0. \tag{2}$$

Wells are very often used as sources/sinks for some components, e.g., water, oil, air. Injection/extraction wells are sources/sinks with very small dimensions with respect to the whole domain in which the transport is considered. That means,

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they are very concentrated. This causes troubles by the modeling as well as by computations.

In this paper we will give some overview how to model wells from mathematical point of view. Some models are taken from the literature and some schemes (mixed finite elements) are new.

2. How to Model Wells

2.1 Classical Approaches

The principle of superposition is in linear case, of course, not limited to adding wells. From this reason we restrict ourselves for a moment to a single extraction well with an infinitely small diameter located at the origin of our coordinate system. Let us suppose that our domain in infinite (in all directions) and we consider a homogeneous unconfined aquifer with the concuctivity K. Then a classical solution (outside the origin) for a single point sink is

$$u(\mathbf{x}) = \begin{cases} -\dfrac{s}{2\pi K} \ln |\mathbf{x}| & \text{in 2-D,} \\ \dfrac{s}{4\pi K|\mathbf{x}|} & \text{in 3-D.} \end{cases}$$

This solution so far has not included any realistic boundary conditions and it generate drawdowns ewerywhere. When a well is pumped near a stream for instance, the heads along the stream will not be affected. But our basic solution cannot satisfy such a constant-head condition along streams and lakes. But there exists a simple technique **method of images** to create some basic boundary conditions. Adding imaginary wells to the real point sink at strategic locations allows to generate infinitely long straight equipotentials or no-flow boundaries (cf. Haitjema [**Hai95**] or Wilson [**Wil95**]).

For the analytical description of single-phase flow caused by a single extraction well for a perfectly layered subsurface we refer the reader to Nieuwenhuizen, Zijl and Van Veldhuizen [**NZV95**]. In many cases, the soil matrix is neither homogeneous nor perfectly layered. The classical methods are not applicable in the case of complicated boundary or inhomogeneous soil matrix. Thus we will try to give reasonable definitions of solutions in a general case as well as we show some numerical schemes for computations.

2.2 Boundary Conditions

Let us start from a model situation. We consider a bounded 3-D domain (unconfined aquifer) with some point sources/sinks inside. A schematic plan is shown in Figure 1 (top view point).

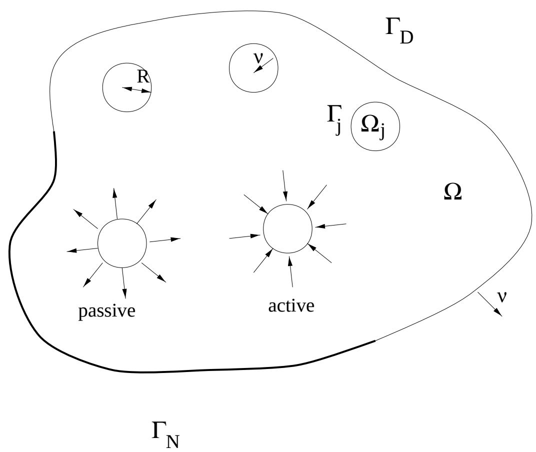

Figure 1. Schematic representation of a domain including active and passive wells.

We would like to model the wells. First of all we must take into account some restrictions and informations: For which component (water, gas,...) the well is constructed. Which data (pressure, discharge) at the well (or somewhere else) are given. What we are interested in: *(i)* to describe the transport at the vicinity of the well (e.g., establishment of the water table) *(ii)* to find out some physical values valid for large subdomains (e.g., to derive the hydraulic conductivity of a layer from pumping experiments).
All these informations are important in order to choose the appropriate boundary conditions. We distinguish the following cases

(P) **Pressure Condition.** Pressure is prescribed on the well. Dirichlet type condition. This type is frequently used for passive wells by soil venting.

(F) **Flux Condition.** Flux through the well boundary is prescribed. Neumann type condition. This type of condition is doubtful in many cases because of the flux distribution is completely unknown. This cannot be used for inhomogeneous vicinity of the well.

(D) **Discharge Condition.** It is assumed that a constant pressure builds up on the well boundary such that the prescribed discharge is obtained. The existence and uniqueness of such solution (for linear elliptic case) can be proved. It remains an open question which scheme should be used for numerical calculations.

(Di) **Dirac Type Condition.** When the well diameter could be neglected, Dirac functions are used for modeling of point sources.

(S) **Signorini Condition.** This type of condition is used for a well with positive diameter. Total discharge of the well is prescribed. Inflow into the well tube is modeled using unilateral (Signorini) boundary condition.

Conditions (P) and (F) are classical and well-known. Discharge condition (D) can be used for air pumping wells with a prescribed discharge but unknown pressure. In the rest of the paper we will consider the cases (Di) and (S).

3. Dirac Type Sinks

Let us consider a bounded domain $\Omega \in \mathcal{C}^{0,1}$ (see Kufner, John and Fučík [**KJF**, p. 270]) in $\mathbb{R}^N$ $(N = 2, 3)$ with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, where Γ_D, Γ_N denote the Dirichlet and Neumann parts, respectively. We assume that Γ_D has a positive measure. Let s_j, $(j = 1, \dots, n)$ be given numbers and $\delta_j(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_j)$. All $\mathbf{x}_j$ are interior points of Ω. We want to solve

Problem 1.

$$\begin{cases} -\nabla \cdot (K\nabla u) = \sum_{j=1}^{n} s_j \delta_j & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ -K\nabla u \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma_N. \end{cases}$$

Remark 1. Applying the method of superposition we can consider nonhomogeneous boundary conditions, too.

The diffusion coefficient K denotes the hydraulic conductivity of the soil matrix, or in the case of soil venting K denotes the air permeability.

Problem 1 is linear, but the right-hand side does not belong to the $\left(W^{1,2}(\Omega)\right)^*$ (dual space to $W^{1,2}(\Omega)$), thus we cannot directly apply the theory of linear elliptic equations.

3.1 Continuous Conductivity at the Well

The soil matrix is inhomogeneous on the pore scale, but on the macro scale it could be different. In many real cases the extraction wells are built in such a way that there is a homogeneous gravel surrounding the extraction tube. This homogeneous neighbourhood serves like a filter which entlarges the suction radius of the well. From this point of view one can suppose that the conductivity is smooth near the sinks (see Figure 2). The regularity of K allows to subtract singularities of the solution at the wells and in this way to give a reasonable definition of solution of Problem 1.

Let us denote the conductivities at the active wells by K_j, $(j = 1, \dots, n)$, i.e., $K_j = K(\mathbf{x}_j)$. Then

$$u_j(\mathbf{x}) = \begin{cases} -\dfrac{s_j}{2\pi K_j} \ln |\mathbf{x} - \mathbf{x}_j| & \text{in 2-D} \\ \dfrac{s_j}{4\pi K_j} \dfrac{1}{|\mathbf{x} - \mathbf{x}_j|} & \text{in 3-D} \end{cases}$$

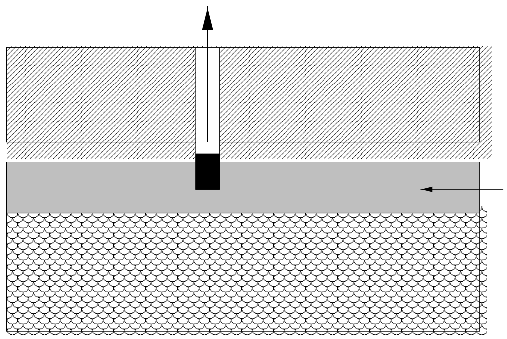

Figure 2. Continuous conductivity near the well (vertical cross section).

are the fundamental solutions of

$$-\nabla \cdot (K_j \nabla u_j) = s_j \delta_j \quad j = 1, \dots, n.$$

Now we give the definition of solution using subtraction of singularities.

Definition 1 (Continuous conductivity at the well). We say that u is a solution of Problem 1 iff

1. $u = \tilde{u} + \sum_{j=1}^{n} u_j$
2. $\tilde{u}$ is the solution of the following problem

$$\begin{cases} -\nabla \cdot (K \nabla \tilde{u}) = \sum_{j=1}^{n} \nabla \cdot ([K - K_j] \nabla u_j) & \text{in } \Omega \\ \tilde{u} = -\sum_{j=1}^{n} u_j & \text{on } \Gamma_D \\ -K \nabla \tilde{u} \cdot \boldsymbol{\nu} = \sum_{j=1}^{n} K \nabla u_j \cdot \boldsymbol{\nu} & \text{on } \Gamma_N. \end{cases} \tag{3}$$

Let the conductivity K be Hölder continuous (with the coefficient α) near each active well ($j = 1, \dots, n$)

$$\begin{cases} \alpha > 0 & \text{in 2-D} \\ \alpha > \frac{1}{2} & \text{in 3-D.} \end{cases} \tag{4}$$

The Hölder continuity of conductivity K at $\mathbf{x}_j$ ($j = 1, \dots, n$) implies $[K - K_j] \nabla u_j \in [L_2(\Omega)]^N$ (Lebesgue space). The existence and uniqueness of $\tilde{u} \in W^{1,2}(\Omega)$ follows from the theory of linear elliptic equations (cf. Gilbarg, Trudinger [**GT83**]), and $u = \tilde{u} + \sum_{j=1}^{n} u_j$.

3.2 Discontinuous Conductivity at the Well

The well can be located at an interface between two layers or at a rock (see Figure 3). Then we cannot apply the theory from Section 3.1. Nevertheless, one can define a **very weak** solution of Problem 1, i.e., the regularity of solution will be worse than in Definition 1. Using the transposition method of Stampacchia [**Sta66**] we explain the way how to give a more general definition of solution for a linear problem with Dirac type sinks without any continuity conditions for the conductivity K. For simplicity we restrict ourselves to the Dirichlet problem ($\Gamma_N = \varnothing$ and $\Gamma = \Gamma_D$), only.

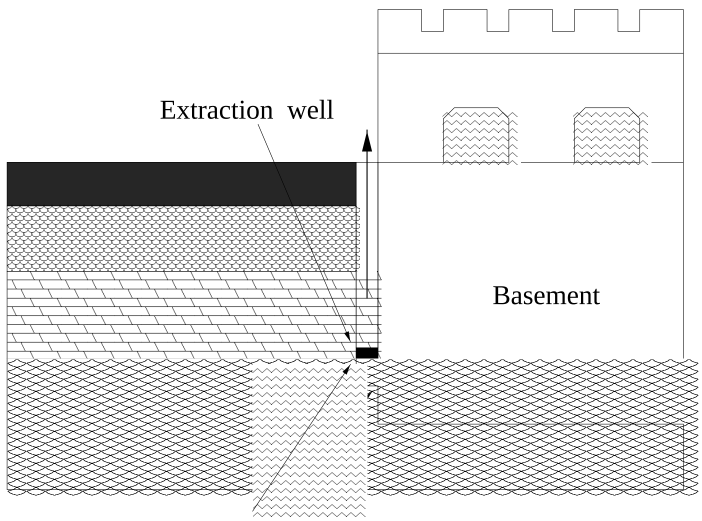

Figure 3. Inhomogeneous neighbourhood of the well (vertical cross section).

First, let us consider

Problem 2 (Adjoint problem). *Find* $U^\phi \in W_0^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ *such that*

$$\int_\Omega K \nabla U^\phi \nabla \psi = \int_\Omega \phi \psi \quad \forall \psi \in W_0^{1,2}(\Omega)$$

for a given $\phi \in L_p(\Omega)$, $p > \frac{N}{2}$.

The theory of linear elliptic equations implies the existence and uniqueness of solution of Problem 2. Gilbarg, Trudinger [**GT83**, Theorem 8.30] guarantees the continuity of the solution up to the boundary, i.e., $U^\phi \in C^0(\overline{\Omega})$. So we are able to define the mapping $T \in \mathcal{L}\left(L_p(\Omega), C^0(\overline{\Omega})\right)$ given by

$$T\colon \phi \longrightarrow U^\phi.$$

Here, $\mathcal{L}(X,Y)$ denotes the space of all linear mappings from X to Y.

Now, we can write for the adjoint operator $T^* \in \mathcal{L}\left((C^0(\overline{\Omega}))^*, L_q(\Omega)\right)$ with $p^{-1}+q^{-1}=1$, i.e., particular $T^* \in \mathcal{L}\left(\mathcal{M}_b, L_q(\Omega)\right)$, where $\mathcal{M}_b$ denotes the space of all Borel measures (see Kufner, John and Fučík [**KJF**, p. 43]).

The considerations above allow us to write

Definition 2 (Discontinuous conductivity). Let U^ϕ be the solution of the adjoint Problem 2 corresponding to $\phi \in L_p(\Omega)$, $p > \frac{N}{2}$. We say that $u \in L_q(\Omega)$, $p^{-1}+q^{-1}=1$ is a solution of Problem 1 iff

$$\int_\Omega u\phi = \int_\Omega \sum_{j=1}^n s_j \delta_j U^\phi = \sum_{j=1}^n s_j U^\phi(\mathbf{x}_j)$$

for all $\phi \in L_p(\Omega)$.

The following theorem implies the well-posedness of Definition 2.

Theorem 1 (Existence and uniqueness). *There exists an unique solution of Problem* 1 *in the sense of Definition* 2.

Proof. The existence follows from the considerations above. The proof of the uniqueness is straightforward, thus we left it to the reader. □

The Definition 2 shows the relation between u and U^ϕ, i.e., the relation between problem and its dual. From this reason we start with the study of the regularity of U^ϕ.

We denote by $||w||_{0,p,\Omega}$, $||w||_{1,p,\Omega}$ the norms in $L_p(\Omega)$, $W_0^{1,p}(\Omega)$, respectively.

Theorem 2 (Adjoint problem regularity). *Let* $\Omega \in \mathcal{C}^{1,0}$, $p > \frac{N}{2}$. *There exists a* P, $2 < P < \infty$, *depending only on the ellipticity constants* λ_0,λ_1

$$\lambda_0 \le |K(\mathbf{x})| \le \lambda_1$$

such that for any p^*, $2 \le p^* < P$ *the adjoint Problem* 2 *has an unique solution* $U^\phi \in W_0^{1,p^*}(\Omega)$ *and*

$$||U^\phi||_{1,p^*,\Omega} \le C||\phi||_{0,\frac{Np^*}{N+p^*},\Omega}.$$

- *if* $\frac{\lambda_1}{\lambda_0}$ *is large then* P *is close to* 2,
- *if* $\frac{\lambda_1}{\lambda_0}$ *is close to* 1 *then* P *is close to* ∞.

Proof. This regularity results follows from Meyers [**Mey63**] and Simader [**Sim72**, p. 90]. □

Remark 2. Let us suppose $\Omega \in \mathcal{C}^{1,0}$, $p > \frac{N}{2}$. Then in 2-D case $\phi \in L_p(\Omega)$ implies $U^\phi \in W_0^{1,p^*}(\Omega)$ for some $p^* > N$. In 3-D case the same is true if $\frac{\lambda_1}{\lambda_0}$ is not very large.

The following theorem says about the comparison of Definition 1 and Definition 2 (cf. Slodička [**Slo97**]).

Theorem 3 (Comparison of definitions). *Let the conductivity K be Hölder continuous near each active well $\mathbf{x}_j$ ($j = 1, \ldots, n$) with a Hölder coefficient α which satisfies* (4)*. Let U^ϕ be the solution of adjoint Problem* 2 *for $\phi \in L_p(\Omega)$, $p > \frac{N}{2}$. We suppose that $\forall \phi \in L_p(\Omega)$ $U^\phi \in W^{1,p^*}(\Omega)$ for some $p^* > N$. If u is the solution in the sense of Definition* 1*, then it is a solution in the sense of Definition* 2*.*

4. Approximation of the δ Function

The use of Dirac functions in approximation schemes could cause troubles (e.g., when a test function in a variational formulation is not continuous). Then, some approximations of δ functions are usefull. The simplest example is

$$\delta_{\varepsilon,j}(\mathbf{x}) = \begin{cases} \dfrac{1}{|S_{\varepsilon,j}|} & \text{for } |\mathbf{x} - \mathbf{x}_j| \le \varepsilon \\ 0 & \text{for } |\mathbf{x} - \mathbf{x}_j| > \varepsilon. \end{cases}$$

Here $S_{\varepsilon,j}$ denotes the ball in $\mathbb{R}^N$ with the center $\mathbf{x}_j$ and the radius ε. Let us note that

$$\int_{\mathbb{R}^N} \delta_{\varepsilon,j} = \int_{S_{\varepsilon,j}} \delta_{\varepsilon,j} = 1,$$

which makes this kind of approximation reasonable. Now, the original Problem 1 can be approximated by

Problem 3 (Approximation of Problem 1).

$$\begin{cases} -\nabla \cdot (K \nabla u_\varepsilon) = \displaystyle\sum_{j=1}^{n} s_j \delta_{\varepsilon,j} & \text{in } \Omega \\ u \qquad = 0 & \text{on } \Gamma_D \\ -K \nabla u \cdot \boldsymbol{\nu} \quad = 0 & \text{on } \Gamma_N. \end{cases}$$

where $\varepsilon > 0$ is sufficiently small.

The right-hand side in Problem 3 belongs to $L_\infty(\Omega)$ for arbitrary fixed $\varepsilon > 0$. Thus, the existence and uniqueness of solution $u_\varepsilon \in W_0^{1,2}(\Omega)$ is guaranted by the theory of linear elliptic equations. The relation between Problem 3 and Problem 1 (for Dirichlet case) is shown in Slodička [**Slo97**]:

Theorem 4 (Approximation of a δ function). *We suppose $p > \frac{N}{2}$, $p^{-1} + q^{-1} = 1$. Then*

$$u_\varepsilon \rightharpoonup u \quad \text{in } L_q(\Omega) \quad \text{as} \quad \varepsilon \to 0$$

and u is the solution of Problem 1 *in the sense of Definition* 2*.*

4.1 Standard Finite Element Scheme

Let Ω be a polyhedral domain in $\mathbb{R}^N$ $(N = 2, 3)$. We consider the simplest piecewise linear standard finite elements. We denote by $\mathcal{T}_h$ a triangulation of Ω with mesh diameter h. All point sinks (located at $\mathbf{x}_j$; $j = 1, \ldots, n$) lie on the interior nodes of the triangulation $\mathcal{T}_h$. We suppose that $\mathcal{T}_h$ is regular (see Ciarlet, Lions [**CL91**, Chapt. III, §16]). We associate the following finite element spaces

$$X_h = \{\psi_h \in C^0(\overline{\Omega});\quad \psi_{h|\mathcal{T}} \text{ is linear } \ \forall \mathcal{T} \in \mathcal{T}_h\},$$
$$V_h = \{\psi_h \in X_h;\quad \psi_h = 0 \text{ on } \Gamma\}$$

with the triangulation $\mathcal{T}_h$.

We consider the following discrete problem

Problem 4. *Find $u_h \in V_h$ such that*

$$\int_\Omega K \nabla u_h \nabla \psi_h = \sum_{j=1}^{n} s_j \psi_h(\mathbf{x}_j) \quad \forall \psi_h \in V_h.$$

Remark 3. In the case when a test function is continuous we do not need to approximate the Dirac function. In the opposite case one can use an approximation cf. Figure 4.

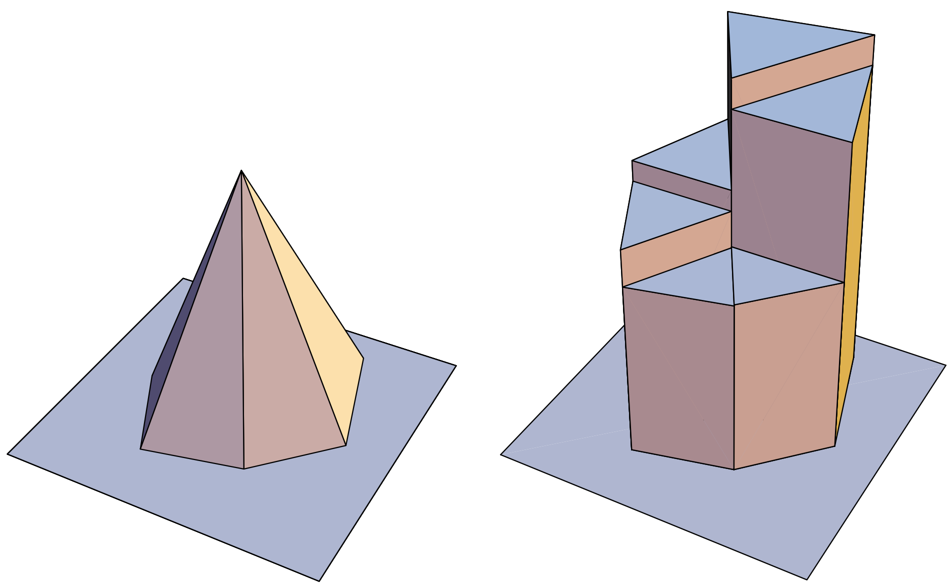

Figure 4. Shape function for approximation of the Dirac function for standard (left) and mixed (right) finite elements.

The convergence of the proposed numerical scheme is proved in Slodička [**Slo97**]:

Theorem 5 (Convergence). *Let $p > \frac{N}{2}$ and U^ϕ be the solution of the adjoint Problem 2 for $\phi \in L_p(\Omega)$. We suppose that $\forall \phi \in L_p(\Omega)$ $U^\phi \in W^{1,p^*}(\Omega)$ for some $p^* > N$. Then*

$$u_h \rightharpoonup u \quad \text{in } L_q(\Omega) \quad \text{as } h \to 0$$

for $p^{-1} + q^{-1} = 1$ and u is the solution of Problem 1 in the sense of Definition 2.

4.2 Mixed Finite Element Scheme

In this section we develop a numerical scheme for the classical mixed finite elements for the Problem 1 without any continuity conditions for the conductivity K. Let Ω be a polyhedral domain in $\mathbb{R}^N$ $(N = 2, 3)$. The triangulation $\mathcal{T}_h$ of Ω (with mesh diameter h) is supposed to be regular. The set of all edges (faces in 3-D) of $\mathcal{T}_h$ is denoted by $\mathcal{E}_h$. We will use the following spaces of test functions

- $RT_0(\mathcal{T}) = (P_0(\mathcal{T}))^N + \mathbf{x}P_0(\mathcal{T}) \subset (P_1(\mathcal{T}))^N$ for each N-simplicial (triangular or tetrahedral) element $\mathcal{T} \in \mathcal{T}_h$. $P_k(\mathcal{T})$ denotes the set of all polynomial functions of order k on $\mathcal{T}$,
- $RT_0^{-1} = \{\mathbf{q} \in (\mathrm{Ł}_2(\Omega))^N; \mathbf{q}|_{\mathcal{T}} \in RT_0(\mathcal{T}), \forall \mathcal{T} \in \mathcal{T}_h\}$,
- $RT_0 = \{\mathbf{q} \in RT_0^{-1}; \mathbf{q} \cdot \boldsymbol{\nu}_e$ is continuous $\forall e \in \mathcal{E}_h\}$,
- $\mathcal{L}_0^0(\mathcal{T}_h)$ the set of all functions constant on each triangle $\mathcal{T} \in \mathcal{T}_h$.

First, we start with the adjoint problem. The classical mixed variational formulation for the adjoint problem (with the right hand-side $f \in L_2(\Omega)$) reads as

Problem 5. *Find $(\mathbf{q}^f, U^f) \in (RT_0 \cap H_{0,N}(div, \Omega), L_2(\Omega))$ such that $(\forall (\boldsymbol{\phi}, \psi) \in (RT_0 \cap H_{0,N}(div, \Omega), L_2(\Omega)))$*

$$\begin{cases} \left(K^{-1}\mathbf{q}^f, \boldsymbol{\phi}\right) - \left(U^f, \nabla \cdot \boldsymbol{\phi}\right) = 0 \\ \left(\nabla \cdot \mathbf{q}^f, \psi\right) = (f, \psi)\,. \end{cases} \tag{5}$$

The corresponding discrete adjoit problem is

Problem 6. *Find $(\mathbf{q}_h^f, U_h^f) \in (RT_0 \cap H_{0,N}(div, \Omega), \mathcal{L}_0^0(\mathcal{T}_h))$ such that $(\forall (\boldsymbol{\phi}_h, \psi_h) \in (RT_0 \cap H_{0,N}(div, \Omega), \mathcal{L}_0^0(\mathcal{T}_h)))$*

$$\begin{cases} \left(K^{-1}\mathbf{q}_h^f, \boldsymbol{\phi}_h\right) - \left(U_h^f, \nabla \cdot \boldsymbol{\phi}_h\right) = 0 \\ \left(\nabla \cdot \mathbf{q}_h^f, \psi_h\right) = (f, \psi_h)\,. \end{cases} \tag{6}$$

We suppose that for all $\mathcal{T} \in \mathcal{T}_h$ and $f \in L_2(\Omega)$ we have

$$\frac{1}{|\mathcal{T}|} \left| \int_{\mathcal{T}} U_h^f - U^f \right| \longrightarrow 0 \quad \text{as} \quad h \to 0. \tag{7}$$

Let us note that this assumption is weaker than the convergence in the L_∞ norm, because of

$$\frac{1}{|\mathcal{T}|}\left|\int_{\mathcal{T}} U_h^f - U^f\right| \le \left|U_h^f - U^f\right|_{L_\infty(\mathcal{T})} \le \left|U_h^f - U^f\right|_{L_\infty(\Omega)}.$$

Remark 4. Assumption (7) is satisfied for standard finite elements (see Ciarlet, Lions [**CL91**, Theorem 21.5]).

We assume that all point sinks (located at $\mathbf{x}_j$; $j = 1, \dots, n$) lie at the interior nodes of the triangulation $\mathcal{T}_h$ and the number n_j of all elements containing $\mathbf{x}_j$ ($j = 1, \dots, n$) is finite and independent of h. We will use the following approximation of δ function ($j = 1, \dots, n$), cf. Figure 4

$$\delta_{h,j} = \begin{cases} \dfrac{1}{|\mathcal{T}|n_j} & \text{iff } \mathbf{x}_j \in \mathcal{T} \\ 0 & \text{else.} \end{cases} \tag{8}$$

Thus, one can easily see

$$\int_\Omega \delta_{h,j} = \sum_{\mathcal{T}} \int_{\mathcal{T}} \delta_{h,j} = 1.$$

We propose the following finite element scheme

Problem 7. *Find* $(\mathbf{q}_h, u_h) \in (RT_0 \cap H_{0,N}(div, \Omega), \mathcal{L}_0^0(\mathcal{T}_h))$ *such that* $(\forall (\boldsymbol{\phi}_h, \psi_h) \in (RT_0 \cap H_{0,N}(div, \Omega), \mathcal{L}_0^0(\mathcal{T}_h)))$

$$\begin{cases} \left(K^{-1}\mathbf{q}_h, \boldsymbol{\phi}_h\right) - (u_h, \nabla \cdot \boldsymbol{\phi}_h) = 0 \\ \qquad (\nabla \cdot \mathbf{q}_h, \psi_h) = \displaystyle\sum_{j=1}^n s_j \left(\delta_{h,j}, \psi_h\right). \end{cases} \tag{9}$$

Now, we are at the position to prove the convergence of the proposed scheme.

Theorem 6. *Let U^f be the solution of the adjoint Problem 5 corresponding to $f \in L_2(\Omega)$ and the condition (7) be satisfied. We suppose that $\forall f \in L_2(\Omega)$ $U^f \in W^{1,p^*}(\Omega)$ for some $p^* > N$. Then*

$$(u_h, f) \longrightarrow (u, f) = \sum_{j=1}^n s_j \left(\delta_j, U^f\right) = \sum_{j=1}^n s_j U^f(\mathbf{x}_j),$$

where u is the solution in the sense of the Definition 2.

Proof. Problem 7 admits an unique solution $(\mathbf{q}_h, u_h)$. Setting $\psi_h = u_h$ into (6) we have

$$(u_h, f) = \left(\nabla \cdot \mathbf{q}_h^f, u_h\right).$$

Now applying (9) we deduce

$$\left(\nabla \cdot \mathbf{q}_h^f, u_h\right) = \left(K^{-1}\mathbf{q}_h, \mathbf{q}_h^f\right) = \left(\mathbf{q}_h, K^{-1}\mathbf{q}_h^f\right).$$

Using (6) and (9) we obtain

$$\left(\mathbf{q}_h, K^{-1}\mathbf{q}_h^f\right) = \left(U_h^f, \nabla \cdot \mathbf{q}_h\right) = \sum_{j=1}^{n} s_j \left(\delta_{h,j}, U_h^f\right).$$

Hence we can write

$$(u_h, f) = \sum_{j=1}^{n} s_j \left(\delta_{h,j}, U_h^f\right) = \sum_{j=1}^{n} s_j \left(\delta_{h,j}, U_h^f - U^f\right) + \sum_{j=1}^{n} s_j \left(\delta_{h,j}, U^f\right).$$

The assumption $U^f \in W^{1,p^*}(\Omega)$ for some $p^* > N$ yields $U^f \in C(\Omega)$. Thus

$$\sum_{j=1}^{n} s_j \left(\delta_{h,j}, U^f\right) \longrightarrow \sum_{j=1}^{n} s_j U^f(\mathbf{x}_j).$$

Applying (7) we can see

$$\sum_{j=1}^{n} s_j \left(\delta_{h,j}, U_h^f - U^f\right) = \sum_{j=1}^{n} \frac{s_j}{n_j} \sum_{\substack{\mathcal{T} \\ \mathbf{x}_j \in \mathcal{T}}} \frac{1}{|\mathcal{T}|} \int_{\mathcal{T}} \left[U_h^f - U^f\right] \longrightarrow 0.$$

Collecting all these considerations we can write

$$(u_h, f) \longrightarrow \sum_{j=1}^{n} s_j U^f(\mathbf{x}_j) = (u, f),$$

where u is the solution in the sense of the Definition 2. □

5. Effect of Storage in a Well of Positive Radius

When a water extraction well of finite diameter is considered, the storage capacity in the well tube must be taken into account. That means that one part of the probe discharge comes from the soil matrix and the other one from the well tube. By this situation the waterhead inside and outside the extraction tube could be different, i.e., the seepage face can exist (cf. Figure 5).

Papadopulos and Cooper [**PCJ67**] have solved the problem of flow into a well of large diameter. They assumed that the drawdown in the aquifer at the face of the well was equal to that at the well at any time, thus they neglected the

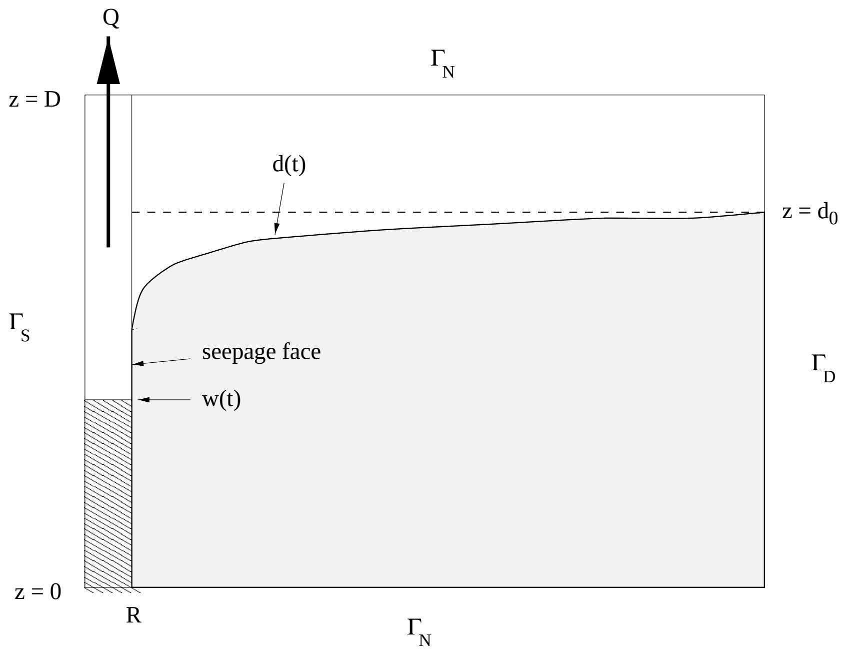

Figure 5. A vertical cross section through a well.

seepage face. Using the Laplace transform Papadopulos and Cooper obtained an explicit formula for the drawdown of the water table depending on time and radial distance.

A more sophisticated model including storage capacity for arbitrary well radius has been described by Neumann and Witherspoon, see [**NW71**]. Compared to Papadopulos and Cooper, they additionally considered the seepage face and computed the water table from a free and moving boundary problem numerically.

A different noniterative model for the direct implementation of well bore boundary conditions has been presented by Sudicky, Unger and Lacombe [**SUL95**]. Here the algorithm is formulated by superimposing conductive one-dimensional line elements representing the well screen onto the three-dimensional matrix elements representing the aquifer. The authors considered the continuity of the waterhead inside and outside the well tube, i.e., they have omitted the seepage face.

All models mentioned about describe the movement of water in the saturated zone. The influence of the unsaturated zone is completelly neglected. Schumacher, Slodička and Jaekel [**SSJ**] have combined the Richards equation together with a van Genuchten model for the description of the unsaturated zone. The water inflow into the well tube is described by unilateral (Signorini) boundary condition (see e.g. Duvaut and Lions [**DL76**] or Baiocchi and Capelo [**BC84**]). This model can be described mathematically as

Problem 8. *Find $\psi(t,\mathbf{x})$ such that*

$$\begin{cases} \partial_t\theta(\psi(t)) + \nabla\cdot\mathbf{q}(t) = 0 & \text{in } \Omega,\ 0<t<T \\ \mathbf{q}(t) = -K(\psi(t))\nabla u(t) & \\ u(t) = \psi(t) + z & \end{cases} \tag{10}$$

with the initial and boundary conditions

$$\begin{array}{ll} u(0) = d_0 & \text{in } \Omega \\ \mathbf{q}(t)\cdot\boldsymbol{\nu} = 0 & \text{on } \Gamma_N \\ u(t) = d_0 & \text{on } \Gamma_D \\ \left.\begin{array}{ll} \psi(t)\le 0,\ \mathbf{q}(t)\cdot\boldsymbol{\nu}\ge 0,\ \psi(t)\mathbf{q}(t)\cdot\boldsymbol{\nu} = 0 & \text{for } z\ge w(t) \\ \psi(t) = w(t) - z & \text{for } z < w(t) \end{array}\right\} & \text{on } \Gamma_S. \end{array} \tag{11}$$

Continuity equation for the water inside the well tube

$$\pi R^2\partial_t w(t) = 2\pi R\int_0^D \mathbf{q}\cdot\boldsymbol{\nu} - Q, \tag{12}$$

where θ denotes the saturation, K conductivity, ψ pressure, $\mathbf{q}$ the mass flow, R the well radius, Q the discharge of the well. D is the thickness of the aquifer. The Neumann, Dirichlet ans Signorini boundaries are denoted by $\Gamma_N, \Gamma_D, \Gamma_S$, respectively.

The van Genuchten model is used to describe the material functions.

$$\begin{cases} \theta(\psi) = \begin{cases} \theta_r + \dfrac{\theta_s-\theta_r}{(1+|\alpha\psi|^n)^m} & \text{for } \psi<0 \\ \theta_s & \text{for } \psi\ge 0 \end{cases} \\ K(\psi) = K_s S^{\frac{1}{2}}\left[1-\left(1-S^{\frac{1}{m}}\right)^m\right]^2, \\ S = \dfrac{\theta(\psi)-\theta_r}{\theta_s-\theta_r} \end{cases} \tag{13}$$

with coefficients $\theta_s, \theta_r, \alpha, n$ and $m = 1-\frac{1}{n}$.

Changes of the water table are large at the beginning of pumping experiment, thus the one must start with small time steps and the magnitude of time steps could increse in time

$$t_1 - t_0 \le t_2 - t_1 \le \cdots \le t_{i_{\max}} - t_{i_{\max}-1}.$$

The numerical scheme for computations is the following

1. **Time loop** $(i = 1,\ldots,i_{\max})$

2. **Signorini loop** ($sig = 1, \ldots, sig_{\max}$) The Signorini outflow condition is approximated by a sequence of Dirichlet (at the saturated part of the screen boundary) and Neumann (at the unsaturated part of the screen boundary) conditions, so that the continuity equation (12) should be fulfilled. In the iteration process the Signorini outflow condition ic checked on corresponding edges of the triangulation and the necessary redeclarations from Dirichlet $\longleftrightarrow$ Neumann boundary condition is made there.

   The water table $w_{i,sig}$ at the well tube is determined by a Newton-like algorithm (using a small perturbation Δh of the argument for numerical determination of the derivative $\dfrac{d\, q(w_{i,sig-1})}{d\, w_{i,sig-1}}$)

$$w_{i,sig} = w_{i,sig-1} + \frac{Q - \left[q(w_{i,sig-1}) - \dfrac{\pi R^2}{t_i - t_{i-1}} (w_{i,sig-1} - w_{i-1}) \right]}{\dfrac{q(w_{i,sig-1}) - q(w_{i,sig-1} - \Delta h)}{\Delta h} - \dfrac{\pi R^2}{t_i - t_{i-1}}}$$

   in order to obtain the prescribed dicharge Q of the well.
3. **Linearization loop** ($j = 1, \ldots, j_{\max}$) Linearization of the nonlinear Richard's PDE. The linear elliptic equation to be solved is

$$\theta' \left(u_{i,sig,j-1} - z\right) \frac{u_{i,sig,j} - u_{i-1}}{t_i - t_{i-1}} - \nabla \cdot \left(K \left(u_{i,sig,j-1} - z\right) \nabla u_{i,sig,j}\right) = 0. \tag{14}$$

with the unknown $u_{i,sig,j}$, where

- $t_i - t_{i-1}$ denotes the length of a time step,
- $q(w)$ denotes the flow through Signorini boundary

$$q(w) = 2\pi R \int_0^D \mathbf{q} \cdot \boldsymbol{\nu}$$

  with the water table w inside the well,
- $u_{i-1} = u_{i-1,sig_{\max},j_{\max}}$ is the solution from the last time step.

For the computations a radial symmetric formulation was used (this reduced the computational effort). The mixed nonconforming finite element method has been used for computations (see Arnold and Brezzi [**AB85**]) together with the refinement strategy for linear elliptic equations developed by Hoppe and Wohlmuth [**HW97**]. The coarsening strategy used by calculations is

An element (father) on the level l removes his sons (elements) on the level $l+1$ iff

- sons do not have sons

- $$\sum_{sons} \int_{son} |u^l - u^{l+1}|^2 < \varepsilon \sum_{sons} \int_{son} |u^{l+1}|^2,$$

  where u^l denotes the solution on the level l and ε is a given tolerance. Integrals are computed using the simplest quadrature rule.

The water table for the well radius $R = 0.162\,\mathrm{m}$ at different time steps is drawn in Figure 6. Gray color denotes the water inside the well tube. The water inflow

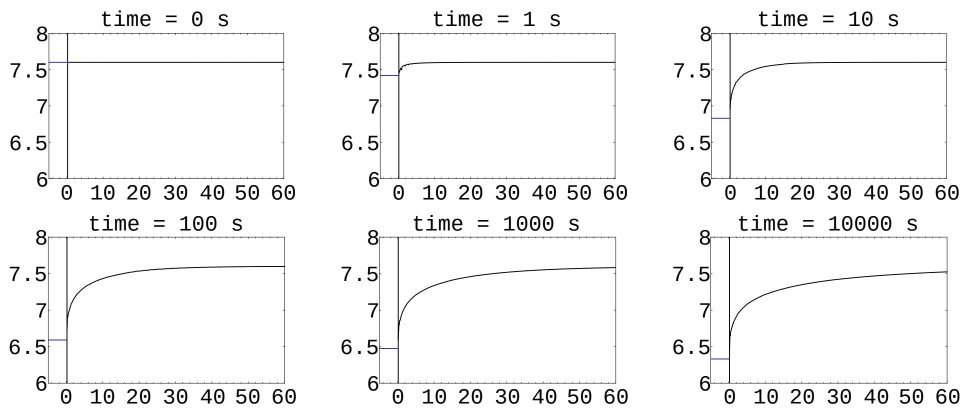

Figure 6. Waterhead in the well and soil. Well radius R = 0.162 m.

along the well screen in drawn in Figure 7. The flux takes its maximum at the bottom of the seepage face. Below this point, the flux decreases, because of the increasing hydrostatic water pressure. The influence of the well diameter on the

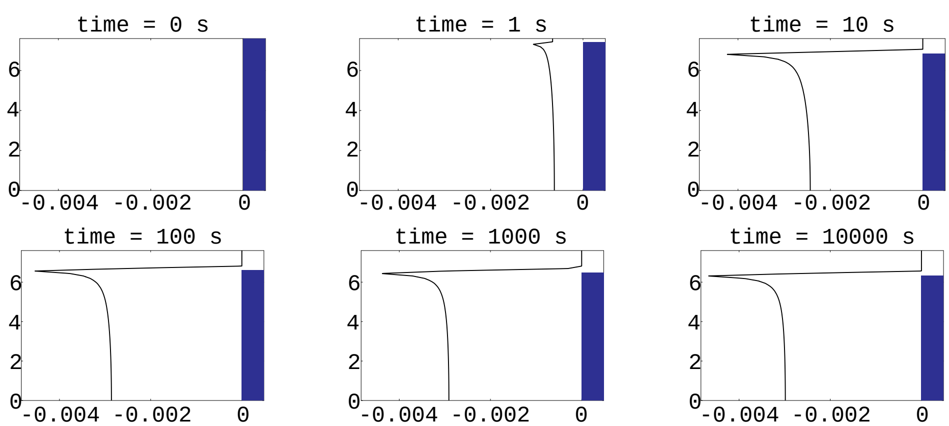

Figure 7. Flux distribution along the screen. Well radius $R = 0.162\,\mathrm{m}$.

length of the seepage face is shown in Figure 8. For large values of the well radius

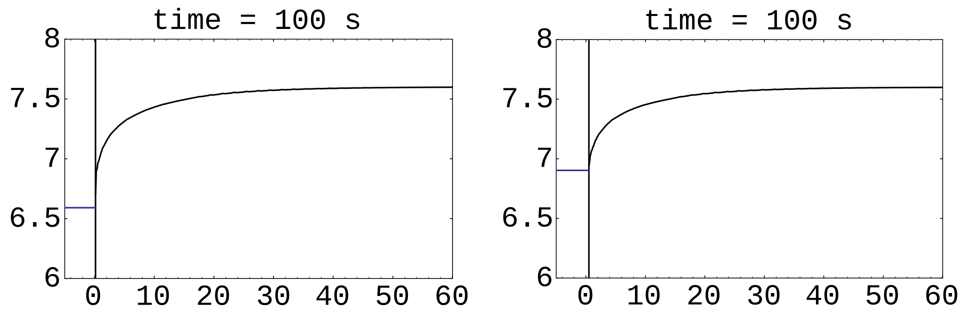

Figure 8. Waterhead in the well and soil. Well radius $R = 0.162\,\mathrm{m}$ (left) and $R = 0.5\,\mathrm{m}$ (right).

the capacity of the well tube is large, and a small change of the water table in the tube gives a large additive to a discharge.

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