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NONDEGENERATE INVARIANT BILINEAR FORMS ON NONASSOCIATIVE ALGEBRAS

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Abstract. A bilinear form f on a nonassociative algebra A is said to be invariant iff $f(ab,c) = f(a,bc)$ for all $a,b,c \in A$. Finite-dimensional complex semisimple Lie algebras (with their Killing form) and certain associative algebras (with a trace) carry such a structure. We discuss the ideal structure of A if f is nondegenerate and introduce the notion of T^*-extension of an arbitrary algebra B (i.e. by its dual space B^*) where the natural pairing gives rise to a nondegenerate invariant symmetric bilinear form on $A := B \oplus B^*$. The T^*-extension involves the third scalar cohomology $H^3(B,\mathbb{K})$ if B is Lie and the second cyclic cohomology $HC^2(B)$ if B is associative in a natural way. Moreover, we show that every nilpotent finite-dimensional algebra A over an algebraically closed field carrying a nondegenerate invariant symmetric bilinear form is a suitable T^*-extension. As a Corollary, we prove that every complex Lie algebra carrying a nondegenerate invariant symmetric bilinear form is always a special type of Manin pair in the sense of Drinfel'd but not always isomorphic to a Manin triple. Examples involving the Heisenberg and filiform Lie algebras (whose third scalar cohomology is computed) are discussed.

1. Introduction

The main subject of this article is the investigation of nonassociative (i.e. not necessarily associative) algebras A over a field $\mathbb{K}$ that carry a nondegenerate invariant bilinear form f. Such a form has the following defining properties:

$$f(ab,c) = f(a,bc) \qquad \forall a,b,c \in A \tag{1}$$

and

$$f(a,b) = 0 \ \forall b \in A \ \Rightarrow \ a = 0 \quad \text{and} \quad f(a,b) = 0 \ \forall a \in A \ \Rightarrow \ b = 0. \tag{2}$$

We shall call the pair (A,f) a **pseudo-metrised algebra** (or **metrised algebra** if f is symmetric) which should not be confused with any metric concepts of topology. A well-known example is any finite-dimensional full matrix algebra with its trace

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form $f(a,b) = \text{trace}\,(ab)$ or any finite-dimensional real or complex semisimple Lie algebra with its Killing form $f(a,b) = \text{trace}\,(ad(a)ad(b))$.

The motivation for studying these algebras comes from the fact that metrised Lie or associative algebras have shown up in several areas of mathematics and physics:

1. **Cartan's criterion** (i.e. the nondegeneracy of the Killing form) for the semisimplicity of a finite-dimensional Lie algebra has been an important tool for developing the structure theory of finite-dimensional semisimple Lie algebras over a field of characteristic zero (see e.g. [**30**] or [**28**]). Hence it seems to be interesting what kind of structure theorems can be derived from the more general conditions (1) and (2).

2. It is known that every nondegenerate (but possibly indefinite) scalar product f on the Lie algebra $\mathcal{G}$ of a finite-dimensional real Lie group G can be extended to a left-invariant (or right-invariant) pseudo-Riemannian metric on G (see e.g. [**53**] or [**49**]). According to Arnol'd's theory of generalized spinning tops (cf. [**5**, Appendix 2]) these metrics are interpreted as the tensor of inertia appearing in the kinetic energy of the top. A **totally symmetric top** would then correspond to a **pseudo-Riemannian metric on G which is both left-invariant and right-invariant**. But this requirement restricts the choice of f, and the crucial condition on f is exactly eqn (1) if G is connected. Hence $(\mathcal{G}, f)$ has to be metrised.

3. A more general situation appears if a reductive homogeneous space G/H together with a G-invariant pseudo-Riemannian metric Q is considered (cf. e.g. [**38**, Ch. X, p. 200]). The metric Q is determined by an $Ad_G(H)$-invariant nondegenerate scalar product q on an $Ad_G(H)$-invariant vector space complement $\mathcal{M}$ to the Lie algebra $\mathcal{H}$ of the closed Lie subgroup H in $\mathcal{G}$. If each geodesic of Q emanating at the point H of G/H has the form of a projected one-parameter-subgroup generated by an element of $\mathcal{M}$ then q has to satisfy further conditions (see [**38**, Ch. X, p. 201, Thm. 3.3.(2)]) and $(G/H, Q)$ is called **naturally reductive**. Now each symmetric invariant nondegenerate bilinear form f on $\mathcal{G}$ whose restriction q to $\mathcal{M}$ is nondegenerate induces such a naturally reductive structure on G/H (see [**38**, p. 203, Thm. 3.5.]). However, if G acts almost effectively on G/H and the space $\mathcal{M}$ generates $\mathcal{G}$, the converse statement is also true, i.e. q induces a symmetric invariant nondegenerate bilinear form f on $\mathcal{G}$ as was shown by Kostant for the compact case (cf. [**43**]) and Forger for the general case (cf. [**22**, p. 106, Thm. 2]). Hence the structure theory of these spaces can again be reduced to the algebraic problem of real metrised finite-dimensional Lie algebras $(\mathcal{G}, f)$.

4. In the theory of those **completely integrable Hamiltonian systems** (see [**5**, p. 271] for definitions) that admit a Lax representation in a finite-dimensional real Lie algebra $\mathcal{G}$ (see e.g. [**17**] or [**11**]) one often has an additional Lie structure on the dual space $\mathcal{G}^*$ of $\mathcal{G}$ by means of a so-called r-matrix which is related to the involutivity of the constructed integrals of motion. Given this situation, there also exists a Lie structure on the vector space direct sum $\mathcal{A} := \mathcal{G} \oplus \mathcal{G}^*$ such that $\mathcal{G}$ and $\mathcal{G}^*$ are both subalgebras of $\mathcal{A}$ and the nondegenerate symmetric bilinear form q induced by the natural pairing of $\mathcal{G}$ and $\mathcal{G}^*$ is invariant. The structure $(\mathcal{A}, \mathcal{G}, \mathcal{G}^*, q)$ is called a **Manin triple** (see [**17**]). Again, $(\mathcal{A}, q)$ is a metrised Lie algebra. A particular case of this with abelian $\mathcal{G}$ appeared in a paper by Kostant and Sternberg on BRS cohomology (cf. [**42**], see also [**46**] and [**39**]).

5. Given a finite group G and a field $\mathbb{K}$ one can always construct a nondegenerate symmetric invariant bilinear form f on the group algebra A of G by declaring $f(g, g')$ to be 1 if gg' equals the unit element of G and 0 otherwise. Hence (A, f) is a particular example of a **symmetric Frobenius algebra**, i.e. a finite-dimensional metrised associative algebra with unit element. These algebras play an important role in the theory of (modular) representations of finite groups, see [**16**] or [**36**] for details.

6. **Statistical models over 2-dimensional graphs** of degree 3 and 4 whose partition function is "almost topological", i.e. invariant under a certain flip move in the graph have recently been classified (cf. [**13**]). The classification uses the observation that the statistical weights attached to the vertices and edges of the graph represent the structure constants of a finite-dimensional complex metrised associative algebra.

In view of this it is not astonishing that several articles on metrised Lie algebras or Frobenius algebras have been published up to now: for the latter see the book of Karpilovsky (cf. [**36**]) and references therein. Metrised Lie algebras have been dealt with by Ruse (cf. [**50**]), Tsou and Walker (cf. [**53**], [**54**]), Zassenhaus and Block (cf. [**56**], [**9**]), and Astrakhantsev (cf. [**6**], [**7**]). More recently, by the independent work of Kac (cf. [**34**, p. 23, Exercise 2.10 and 2.11]), Favre and Santharoubane (cf. [**20**]), Medina and Revoy (cf. [**47**], [**48**]), and Hilgert, Hofmann and Keith (cf. [**24**], [**25**], [**37**]) a major result, namely the so-called **double extension construction** had been developed: the simplest case of this method consists of a one-dimensional central extension followed by the semidirect addition of the scalar multiples of an antisymmetric derivation. Moreover, starting with an abelian Lie algebra of dimension zero or one one can construct every finite-dimensional solvable metrised Lie algebra by repeated application of this technique. The basic information which is needed for this procedure is the second scalar cohomology group $H^2(\mathcal{G}, \mathbb{K})$ of the Lie algebra $\mathcal{G}$ constructed at each step.

Now, if the proof of a theorem on the structure of a metrised Lie algebra or a Frobenius algebra is analysed it will often turn out that the Jacobi identity or associativity is not needed. Therefore, it seems to be natural to look for a structure theory of pseudo-metrised algebras that do not a priori satisfy any prescribed identity. This can for instance be used to get more information on metrised associative algebras by transferring methods used for Lie algebras and vice versa. As a further spin-off one gets theorems about other classes of nonasssociative algebras like Jordan or alternative algebras (see Schafer's book [**52**] or Appendix A for definitions).

Hence it is one of the purposes of the present paper to give a generalized review of the orthogonal structure of ideals developed for metrised Lie and associative algebras that is valid for arbitrary nonassociative algebras carrying a nondegenerate (not necessarily symmetric) invariant bilinear form (Section 2).

The above-mentioned method of double extension helps to construct finite-dimensional metrised Lie algebras from metrised Lie algebras of smaller dimension. However, there are two principal disadvantages of this technique: at least for the solvable Lie algebras it is a multistep procedure which can be very clumsy when it comes to higher dimensions. Furthermore, there does not seem to be any reasonable analogue of a double extension in other classes of algebras because every doubly extended Lie algebra is a nontrivial semidirect sum which is not the case for certain metrised associative algebras (compare the discussion following Thm. 4.2).

Therefore, it is the second purpose of this paper to introduce a different extension technique called T^*-extension (Section 3). This method is a one-step procedure and applies to all known classes of nonassociative algebras. Now, the main result of this paper is the proof of an important feature of this extension: all finite-dimensional nilpotent metrised algebras in these classes can be constructed by this method if the field is algebraically closed and of characteristic not two (see Cor. 3.1 in Section 3). One starts with an arbitrary algebra B and constructs an abelian extension by its dual space B^*. The natural pairing on $A = B \oplus B^*$ will give rise to a nondegenerate symmetric invariant bilinear form on the extended algebra A if a certain cyclic condition on the extension is satisfied. The case where the extension is split had been discussed in the literature before (see [**47**], [**42**], [**17**] for Lie algebras and Tachikawa for Frobenius algebras). However, split T^*-extensions alone do not exhaust all finite-dimensional nilpotent metrised algebras as can be seen by counterexamples (cf. Example 4.2 or 4.3 in Section 4). The basic information needed to construct these extensions is contained in the third scalar cohomology $H^3(B,\mathbb{K})$ of B if B is a Lie algebra and in the second cyclic cohomology $HC^2(B)$ of B if B is an associative algebra. For certain Lie algebras we compute this cohomology and construct some T^*-extensions explicitly (Section 4). In this way we get an example of a metrised Lie algebra of even dimension which is no Manin triple. On the other hand every metrised Lie algebra can be shown to be a certain Manin pair in the sense of Drinfel'd (cf. [**18**]).

The paper is organized as follows:

Section 2 contains information on the orthogonal structure of ideals (Prop. 2.1), and some homomorphism statements (Prop. 2.3). Every antisymmetric invariant bilinear form on an algebra A degenerates on the first derived ideal A^2 whence every (anti)commutative pseudo-metrised algebra is metrisable (Prop. 2.4). Thm. 2.1

shows that the notion of decomposition of any finite-dimensional pseudo-metrised algebra (A, f) into a direct sum of indecomposable ideals and decomposition into an orthogonal direct sum of f-indecomposable ideals is (up to annihilating abelian ideals) the same. In Thm. 2.2 the above-mentioned double extension method for Lie algebras is restated.

In Section 3 the method of T^*-extension is introduced (eqs. (5)–(9) and Lemma 3.1). Thm. 3.1(i) and (ii) shows that T^*-extension is compatible with nilpotency, solvability, and all well-known classes of nonassociative algebras. Every trivial T^*-extension preserves in a certain way the above-mentioned decomposition properties (Thm. 3.1(iii)). The basic recognition criterion for T^*-extensions is the existence of a maximally isotropic ideal (Thm. 3.2). Then we investigate the equivalence of T^*-extensions in the sense of cohomology (Prop. 3.1) and discuss the case of Lie algebras ($H^3(\mathcal{G}, \mathbb{K})$, see eqs. (12) and (13)) and associative algebras (cyclic cohomology, see eqs. (18) and (19)). By proving a Lemma on the existence of maximally isotropic subspaces in a metrised vector space which are invariant under the "transposition invariant" action of a nilpotent Lie algebra (see Lemma 3.2) we are able to show the above-mentioned main result (see Cor. 3.1) that every finite-dimensional nilpotent metrised algebra A "is" a suitable T^*-extension. A natural candidate for an isotropic ideal is constructed out of the central descending and ascending series (see eqn (20)).

In Section 4 we prove (Thm. 4.1) that every finite-dimensional metrised Lie algebra over an algebraically closed field of characteristic zero is a **Manin pair** in the sense of Drinfel'd (see [**18**]). A similar theorem can be derived for associative algebras (Thm. 4.2). Example 4.1 shows that nonisomorphic Lie algebras could have isometric T^*-extensions which raises the question whether every T^*-extension can be rewritten as the T^*-extension of a particularly "nice" algebra i.e. whose structure and cohomology are computable and/or classifiable. The Heisenberg and filiform Lie algebras (see Example 4.2 and 4.3) illustrate some features of the T^*-extension, notably that not every even-dimensional metrised Lie algebra over an algebraically closed field of characteristic zero is isometric to some Manin triple, in spite of the seemingly well-known fact that every semisimple such algebra is (see Example 4.2 and Thm. 4.1(iv)).

Appendix A contains a compilation of definitions and facts in the theory of nonassociative algebras and bilinear forms mainly based on Schafer's book [**52**] and a few other sources and may be used as a dictionary for notations appearing in the main sections.

The computation of $H^3(\mathcal{G}, \mathbb{K})$ of the filiform Lie algebras is done in Appendix B by applying the representation theory of $sl(2, \mathbb{K})$.

This article is a somewhat extended version of parts of my Diplomarbeit [**10**].

2. Invariant Bilinear Forms on Nonassociative Algebras

Let A be a (nonassociative) algebra over a field $\mathbb{K}$ (compare Appendix A for definitions). A bilinear form $f\colon A \times A \to \mathbb{K}$ is called **invariant** (or **associative**) iff it satisfies the following condition:

$$f(ab, c) = f(a, bc) \qquad \forall a, b, c \in A. \tag{3}$$

Any algebra A admits invariant bilinear forms (e.g. $f = 0$). However, there will be restrictions on the structure of A **if it admits a nondegenerate invariant bilinear form** f. If this is the case we shall call A **pseudo-metrisable** and the pair (A, f) a **pseudo-metrised algebra**. If in addition f is symmetric, we shall call A **metrisable** and the pair (A, f) a **metrised algebra** following Astrakhantsev (cf. [**6**]), Tsou and Walker (cf. [**53**]), and Ruse (cf. [**50**]).

For computational purposes it is often more convenient to have the following analogous formula for invariant f and three subspaces V,W, and X of A:

$$f(VW, X) = f(V, WX). \tag{4}$$

This is clear: if there are $v \in V$, $w \in W$, and $x \in X$ such that $f(vw, x)$ $(= f(v, wx))$ is nonzero then both sides of the above relation equal $\mathbb{K}$; if not they both vanish.

Some facts about the ideal structure of such algebras are contained in the following

Proposition 2.1. *Let (A, f) be a pseudo-metrised algebra over a field $\mathbb{K}$ and V an arbitrary vector subspace of A.*

(i) *Let I be an arbitrary ideal of A. Then ${}^{\perp}I$ and $I^{\perp}$ are again ideals of A satisfying $I(I^{\perp}) = 0 = ({}^{\perp}I)I$.*

(ii) $Z(V) = (VA) \cap (AV)$.
In particular, if f is (anti)symmetric or $AV = VA$ (e.g. for (anti)commutative A) one has

$$Z(V) = {}^{\perp}(AV + VA) = (AV + VA)^{\perp}$$

in which case $Z(V)$ is an ideal if V is an ideal. In particular:

$$Z = {}^{\perp}(A^2) = (A^2)^{\perp}.$$

In what follows assume that A has finite dimension:

(iii) $C(V) = (A({}^{\perp}V) + ({}^{\perp}V)A)^{\perp} = {}^{\perp}(A(V^{\perp}) + (V^{\perp})A)$

(iv) $C^i(A) = {}^{\perp}C_i(A) = C_i(A)^{\perp}$ $\forall i \in \mathbb{N}$

Proof. (i) Since $IA \subset I$ it follows $0 = f(IA, I^\perp) = f(I, A(I^\perp))$ implying $A(I^\perp) \subset I^\perp$. Because $AI \subset I$ it follows $0 = f(AI, I^\perp) = f(A, I(I^\perp))$ hence $I(I^\perp) = 0$. Therefore $0 = f(I(I^\perp), A) = f(I, (I^\perp)A)$ whence $(I^\perp)A \subset I^\perp$. ${}^\perp I$ is treated in an analogous way.

(ii) According to the definition of the annihilator, $z \in Z(V)$ iff $zV = 0$ and $Vz = 0$. This is equivalent to $f(zV, A) = 0$ and $f(A, Vz) = 0$ iff $f(z, VA) = 0$ and $f(AV, z) = 0$ which implies the first assertion. If f is (anti)symmetric right and left orthogonal spaces coincide and the second assertion follows by the duality relation (23). If $VA = AV$ then $z \in Z(V)$ iff $zV = 0$ iff $f(zV, A) = 0$ iff $f(z, VA) = 0$ iff $z \in {}^\perp(VA)$ iff $z \in {}^\perp(VA + AV)$ and in an analogous manner iff $z \in (VA + AV)^\perp$. If V is an ideal then $AV + VA$ is again an ideal and so is its left or right orthogonal space by (i). The last assertion is the particular case $V = A$.

(iii) Set $W := {}^\perp V$ and $J := (AW + WA)^\perp$. Since $WA \subset AW + WA$ it follows that $0 = f(WA, J) = f(W, AJ)$, hence: $AJ \subset W^\perp$. Since $AW \subset AW + WA$ one has $0 = f(AW, J) = f(A, WJ)$, hence: $WJ = 0$ and thus: $0 = f(WJ, A) = f(W, JA)$ implying $JA \subset W^\perp$. Both relations imply $AJ + JA \subset W^\perp$ giving $J \subset C(W^\perp)$. Conversely: $f(WA, C(W^\perp)) = f(W, A(C(W^\perp))) \subset f(W, W^\perp) = 0$ implying that $C(W^\perp) \subset (WA)^\perp$. Furthermore, we have $f(W(C(W^\perp)), A) = f(W, (C(W^\perp))A)$ which is contained in $f(W, W^\perp) = 0$ implying $W(C(W^\perp)) = 0$, hence $0 = f(A, W(C(W^\perp))) = f(AW, C(W^\perp))$ which gives $C(W^\perp) \subset (AW)^\perp$. From both relations it follows that $C(W^\perp)$ is contained in $(AW)^\perp \cap (WA)^\perp = (AW + WA)^\perp = J$ by eqn (23). Therefore, $C(W^\perp) = J$ proving the first assertion because $W^\perp = ({}^\perp V)^\perp = V$ by eqn (25) since A is finite-dimensional. The second assertion is proved in a completely analogous way starting with $W := V^\perp$ and $J := {}^\perp(AW + WA)$.

(iv) We shall use induction w.r.t. i: The case $i = 0$ is clear because of the relation $A = C^0(A) = {}^\perp 0 = {}^\perp(C_0(A)) = 0^\perp = (C_0(A))^\perp$. Assume that $C^i(A) = {}^\perp(C_i(A))$. It follows that $C^{i+1}(A) = A(C^i(A)) + (C^i(A))A$ by definition, and this is equal to $A({}^\perp(C_i(A))) + ({}^\perp(C_i(A)))A = {}^\perp((A({}^\perp(C_i(A))) + ({}^\perp(C_i(A)))A)^\perp)$ by the inversion formula (25), and this is equal to ${}^\perp(C(C_i(A)))$, hence to ${}^\perp(C_{i+1}(A))$ by (iii). The second assertion is proved in an analogous manner. □

Some consequences can be drawn from this Proposition: Firstly, any solvable nonzero pseudo-metrisable algebra of finite dimension must have a nonzero annihilator, because the codimension of A^2 equals $\dim Z$ by (ii). For example, the two-dimensional nonabelian Lie algebra cannot be pseudo-metrisable. Secondly, assertion (iv) shows that the central ascending series and the central descending series of any finite-dimensional pseudo-metrisable algebra are strongly related which is important for nilpotent algebras.

The following Proposition contains further properties of pseudo-metrisable algebras which are anticommutative (e.g. Lie algebras):

Proposition 2.2. *Let (A, f) be an anticommutative pseudo-metrised algebra over a field $\mathbb{K}$.*

(i) *If for two nonzero elements x and y of A and a 1-form g in A^* the following equation $xa = (g(a))y$ holds for all $a \in A$ then g must vanish.*
(ii) *For each one-dimensional ideal I the ideal $C(I)$ is equal to the annihilator Z of A. In particular, I is contained in Z.*

Proof. (i) Suppose there is an element $b \in A$ satisfying $g(b) \neq 0$. Clearly, one has the direct sum $A = \mathrm{Ker}\ g \oplus \mathbb{K}\,b$. For $a \in \mathrm{Ker}\ g$ it follows that $g(b)f(a, y) = f(a, xb) = f(ax, b) = -f(xa, b) = -g(a)f(y, b) = 0$. On the other hand, since A is anticommutative: $g(b)f(b, y) = f(b, xb) = -f(b, bx) = -f(bb, x) = 0$. As $g(b)$ was assumed to be nonzero it follows that $f(A, y) = 0$ contradicting the nondegeneracy of f. Hence g must be zero.

(ii) There is a nonzero element $y \in A$ such that $I = \mathbb{K}\,y$. Then for any $x \in C(I)$ there is a 1-form $g \in A^*$ such that $xa = g(a)y$ because I is an ideal. Using (i) one infers that $g = 0$ which implies $x \in Z$. Conversely, since $ZA = 0 \subset I$ it is clear that $Z \subset C(I)$. Since each ideal I is contained in $C(I)$ the Proposition is proved. □

A direct consequence of assertion (ii) of this Proposition is the fact that the annihilator Z of any anticommutative nilpotent pseudo-metrisable algebra of finite dimension ≥ 2 must be at least two-dimensional, for otherwise $Z = C_i(A)$ would be one-dimensional and consequently $C_1(A) = C(C_0(A)) = C(Z)$ would be equal to Z contradicting the nilpotency of A.

The following Proposition collects some facts about the transfer of invariant bilinear forms from one algebra to another one (consult Appendix A for definitions):

Proposition 2.3. *Let A (resp. A') be an algebra over a field $\mathbb{K}$ and f (resp. g) be an invariant bilinear form on A (resp. on A'). Let $m\colon A \to A'$ be a homomorphism of algebras.*

(i) *The pull-back m^*g of g is an invariant bilinear form on A.*
(ii) *Assume that m is surjective and that $\mathrm{Ker}\ m$ is contained in the kernel of f. Then the projection f^m is an invariant bilinear form on A'.*
(iii) *Let B be a subalgebra of A and assume that $B \cap {}^{\perp}B = B \cap B^{\perp}$ (this is the case if for instance f is (anti)symmetric). Then $B \cap B^{\perp}$ is an ideal of B. Let $p\colon B \to B/(B \cap B^{\perp})$ denote the canonical projection and f_B the restriction of f to $B \times B$. Then the projection $(f_B)^p$ is a nondegenerate invariant bilinear form on the factor algebra $B/(B \cap B^{\perp})$.*
(iv) *The bilinear form $f \perp g$ (resp. $f \otimes g$) on the direct sum $A \oplus A'$ (resp. the tensor product $A \otimes A'$) is invariant. Moreover, $f \perp g$ (resp. $f \otimes g$) is nondegenerate if and only if f and g are nondegenerate.*

Proof. The proofs of (i), (ii), and (iv) are completely straight forward (using Appendix A) and are left to the reader.

(iii) Since the canonical inclusion $B \to A$ is a homomorphism of algebras and f_B is equal to the pull-back of f to B it follows from (i) that f_B is invariant. B being a subalgebra of A we have $B^2 \subset B$, hence: $0 = f(BB, B^\perp) = f(B, B(B^\perp))$ implying $B(B^\perp) \subset B^\perp$. Analogously: $({}^\perp B)B \subset {}^\perp B$. Hence: $B(B \cap B^\perp) \subset (B \cap B^\perp)$ and $(B \cap {}^\perp B)B \subset (B \cap {}^\perp B)$. By assumption: $B \cap {}^\perp B = B \cap B^\perp$ whence $B \cap B^\perp$ is an ideal of B. Clearly, $f_B(B, B \cap B^\perp) = 0 = f_B(B \cap {}^\perp B, B) = f_B(B \cap B^\perp, B)$ hence Ker $p = B \cap B^\perp$ is equal to the kernel of f_B and consequently the projection $(f_B)^p$ is well-defined and nondegenerate on the factor algebra $B/(B \cap B^\perp)$. □

Part (i) of Prop. 2.3 gives rise to the following definition: let (A, f) and (B, g) two pseudo-metrised algebras. A linear map $\Phi\colon A \to B$ is said to be an **isometry** or an **isomorphism of pseudo-metrised algebras** iff Φ is an isomorphism of algebras and $f = \Phi^* g$.

The last assertion (iv) of the preceding Proposition can be used to construct pseudo-metrisable algebras: For instance, observing that for each integer $n \geq 1$ the commutative associative algebra $K(n, 1)$ (resp. $K(n)$) defined by the quotient of the polynomial algebra $\mathbb{K}[x]$ modulo the ideal (x^n) generated by x^n (resp. $\mathbb{K}[x]^+/(x^n)$ where $\mathbb{K}[x]^+$ is the ideal generated by x) is metrised by setting $f(x^i, x^j) := \delta_{i+j,n-1}$, $(0 \leq i, j \leq n-1$ and $x^0 := 1)$ (resp. $f(x^i, x^j) := \delta_{i+j,n}$, $(1 \leq i, j \leq n-1))$ one can form the tensor product $A \otimes K(n, 1)$ (resp. $A \otimes K(n)$) with a finite-dimensional semisimple Lie algebra A whose Killing form is nondegenerate to get a metrised Lie algebra with a nilpotent radical $A \otimes K(n) \subset A \otimes K(n, 1)$ (resp. a metrisable nilpotent Lie algebra) of arbitrary length n. In particular, the Lie algebra $TA := A \otimes K(2, 1)$ which as a vector space is isomorphic to $A \oplus A$ deserves special attention: if A is a finite-dimensional real Lie algebra belonging to a Lie group G then TA will be the Lie algebra of its tangent bundle: indeed, the map $TL\colon TG \to G \times A : v_g \mapsto (g, (T_e L_g)^{-1} v_g)$ (where v_g is a tangent vector at $g \in G$, e is the unit element of G, and $T_e L_g$ is the tangent map of the left multiplication map L_g at e) is a vector bundle isomorphism onto the semidirect product $G \times A$ where A is the abelian normal subgroup and the subgroup G acts on A by the adjoint (group) representation.

In the next Proposition we shall investigate the symmetry of invariant bilinear forms. As it will turn out antisymmetric bilinear forms have quite a large kernel:

Proposition 2.4. *Let A be an algebra over a field $\mathbb{K}$ and f an invariant bilinear form on A.*

(i) *If f is antisymmetric it obeys the equation*

$$2f(ab, c) = 0 \qquad \forall a, b, c \in A.$$

(ii) *Assume that the characteristic of $\mathbb{K}$ is different from 2 and that A is (anti)commutative and pseudometrisable. Then A is metrisable.*

Proof. (i) We use three times antisymmetry and invariance of f in interchanging order:

$$f(ab,c) = -f(c,ab) = -f(ca,b) = f(b,ca) = f(bc,a) = -f(a,bc) = -f(ab,c).$$

(ii) Choose a nondegenerate invariant bilinear form f on A. We set $f^t(a,b) := f(b,a)$ for $a,b \in A$. Clearly, f^t is a nondegenerate bilinear form on A. Moreover, f^t is invariant: Indeed, observing that for $a,b \in A$ we have $ab = \epsilon_A ba$ with either $\epsilon_A = 1$ (A commutative) or $\epsilon_A = -1$ (A anticommutative) we get for all $a,b,c \in A$:

$$f^t(ab,c) = f(c,ab) = \epsilon_A f(c,ba) = \epsilon_A f(cb,a) = f(bc,a) = f^t(a,bc).$$

It follows that the **symmetric part** f_s (resp. the **antisymmetric part** f_{as}) of f defined by $f_s(a,b) := (1/2)(f(a,b) + f^t(a,b))$ (resp. $f_{as}(a,b) := (1/2)(f(a,b) - f^t(a,b))$ is a symmetric (resp. antisymmetric) invariant bilinear form on A and clearly $f = f_s + f_{as}$. Because of assertion (i) of this Proposition we get the relation

$$f(A,A^2) = f_s(A,A^2). \tag{$*$}$$

Let N denote the kernel of f_s. Since $N = {}^{\perp'}A = A^{\perp'}$ (${}^{\perp'}$ denoting orthogonal space w.r.t. f_s) because of the symmetry of f_s it follows from Prop. 2.3(iii) that N is an ideal of A. As a particular case of $(*)$ we get

$$f(A, N \cap A^2) = f_s(A, N \cap A^2) \subset f_s(A,N) = 0$$

implying

$$N \cap A^2 = 0 \tag{$**$}$$

since f is nondegenerate. Clearly $NA \subset A^2 \cap N = 0$ whence $N \subset Z$. Now take any vector subspace V of A such that $A = V \oplus (N \oplus A^2)$. Clearly $B := V \oplus A^2$ is an ideal of A for which the restriction of the canonical projection $p\colon A \to A/N$ is an isomorphism of algebras. Again using Proposition 2.3(iii) for f_s we can conclude that f_s restricted to B is nondegenerate. Now, choose a vector space base (e_i) of N and define $g(e_i,e_j) := \delta_{ij}$. Then g is a nondegenerate symmetric invariant bilinear form on the abelian algebra N. Since it has been shown above that A is the direct algebra sum $A = B \oplus N$ it is clear from Prop. 2.3(i) that the orthogonal sum $f_s \perp g$ is a nondegenerate symmetric invariant bilinear form on A. Hence A is metrisable. □

The second part of this Proposition is applicable to the particular case of those finite-dimensional Lie algebras A over a field of characteristic $\neq 2$ for which **the**

adjoint and the coadjoint representation are equivalent: This means that there is a linear isomorphism $\phi : A \to A^*$ such that for all $a, b, c \in A$

$$(*) \qquad \phi(ab)(c) = (ad(a)(\phi(b)))(c) := \phi(b)(ca).$$

Defining $f(a,b) := \phi(b)(a)$ it follows that f is a nondegenerate bilinear form on A which is invariant because of $(*)$. Hence (A, f) is pseudo-metrised. Conversely, assuming that (A, f) is a pseudo-metrised finite-dimensional Lie algebra one can by the same definition construct a linear isomorphism $\phi\colon A \to A^*$ having property $(*)$. Hence pseudo-metrisability is an equivalent notion to the equivalence of adjoint and coadjoint representation. Now, the above Proposition says that this is even equivalent to the metrisability of A.

If an algebra is neither commutative nor anticommutative pseudo-metrisability and metrisability are no longer equivalent as the following example will show:

Assume that char $\mathbb{K} \neq 2$. For some positive integer n let $\wedge(\mathbb{K}^n)$ denote the **Grassmann algebra** over the vector space $\mathbb{K}$ and let $(e_1, \ldots, e_n)$ be the standard basis of $\mathbb{K}^n$. Define the volume $\Omega := e_1 e_2 \cdots e_n$ and the following bilinear form

$$f_0(e_{i_1} \cdots e_{i_r}, e_{j_1} \cdots e_{j_s})\Omega := \begin{cases} 0 & \text{if } r + s \neq n, \\ e_{i_1} \cdots e_{i_r} e_{j_1} \cdots e_{j_s} & \text{if } r + s = n. \end{cases}$$

Clearly f_0 is invariant and nondegenerate because $\dim \wedge^r(\mathbb{K}^n) = n!/(r!(n-r)!) = \dim \wedge^{n-r}(\mathbb{K}^n)$. Now let $n \geq 2$ and suppose there is a symmetric invariant bilinear form q on $\wedge(\mathbb{K}^n)$. Then for all $1 \leq i, j_1, \ldots, j_n \leq n$ the following holds:

$$\begin{aligned} q(e_i, e_{j_1} \cdots e_{j_{n-1}}) &= q(e_{j_1} \cdots e_{j_{n-1}}, e_i) = (-1)^{n-2} q(e_{j_2} \cdots e_{j_{n-1}} e_{j_1}, e_i) \\ &= (-1)^{n-2} q(e_{j_2} \cdots e_{j_{n-1}}, e_{j_1} e_i) = (-1)^{n-1} q(e_{j_2} \cdots e_{j_{n-1}}, e_i e_{j_1}) \\ &= (-1)^{n-1} q(e_i e_{j_1}, e_{j_2} \cdots e_{j_{n-1}}) = (-1)^{n-1} q(e_i, e_{j_1} e_{j_2} \cdots e_{j_{n-1}}). \end{aligned}$$

If n is even then $0 = q(e_i, e_{j_1} e_{j_2} \cdots e_{j_{n-1}}) = q(1, e_i e_{j_1} e_{j_2} \cdots e_{j_{n-1}})$. In particular: $q(1, \Omega) = 0$. But q being invariant we have $0 = q(e_{i_1} \cdots e_{i_r}, \Omega)$ for $r \geq 0$ whence Ω lies in the kernel of q. Therefore, $\wedge(\mathbb{K}^n)$ is pseudo-metrisable but not metrisable for even n. In order to get a nilpotent example consider the radical R of the Grassmann algebra $\wedge(\mathbb{K}^n)$ which is spanned by all elements of positive degree. Its left and right orthogonal space w.r.t. f_0 above is equal to the ideal $\mathbb{K}\,\Omega$. By Proposition 2.3(iii) the factor algebra $A := \wedge(\mathbb{K}^n)/\mathbb{K}\,\Omega$ is pseudo-metrisable w. r. t. the projection of the restriction of f_0 to R. By the same reasoning as above applied to cosets modulo $\mathbb{K}\,\Omega$ one can conclude that for even n the coset of $e_1 \cdots e_{n-1}$ lies in the radical of any symmetric invariant bilinear form on A. Hence A is not metrisable.

Next, we shall discuss **decomposability properties** of a finite-dimensional pseudo-metrised algebra (A, f): We call an ideal I of A **f-nondegenerate** iff $I \cap I^{\perp} = 0$. This is equivalent to $I \cap {}^{\perp}I = 0$ which in turn holds iff $A = I \oplus I^{\perp}$ iff $A = I \oplus {}^{\perp}I$ (compare Appendix A). Clearly, I is f-nondegenerate iff $I^{\perp}$ is nondegenerate iff ${}^{\perp}I$ is nondegenerate iff the restriction of f to $I \times I$ is nondegenerate iff the restriction of f to $I^{\perp} \times I^{\perp}$ is nondegenerate. Now, (A, f) is called **f-decomposable** iff $A = 0$ or A contains a nonzero f-nondegenerate ideal $I \neq A$. Otherwise, (A, f) is called **f-indecomposable**. Suppose that (A, f) decomposes into the direct sum $I \oplus I^{\perp}$ of an f-nondegenerate ideal I and its right orthogonal space $I^{\perp}$. Since the restriction of f to $I \times I$ is nondegenerate we can try to find a nontrivial f-nondegenerate ideal J of I. Because $I(I^{\perp}) \subset I \cap I^{\perp} \supset (I^{\perp})I$ and $I \cap I^{\perp} = 0$ we see that J is an f-nondegenerate ideal of A whose right orthogonal space $J^{\perp'}$ in I is again an f-nondegenerate ideal. Hence A decomposes into the direct sum $J \oplus J^{\perp'} \oplus I$. Proceeding in this way we end up with a decomposition of A into a finite direct sum of f-nondegenerate ideals of A which are all f-indecomposable. The following Theorem shows that this notion of f-decomposition into f-indecomposables is almost equivalent to the more general notion of decomposition of A into a direct sum of indecomposable ideals mentioned in the Appendix (cf. Thm. 4.3):

Theorem 2.1. *Let (A, f) be a finite-dimensional pseudo-metrised algebra over a field $\mathbb{K}$.*

(i) *Suppose that for two ideals I and J of A one has the (not necessarily direct) decomposition $A = I + J$ and, in addition: $IJ = 0 = JI$. Then $A^2 = I^2 \oplus J^2$ (direct sum of ideals). Moreover, let A be f-indecomposable. If $A^2 \neq 0$ then either $I = A$ and $J \subset Z \subset I^2$ or $J = A$ and $I \subset Z \subset J^2$ where Z denotes the annihilator of A. In particular, A is indecomposable. If $A^2 = 0$ then either A is one-dimensional (and hence indecomposable) or A is two-dimensional and f is antisymmetric.*

(ii) *Let f' be another nondegenerate invariant bilinear form on A. Moreover, assume that there is a decomposition $A = I_1 \oplus \cdots \oplus I_k \oplus \cdots \oplus I_K$ (resp. $A = J_1 \oplus \cdots \oplus J_m \oplus \cdots \oplus J_M$) of A into a direct sum of f-indecomposable (resp. f'-indecomposable) ideals where k, K, m, M are integers s. t. $0 \leq k \leq K$ and $0 \leq m \leq M$ and the ideals I_i (resp. J_j) are non-abelian for $1 \leq i \leq k$ (resp. $1 \leq j \leq m$) and abelian otherwise.*
Then $k = m$ and there is a permutation $'$ of the set $\{1, 2, \ldots, m\}$ such that the canonical projection $p_{j'} : A \to I_{j'}$ restricted to the ideal J_j is an isomorphism of algebras. The permutation of $\{1, 2, \ldots, m\}$ is uniquely defined by the condition $J_j \cap I_{j'} \neq 0$. Furthermore, one has $J_j + Z = I_{j'} + Z$ and $J_j^2 = I_{j'}^2$ for all $1 \leq j \leq m$. In particular, if A is perfect or has vanishing annihilator it follows that $m = M$ and the above decomposition is unique up to permutations.

If f and f' are symmetric and char $\mathbb{K} \neq 2$ *then one also has the relation $K = M$ and all f-indecomposable (resp. f'-indecomposable) abelian ideals are one-dimensional.*

(iii) *Let $A = G_1 \oplus \cdots \oplus G_N$ be a decomposition of A into a direct sum of indecomposable ideals G_r where N is a positive integer and $1 \leq r \leq N$. Then there is a nondegenerate invariant bilinear form g on A such that each ideal G_r is g-nondegenerate.*

Proof. (i) Because of $IJ = 0 = JI$ we conclude $I \subset Z(J)$ and $J \subset Z(I)$. Moreover, $AI = (I+J)I = I^2 = I(I+J) = IA$ and likewise: $AJ = J^2 = JA$. Consequently, I^2 and J^2 are ideals of A and, using Prop. 2.1(ii) we can conclude that ${}^{\perp}(I^2) = {}^{\perp}(AI + IA) = Z(I) = (AI + IA)^{\perp} = (I^2)^{\perp}$ hence $J \subset (I^2)^{\perp}$ and taking orthogonal spaces, we have $I^2 \subset {}^{\perp}J$. Likewise: $I^2 \subset J^{\perp}$, and of course: $J^2 \subset {}^{\perp}I \cap I^{\perp}$. Since $A = I + J$ we get $I^{\perp} \cap J^{\perp} = A^{\perp} = 0 = {}^{\perp}A = {}^{\perp}I \cap {}^{\perp}J$ which implies $I^2 \cap J^2 = 0$. On the other hand, $A^2 = (I+J)(I+J) = I^2 + J^2$ whence $A^2 = I^2 \oplus J^2$. Now let A be f-indecomposable and $A^2 \neq 0$. We shall show that $Z \subset A^2$: Indeed, let Z_0 be a vector subspace of Z such that $Z = Z_0 \oplus (Z \cap A^2)$. Using Prop. 2.1(ii) we get $A^2 = Z^{\perp} = Z_0^{\perp} \cap (A^2 + Z)$ and consequently $0 = Z_0 \cap A^2 = Z_0 \cap Z_0^{\perp} \cap (A^2 + Z) = Z_0 \cap Z_0^{\perp}$ because $Z_0 \subset Z \subset Z + A^2$. Hence Z_0 is a f-nondegenerate ideal of A being a vector subspace of Z. Consequently: $Z_0 = 0$ hence: $Z \subset A^2$. Without loss of generality we can now assume that $I^2 \neq 0$. We shall show next that $I \cap {}^{\perp}I = I \cap I^{\perp}$: Indeed, since obviously $I^2 \subset I$ we get ${}^{\perp}I \subset {}^{\perp}(I^2) = Z(I) = (I^2)^{\perp} \supset I^{\perp}$. Hence both ideals $I \cap I^{\perp}$ and $I \cap {}^{\perp}I$ are in $Z(I)$ whence it follows that they are contained in Z because $I \subset Z(J)$. Consequently: $I \cap I^{\perp} \subset Z \subset A^2 = I^2 \oplus J^2$. Since $I^2 \cap I^{\perp} \subset J^{\perp} \cap I^{\perp} = 0$ and $J^2 \subset I^{\perp}$ it follows that $I \cap I^{\perp} \subset J^2$ and obviously $I \cap I^{\perp} \subset J^2 \cap I$. Likewise: $I \cap {}^{\perp}I \subset J^2 \cap I$. Conversely, $J^2 \subset I^{\perp}$ and $J^2 \subset {}^{\perp}I$ whence: $I \cap J^2 \subset I \cap I^{\perp}$ and $I \cap J^2 \subset I \cap {}^{\perp}I$. This proves $I \cap I^{\perp} = I \cap J^2 = I \cap {}^{\perp}I$. Because of $0 = I^{\perp} \cap I^2 = I \cap I^{\perp} \cap I^2$ we have $(I \cap I^{\perp}) + I^2 = (I \cap I^{\perp}) \oplus I^2$. Choose a vector subspace V of I such that $I = V \oplus (I \cap I^{\perp}) \oplus I^2$. Clearly, $I' := V \oplus I^2$ is an ideal of I. But since $IJ = 0 = JI$ we can conclude that I' is an ideal of A. We shall show now that I' is f-nondegenerate which (together with $0 \neq I^2 \subset I'$) will imply that $I' = A$ hence $I = A$: Indeed, let $x \in I'$ such that $f(x, I') = 0$. Obviously, $f(x, I \cap I^{\perp}) = 0$, hence: $0 = f(x, I' \oplus (I \cap I^{\perp})) = f(x, I)$ which implies that $x \in I \cap {}^{\perp}I$. As was shown above, it follows that $x \in I \cap I^{\perp}$. But then $x \in I' \cap (I \cap I^{\perp}) = 0$. Therefore $I = A$ and $J \subset Z \subset I^2 = A^2$. In case A is abelian every f-nondegenerate one-dimensional subspace of A will be a f-nondegenerate ideal of A. Therefore, either there is such a subspace implying A to be one-dimensional or there is no such subspace implying f to be antisymmetric. In this last case, pick a nonzero vector a in A. Since f is nondegenerate there is another nonzero vector b linearly independent on a such that $f(a, b) \neq 0$. Since $f(b, a) = -f(a, b)$ the restriction of f to the two-dimensional ideal B of A spanned by a and b is nondegenerate implying that $A = B$.

(ii) As was proved in (i) every nonabelian f-indecomposable (resp. f'-indecomposable) ideal in this decomposition is indecomposable. Therefore, the first part of the assertion follows from the general decomposition Theorem 4.3 mentioned in Appendix A. Moreover, since no nonzero symmetric bilinear form can be antisymmetric if char $\mathbb{K} \neq 2$ the second part of the assertion follows from (i) and Theorem 4.3 because every f-indecomposable (resp. f'-indecomposable) abelian ideal must be one-dimensional, hence indecomposable.

(iii) Assume that for an integer n $(0 \leq n \leq N)$ the first n ideals G_r are the nonabelian ideals in that decomposition. Consider the decomposition $A = I_1 \oplus \cdots \oplus I_k \oplus \cdots \oplus I_K$ of A into a direct sum of f-indecomposable ideals mentioned in (ii). On the direct sum $Z_0 := I_{k+1} \oplus \cdots \oplus I_K$ of the abelian ideals choose a vector space base (z_i), $(k+1 \leq i \leq K' := \dim Z + k)$ of Z_0 and define $f_0\colon Z_0 \times Z_0 \to \mathbb{K}$ to be the bilinear form $f_0(z_i, z_j) := \delta_{ij}$. Clearly, f_0 is a nondegenerate invariant bilinear form on Z_0 where the one-dimensional ideals $\mathbb{K}z_i$ are f_0-indecomposable and indecomposable. Denote by f_1 the restriction of f to the direct sum $I_1 \oplus \cdots \oplus I_k$ of the nonabelian ideals. It follows easily that the orthogonal sum $h := f_0 \perp f_1$ is a nondegenerate invariant bilinear form on A. Using (i) and the Decomposition Theorem 4.3 we can infer that $k = n$ and also $K' = N$ since the indecomposable abelian ideals $G_{n+1}, \ldots, G_N$ are one-dimensional as well as the indecomposable abelian ideals $\mathbb{K}z_{k+1}, \ldots, \mathbb{K}z_{K'}$. Denote by h_i $(1 \leq i \leq K')$ the restriction of h to the ideal I_i. Clearly, each h_i is a nondegenerate invariant bilinear form on I_i. Now we take the restrictions of the canonical projections $p_{r'}\colon G_r \to I_{r'}$ (compare Thm. 4.3) which are isomorphisms of algebras and form the pulled-back bilinear forms $g_r := p_{r'}^*(h_{r'})$ on G_r. According to Prop. 2.3(i) each g_r is an invariant bilinear form on G_r which is nondegenerate since $p_{r'}$ is an isomorphism. The orthogonal sum $g := g_1 \perp \cdots \perp g_N$ will then be a nondegenerate invariant bilinear form on A such that each G_r is g-nondegenerate. □

We conclude this section with the method of double extension which gives rise to an inductive classification of metrised Lie algebras over a field of characteristic zero:

Theorem 2.2 (Double Extension). *Let (A, f) be a finite-dimensional metrised Lie algebra over a field $\mathbb{K}$. Let furthermore B be another finite-dimensional Lie algebra over $\mathbb{K}$ and suppose that there is a Lie homomorphism $\phi\colon B \to \mathrm{Der}_f(A)$ which denotes the space of all f-antisymmetric derivations of A (i.e. the derivations d of A for which $f(da, a') + f(a, da') = 0$ holds for all $a, a' \in A$). Let B^* denote the dual space of B. Denote by $w\colon A \times A \to B^*$ the bilinear antisymmetric map $(a, a') \mapsto (b \mapsto f(\phi(b)a, a'))$ and for $b \in B$ and $\beta \in B^*$ denote by $b \cdot \beta$ the coadjoint representation (i.e. $(b \cdot \beta)(b') := -\beta(bb')$). Take the vector space direct sum $A_B := B \oplus A \oplus B^*$ and define the following multiplication for $b, b' \in B$,*

$a, a' \in A$, and $\beta, \beta' \in B^*$:

$$(b + a + \beta)(b' + a' + \beta') := bb' + \phi(b)a' - \phi(b')a + aa' + w(a, a') + b \cdot \beta' - b' \cdot \beta.$$

Moreover, define the following symmetric bilinear form f_B on A_B:

$$f_B(b + a + \beta, b' + a' + \beta') := \beta(b') + \beta'(b) + f(a, a').$$

(i) *The pair (A_B, f_B) is a metrised Lie algebra over $\mathbb{K}$ and is called the* **double extension of** A **by** (B, ϕ).

(ii) *Suppose that $\mathbb{K}$ is of characteristic different from 2 and assume that the intersection of the ideal A^2 with the centre Z of A (i.e. the annihilator of A) is nonzero. Then there is a one-dimensional isotropic ideal I (i.e. $I \subset I^\perp$) contained in Z and an element $b \in A$ such that $A = \mathbb{K}\, b \oplus I^\perp$ and (A, f) is isomorphic to the double extension (A'_B, f'_B) of the metrised factor algebra (A', f') by B where $A' := I^\perp / I$, f' is the projection to A' of the restriction of f to $I^\perp \times I^\perp$ (cf. Prop. 2.3(iii)), and $B := \mathbb{K}\, b$. In particular, this applies to every nonabelian solvable Lie algebra.*

(iii) *Suppose that the characteristic of $\mathbb{K}$ is equal to 0. Let A_0 denote the largest semisimple ideal of A and A_1 its orthogonal space. Then (A, f) is given by the orthogonal direct sum $(A_0 \oplus A_1, f_0 \perp f_1)$ where f_0 (f_1) denotes the restriction of f to $A_0 \times A_0$ ($A_1 \times A_1$). Moreover, A_1 does not contain any nonzero semisimple ideal and the radical R of A is contained in A_1.*
*Let L be a Levi subalgebra of A_1 (cf. [**30**, p. 91]). Denote by f^L the restriction of f to $L \times L$ and let p_L denote the canonical projection $A_1 \to A_1/R \cong L$.*
Then the orthogonal space $R^\perp$ (w. r. t. f_1) of R is contained in R. Moreover, (A_1, f_1) is isomorphic to the double extension $(A'_L, f'_L + p_L^* f^L)$ of the solvable metrised factor algebra (A', f') by L where $A' := R/R^\perp$ and f' denotes the projection to A' of the restriction of f_1 to $R \times R$.

Proof. For a detailed proof the reader is referred to the papers of Medina, Revoy; Keith; Hofmann, Keith; Favre, Santharoubane (cf. [**47**], [**48**], [**37**], [**25**], [**20**]). In (i), the proof of the Jacobi identity for the multiplication in A_B and the invariance of f_B is lengthy, but straight forward. In (ii) and (iii) the action of the algebra B is the induced adjoint representation of b (resp. the elements of L) on A' which is well-defined since $I^\perp$ and I (resp. R and $R^\perp$) are ideals of A and therefore invariant under the adjoint action of all the elements in A. If for solvable A the intersection $A^2 \cap Z$ was zero one would have $A = A^2 \oplus Z$ according to Prop. 2.1(ii). But then it would clearly follow that $A = A^2 A^2$ which would contradict the solvability of A. For (iii) note that every semisimple ideal of A is

f-nondegenerate since its intersection with its orthogonal space is semisimple and abelian hence zero. Hence $A = A_0 \oplus A_1$ and every nonzero semisimple ideal of A_1 would be one of A and hence contained in A_0. Using the Levi-Mal'cev-Harish-Chandra Theorem on A_1 and putting the semisimple part L_0 of $R^\perp$ into a suitable Levi subalgebra of A_1 one easily sees that L_0 is a semisimple ideal of A_1. Hence it must vanish because of the above assumption which implies: $R^\perp \subset R$. □

Observe that part (ii) of this Theorem can be used to build up any finite-dimensional metrised solvable Lie algebra by successive double extensions with one-dimensional algebras starting from the zero or the one-dimensional Lie algebra. Therefore, an inductive classification of these Lie algebras in characteristic zero is thereby achieved. However, in prime characteristic not every metrised Lie algebra is isomorphic to some double extension (compare [**10**, Satz 4.3.25]) for a counterexample in characteristic five).

Notes and Further results

Section 2 and Appendix A of this paper are short versions of Chapter 1 and Chapter 2, Sections 2.1, 2.2, and 2.5, of my Diplomarbeit [**10**].

Most of the statements of Prop. 2.1 are classical results for (Lie) algebras with a symmetric nondegenerate invariant bilinear form, compare e. g. [**30**, p. 71], [**35**, p. 30–31], or [**52**, p. 24–25], and are also contained in the following articles on metrised Lie algebras: [**6**], [**7**], [**20**], [**25**], [**37**], [**47**], [**48**], [**53**] and [**54**]. The mutual orthogonality of the central ascending and the central descending series (Prop. 2.1(iv)) had been proved for finite-dimensional metrised Lie algebras in [**47**, p. 159], and [**37**, p. 32], see also [**25**, p. 28], where in addition the mutual orthogonality of the derived series $(D^nA)_{n\geq 0}$ and a series $(K^nA)_{n\geq 0}$, inductively defined by $K^0A := 0$, $K^nA := \{a \in A \mid a(D^{n-1}A) \subset K^{n-1}A\}$ has been stated. The assertion (iii) of Prop. 2.3 also appears in [**20**], [**25**], [**37**], [**47**], and [**48**] for Lie algebras. If B is an ideal of A containing $B^\perp$ Keith calls A a **bi-extension** of $B/B^\perp$ (cf. [**37**, p. 56], or [**25**, p. 30]). The use of tensor products of metrised Lie algebra and metrised commutative associative algebras to construct metrised Lie algebras with radicals of large nilindex is also due to Hofmann and Keith (cf. [**25**, p. 23]). Assertion (i) of Prop. 2.4 was motivated by a similar Lie algebraic statement of Koszul (cf. [**44**, p. 95], proof of Lemme 11.I., see also [**21**, p. 44, Lemma 2.8]). The orthogonal decomposition of finite-dimensional metrised real Lie algebras was systematically investigated by V. V. Astrakhantsev in [**6**], and assertions (i) and (ii) of Thm. 2.1 are generalizations of his Theorem 1 and Theorem 4 in [**6**]. Similar decomposition statements had been proved in [**53**, Thm. 8.1], and in [**9,** Lemma 2.1]. L. J. Santharoubane had let me know that V. G. Kac had given his students two exercises (cf. [**34**, p. 23, Exercise 2.10 and 2.11]) around 1980 where the double extension with a one-dimensional derivation algebra has been defined and the fact that every finite-dimensional solvable metrised Lie algebra can be constructed thereby has been mentioned. G. Favre and L. J. Santharoubane had worked this out in [**20**] and got a classification of low-dimensional nilpotent metrised Lie algebras. Independently, A. Medina and P. Revoy in [**47**] and [**48**] and Hofmann and Keith in [**37**], [**24**], and [**25**] have also developed the double extension technique and have taken into account a Levi algebra of a metrised Lie algebra which requires double extension by a higher dimensional derivation algebra. The fact that the radical of a finite-dimensional metrised Lie algebra in characteristic zero will contain its orthogonal space if all of its semisimple ideals vanish has been proved in [**25**, Lemma 2.8].

For a finite-dimensional metrised algebra (A, q) the **space of all invariant bilinear forms** is isomorphic to its commutant $K(A)$ (see Appendix A for the definition) by mapping $\phi \in K(A)$ to $(a, b) \mapsto q(\phi a, b)$, see Section 2.3 of [**10**] for details. This has also been noted by Kaplansky (cf. [**35**, p. 30, Ex. 15(a)]); and for Lie algebras by Medina and Revoy (cf. [**48**, Lemme 3.1]), and by Tsou and Walker (cf. [**53**, Section 9], where estimates for the dimension of $K(A)$ are given).

Given a faithful representation ρ of a finite-dimensional Lie or associative algebra A in a finite-dimensional vector space one can construct a **trace form** defined by $(a, b) \mapsto trace(\rho(a)\rho(b))$ which is symmetric and invariant because of the cyclic properties of the trace and the Jacobi or associative identity (see also [**10**, Section 2.4]). For associative algebras such a trace form can only be nondegenerate if its radical vanishes since $\rho(n)$ is nilpotent if n is contained in the radical. For Lie algebras in characteristic zero it is a classical result that the radical is central if a trace form is nondegenerate. This is also true for finite-dimensional Lie algebras in characteristic $p > 3$ as has been shown by Zassenhaus (cf. [**56**] and [**9**]).

Finite-dimensional **Hopf algebras** (with an antipode) carry a nondegenerate (not necessarily symmetric) invariant bilinear form (cf. [**45**]). A particular case of this is Berkson's result [**8**] that the restricted universal enveloping algebra of a finite-dimensional restricted Lie algebra over a field of characteristic $p > 0$ (cf. [**30**, p. 190] for definitions) is pseudo-metrised. On the other hand, the (infinite-dimensional) universal enveloping algebra of a finite-dimensional metrised Lie algebra over a field of characteristic zero is metrised (cf. [**12**]).

Finite-dimensional metrised Lie algebras whose bilinear form is in addition invariant under all of its derivations have been investigated in [**19**] and [**10**, Chapter 4]. There exist nonsemisimple Lie algebras with this property (see [**10**, p. 150, Satz 4.3.25] for an example in characteristic five, and the article [**4**] by Angelopoulos and Benayadi for an example in characteristic zero).

3. The Method of T^*-extension

In this Section we shall introduce a new technique of constructing metrisable algebras out of arbitrary ones. This method which we shall call **T^*-extension** is closely related to the double extension technique mentioned at the end of the last section (cf. Thm. 2.2). However, in contrast to the double extension the method to be described applies not only to Lie algebras, but to arbitrary nonassociative algebras and is a one-step rather than a multi-step extension.

Let A be an arbitrary nonassociative algebra over a field $\mathbb{K}$ and consider its dual space A. Define the following **dual left and right multiplications** for an $a \in A$:

$$L^*(a)\colon A^* \to A^* : \alpha \mapsto (R(a))^*\alpha\colon a' \mapsto \alpha(a'a) := (L^*(a)\alpha)(a'), \tag{5}$$

$$R^*(a)\colon A^* \to A^* : \alpha \mapsto (L(a))^*\alpha\colon a' \mapsto \alpha(aa') := (R^*(a)\alpha)(a'). \tag{6}$$

We shall often make the abbreviation $L^*(a)\alpha =: a \cdot \alpha$ and $R^*(a)\alpha =: \alpha \cdot a$. Note the exchange of left and right multiplication in this dualisation. For Lie algebras we have $L^*(a) = ad^*(a) = -R^*(a)$ which is the well-known **coadjoint representation** mentioned in the last section after Prop. 2.4. Consider now an arbitrary bilinear map w (which will be specified later)

$$w\colon A \times A \to A^* : (a, a') \mapsto w(a, a') \tag{7}$$

and define the following multiplication on the vector space direct sum $A \oplus A^*$ for all $a, a' \in A$ and $\alpha, \alpha' \in A$:

$$(a + \alpha) \cdot (a' + \alpha') := aa' + w(a, a') + a \cdot \alpha' + \alpha \cdot a' \tag{8}$$

Clearly, the subspace A^* of $A \oplus A^*$ is an abelian ideal of $A \oplus A^*$ and A is isomorphic to the factor algebra $(A \oplus A^*)/A^*$. Moreover, consider the following symmetric bilinear form q_A on $A \oplus A^*$ defined for all $a, a' \in A$ and $\alpha, \alpha' \in A^*$:

$$q_A(a + \alpha, a' + \alpha') := \alpha(a') + \alpha'(a). \tag{9}$$

We then have the following simple

Lemma 3.1. *Let A, A^*, w, and q_A as above. Then the pair $(A \oplus A^*, q_A)$ is a metrised algebra if and only if w is cyclic in the following sense:*

$$w(a, b)(c) = w(c, a)(b) = w(b, c)(a) \text{ for all } a, b, c \in A.$$

Proof. The symmetric bilinear form q_A is nondegenerate: For if $a' + \alpha'$ is orthogonal on all elements of $A \oplus A^*$ then in particular $\alpha(a') = 0$ for all $\alpha \in A^*$ and $\alpha'(a) = 0$ for all $a \in A$ which implies $a' = 0$ and $\alpha' = 0$. Now let $a, b, c \in A$ and $\alpha, \beta, \gamma \in A^*$. Then:

$$\begin{aligned} q_A((a + \alpha) \cdot (b + \beta), c + \gamma) &= q_A(ab + w(a, b) + a \cdot \beta + \alpha \cdot b, c + \gamma) \\ &= \gamma(ab) + w(a, b)(c) + (a \cdot \beta)(c) + (\alpha \cdot b)(c) \\ &= \alpha(bc) + \beta(ca) + \gamma(ab) + w(a, b)(c). \end{aligned}$$

On the other hand:

$$\begin{aligned} q_A(a+\alpha,(b+\beta)\cdot(c+\gamma)) &= q_A(a+\alpha, bc+w(b,c)+b\cdot\gamma+\beta\cdot c) \\ &= \alpha(bc)+w(b,c)(a)+(b\cdot\gamma)(a)+(\beta\cdot c)(a) \\ &= \alpha(bc)+\beta(ca)+\gamma(ab)+w(b,c)(a). \end{aligned}$$

This proves the Lemma. □

Now, for cyclic w we shall call the metrised algebra $(A\oplus A^*, q_A)$ the **T^*-extension of A (by w)** and denote the algebra $A\oplus A^*$ by T^*_wA or, more simply, by T^*A if it is clear from the context how the map w looks like. In the special case where A is a finite-dimensional Lie algebra and w vanishes one easily sees that T^*A is nothing but the double extension of the zero algebra by A with the zero map as homomorphism $\phi\colon A\to 0$ (compare Thm. 2.2). If A is a real finite-dimensional Lie algebra belonging to a Lie group G then T^*A will be the Lie algebra of the cotangent bundle T^*G of G: indeed, the map $T^*L\colon T^*G\to G\times A^* : \alpha_g\mapsto (g,\alpha_g\circ T_eL_g)$ (where α_g is a one-form in the cotangent space of G at $g\in G$, e is the unit element of G, and T_eL_g is the tangent map of the left multiplication map L_g at e) is a vector bundle isomorphism onto the semidirect product $G\times A^*$ where A^* is the abelian normal subgroup and the subgroup G acts on A^* by the coadjoint (group) representation. This differential geometric fact motivates the notation "T^*-extension".

If A is infinite-dimensional then the dimension of its dual space A^* will always be strictly larger than the dimension of A (cf. e.g. [**29**, p. 68, Thm. 1]). In order to get T^*-extensions of A having "smaller dimensions" one could replace the full dual space A^* of A by any subspace A' of A^* that is stable under all dual left and right multiplications (cf. eqs. (5) and (6)) and is **total** in the sense that for each nonzero $a\in A$ there is an $\alpha\in A'$ such that $\alpha(a)\neq 0$ (cf. [**29**, p. 68–69]). Moreover, the map w should take its values in A'. For instance, this applies to any nonassociative algebra A that is $\mathbb{Z}$-graded in the sense that it is equal to a direct sum $\oplus_{i\in\mathbb{Z}}A_i$ of finite-dimensional subspaces A_i of A such that $A_iA_j\subset A_{i+j}$ for all $i,j\in\mathbb{Z}$: If A_i^* is identified with the space of all linear maps in A^* that vanish on the direct sum of all A_j, $j\neq i$, then the subspace $A' := \oplus_{i\in\mathbb{Z}}A_i^*$ will clearly be total and invariant by all dual left and right multiplications and has the same dimension as A. Prominent examples of $\mathbb{Z}$-graded algebras are the well-known **Kac-Moody-Lie algebras** (cf. [**34**]).

We shall show in the following Theorem how certain properties of an algebra A are transferred to a T^*-extension of A:

Theorem 3.1. *Let A be a nonassociative algebra over a field $\mathbb{K}$.*

(i) *If A is solvable (nilpotent) of length $k\in\mathbb{N}$ (nilindex $k\in\mathbb{N}$) then for each bilinear cyclic map $w\colon A\times A\to A^*$ the T^*-extension T^*_wA will be* *solvable (nilpotent) of length $r \in \mathbb{N}$ (nilindex $r \in \mathbb{N}$) where $k \le r \le k+1$ ($k \le r \le 2k-1$).*

(ii) *Suppose A has one of the following properties: nilpotent of nilindex $k \in \mathbb{N}$, commutative, anticommutative, associative, alternative, Lie or Jordan. Then the trivial T^*-extension T_0^*A has the same property.*

(iii) *If A is decomposable so is the trivial T^*-extension T_0^*A. If A is finite-dimensional, indecomposable and nonabelian so is the trivial T^*-extension T_0^*A.*

Proof. (i) Suppose first that A is solvable of length k. Since the derived series $D^n(T_w^*A)$ of T_w^*A modulo the ideal A^* is isomorphic to the derived series $D^n(A)$ of A it follows that $D^k(T_w^*A) \subset A^*$. This implies $D^{k+1}(T_w^*A) = 0$ because A^* is abelian, and T_w^*A is solvable of length k or $k+1$. Suppose now that A is nilpotent of nilindex k. Since the central descending series $C^n(T_w^*A)$ of T_w^*A modulo the ideal A^* is isomorphic to the central descending series $C^n(A)$ of A it follows that $C^k(T_w^*A) \subset A^*$. Let $x_1, \dots, x_{k-1} \in T_w^*A$ and set $x_i = a_i + \alpha_i$ for all $1 \le i \le k-1$ with $a_i \in A$ and $\alpha_i \in A^*$. Now, if $S^T(x)$ denotes right or left multiplication with $x \in T_w^*A$ and $a \in A$, $\beta \in A^*$ then it follows that $(S^T(x_1) \cdots S^T(x_{k-1})\beta)(a) = (S^*(a_1) \cdots S^*(a_{k-1})b)(a)$ because A^* is abelian, and this in turn is equal to $\beta(S(a_{k-1}) \cdots S(a_1)a) \in \beta(C^k(A)) = 0$. This proves that $C^{2k-1}(T_w^*A)$ vanishes, and T_w^*A is nilpotent of nilindex at least k and at most $2k-1$.

(ii) Suppose that A is nilpotent of nilindex k. Adopting the notations of the proof of part (i) we see that for $x_k = a_k + \alpha_k \in T_0^*A$ one has

$$\begin{aligned} S^T(x_1) \cdots S^T(x_{k-1})x_k &= S(a_1) \cdots S(a_{k-1})a_k \\ &+ \sum_{i=1}^{k-1} S^*(a_1) \cdots S^*(a_{i-1}) S^T(\alpha_i) S(a_{i+1}) \cdots S(a_{k-1})a_k \\ &+ S^*(a_1) \cdots S^*(a_{k-1})\alpha_k \end{aligned}$$

because w vanishes. The first summand on the r. h. s. of this equation vanishes because it is contained in $C^k(A) = 0$. All the other summands are of the type $a \mapsto \alpha_i(S(b_1) \cdots S(b_{k-1})a)$ with $(b_1, \dots, b_{k-1})$ denoting a permutation of the set $\{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_k\}$. Again, they vanish because $C^k(A) = 0$.
The verification of the above algebra identities in T_0^*A is completely straight forward and is left to the reader (see also [**10**, p. 85–87]). As an example we shall give a proof for the Jordan identity under the assumption that the commutativity has already been shown. Let $a, b, c \in A$ and $\alpha, \beta \in A^*$:

$$\begin{aligned} ((a+\alpha)\cdot(b+\beta))\cdot((a+\alpha)\cdot(a+\alpha)) - (a+\alpha)\cdot((b+\beta)\cdot((a+\alpha)\cdot(a+\alpha))) \\ = (ab)(aa) - a(b(aa)) + (aa)\cdot(a\cdot\beta) - a\cdot((aa)\cdot\beta) \\ + (aa)\cdot(b\cdot\alpha) - ((aa)b)\cdot\alpha + 2(ab)\cdot(a\cdot\alpha) - 2a\cdot(b\cdot(a\cdot\alpha)) \end{aligned}$$

The first two summands vanish because of the Jordan identity in A. If the sum of the third and the forth summand is applied to c one gets

$$\beta\big(a((aa)c) - (aa)(ac)\big) = -\beta\big((ac)(aa) - a(c(aa))\big) = 0$$

because of the Jordan identity in A. If the sum of the last four summands is applied to c one gets

$$\alpha((L(b)L(aa) - L((aa)b) + 2L(a)L(ab) - 2L(a)L(b)L(a))(c)) = 0$$

because of a linearized version of the Jordan identity in A (cf. [**52**, p. 92, eqn 4.5]). This proves the Jordan identity in T^*_wA.

(iii) Suppose that $0 \neq A = I \oplus J$ where I and J are two nonzero ideals of A. Let I^* (J^*) denote the subspace of all linear forms in A^* that vanish on J (I). Clearly, I^* and J^* can canonically be identified with the dual spaces of I and J. Because of $IJ = 0 = JI$ we have $(I \cdot I^*)(J) = I^*(JI) = 0 = I^*(IJ) = (I^* \cdot I)(J)$ and $(J \cdot I^*)(A) = I^*(AJ) \subset I^*(J) = 0 = I^*(J) \supset I^*(JA) = (I^* \cdot J)(A)$. This and a completely analogous reasoning for J^* replacing I^* imply that the subspaces $T^*I := I \oplus I^*$ and $T^*J := J \oplus J^*$ are ideals of $T^*A := T^*_0A$ such that T^*A is given by the orthogonal direct sum $T^*I \oplus T^*J$. Here the symmetric bilinear form q_A (cf. eqn (9)) restricted to T^*I is equal to q_I and $T^*I \cong T^*_0I$ where a similar statement holds for J replacing I. This proves the first assertion of (iii).

Conversely, suppose that A is finite-dimensional, indecomposable and non-abelian. Assume that $T^*A := T^*_0A$ decomposes into the direct sum $I' \oplus J'$ of two nonzero ideals I' and J' of T^*A. According to Thm. 2.1 we can assume that T^*A is q_A-decomposable, i.e. $I' = J'^{\perp}$. Denote by p_A (p_{A^*}) the canonical projection $T^*A = A \oplus A^* \to A$ ($T^*A \to A^*$). Let I and J denote the subspaces p_AI' and p_AJ' of A, respectively. Since p_A is a homomorphism of algebras and $I'J' = 0 = J'I'$ it follows that I and J are ideals of A which annihilate each other, i.e. $IJ = 0 = JI$ and whose (not necessarily direct) sum $I + J$ is equal to A.
As a first step we show that the subspaces $I+I'$ and $J+J'$ of T^*A are ideals of T^*A annihilating each other, i.e. $(I + I') \cdot (J + J') = 0 = (J + J') \cdot (I + I')$: Indeed, let $i \in I'$ and $j \in J'$. Then it follows that $(p_Ai) \cdot A^* = (p_Ai + p_{A^*}i) \cdot A^* = i \cdot A^* \subset I' \cap A^*$ because A^* is an abelian ideal. Likewise, $A^* \cdot (p_Ai) \subset I' \cap A^*$. This entails

$$(*) \qquad I \cdot A^* + A^* \cdot I \subset I' \cap A^* \text{ and } J \cdot A^* \cap A^* \cdot J \subset J' \cap A^*$$

after a completely analogous reasoning for J replacing I. Now this relation and the fact that I and J are ideals of A imply that $I \cdot (T^*A) + (T^*A) \cdot I \subset I + (I' \cap A^*)$ and $J \cdot (T^*A) + (T^*A) \cdot J \subset J + (J' \cap A^*)$, hence $(I + I')$ and $(J + J')$ are ideals of T^*A. Moreover:

$$\begin{aligned} 0 = i \cdot j &= (p_Ai)(p_Aj) + (p_Ai) \cdot (p_{A^*}j) + (p_{A^*}i) \cdot (p_Aj) + (p_{A^*}i) \cdot (p_{A^*}j) \\ &= 0 + (p_Ai) \cdot (p_{A^*}j) + (p_{A^*}i) \cdot (p_Aj) + 0 \end{aligned}$$

because A^* is abelian and $p_A(i \cdot j) = 0$. But according to $(*)$ the first remaining summand on the r.h.s. of this equation lies in $I' \cap A^*$ whereas the second remaining summand on the r.h.s. of this equation is contained in $J' \cap A^*$: Since $I' \cap J' = 0$ they have to vanish separately. Therefore:

$$(p_A i) \cdot j = (p_A i)(p_A j) + (p_A i) \cdot (p_{A^*} j) = 0 + 0.$$

Analogously, $0 = j \cdot (p_A i)$ and $(p_A j) \cdot i = 0 = i \cdot (p_A j)$. Hence it follows that

$$(**) \qquad I \cdot J' + J' \cdot I = 0 = J \cdot I' + I' \cdot J.$$

Now this relation and the fact that $IJ = 0 = JI$ and $I'J' = 0 = J'I'$ entail that the ideals $(I + I')$ and $(J + J')$ annihilate each other.
By Thm. 2.1(i) it follows that $(I + I')^2 \cap (J + J')^2 = 0$. In particular:

$$(***) \qquad I^2 \cap J^2 = 0, \text{ hence: } A^2 = I^2 \oplus J^2.$$

In a second step we show the existence of ideals I_1 and J_1 of A such that $A = I_1 \oplus J_1$ and $I \supset I_1 \supset I^2$ and $J \supset J_1 \supset J^2$: Clearly, $I \supset I \cap A^2 \supset I^2$. Let V be a vector space complement to $I \cap A^2$ in I, i.e. $I = I \cap A^2 \oplus V$. Define

$$I_1 := I^2 \oplus V.$$

Since $I_1 \supset I^2$ it is an ideal of I and because of $IJ = 0 = JI$ it is an ideal of A. Obviously, $J \supset J \cap (I + J^2) \supset J^2$. Let W be a vector space complement in J to $J \cap (I + J^2)$, i.e. $J = J \cap (I + J^2) \oplus W$. Define

$$J_1 := J^2 \oplus W.$$

A similar reasoning as above entails that J_1 is an ideal of A contained in J. Since the spaces A^2, V, and W are contained in $I_1 + J_1$ by definition it follows that $A = I + J \subset I_1 + J_1$, hence $I_1 + J_1 = A$. Let x be an element of the intersection $I_1 \cap J_1$. Then $x = i + v = j + w$ where $i \in I^2$, $v \in V$, $j \in J^2$, and $w \in W$. But clearly, the element $w = i - j + v$ lies both in W and in $J \cap (I + J^2)$ and must vanish by definition of W. Moreover, the element $v = j - i$ clearly lies both in V and in $I \cap A^2$ and must vanish by definition of V. Furthermore, since I^2 and J^2 have zero intersection we must have $i = 0 = j$ which implies $I_1 \cap J_1 = 0$ showing

$$A = I_1 \oplus J_1.$$

Now, since A is indecomposable one of these ideals, say J_1, must be zero. But then $A = I_1 = I$, hence $T^*A = I \oplus A^* = I' + A^*$ because $p_A I' = A = I = p_A I$. This implies $J' \cong (I' \oplus J')/I' = (T^*A)/I' = (A^* + I')/I' \cong A^*/(A^* \cap I')$, and J' must be abelian being a homomorphic image of the abelian ideal A^*.

Because of $I' \cdot J' = 0 = J' \cdot I'$ the ideal J' lies in the annihilator $Z(T^*A)$ of T^*A. Since $(T^*A)^2$ is equal to $A^2 \oplus (A \cdot A^* + A^* \cdot A)$ a simple computation of the orthogonal space of this (cf. Prop. 2.1(ii)) shows that $Z(T^*A) = Z(A) \oplus (A^2)^{ann}$ the latter space denoting the space of one-forms in A^* that vanish on A^2. But since A is indecomposable and nonabelian its annihilator $Z(A)$ is contained in A^2: take any vector space complement S to $A^2 + Z(A)$ in A, take any vector space complement Z_0 to $Z(A) \cap A^2$ in $Z(A)$, define $I_2 := A^2 \oplus S$, and A is equal to the direct sum of ideals $I_2 \oplus Z_0$. Since A is nonabelian and indecomposable $Z_0 = 0$. But then $q_A(Z(A), (A^2)^{ann}) = (A^2)^{ann}(Z(A)) \subset (A^2)^{ann}(A^2) = 0$, and $Z(T^*A)$ is isotropic. But so is J' being contained in $Z(T^*A)$, hence J' must be zero because it is assumed to be nondegenerate. It follows that $T^*A = I'$ is indecomposable. □

The upper bound $2k-1$ for the nilindex of the T^*-extension in Thm. 3.1(i) does actually occur: see e.g. Example 4.3 in the next section. Moreover, note that both statements of Thm. 3.1(iii) are wrong if the T^*-extension is nontrivial, i.e. $w \neq 0$: for instance, Example 4.2 in Section 4 is a counterexample.

The following criterion is central in recognizing T^*-extensions:

Theorem 3.2. *Let (A, f) be a metrised algebra of finite dimension n over a field $\mathbb{K}$ of characteristic not equal to two. Then (A, f) will be isometric to a T^*-extension $(T^*_w B, q_B)$ if and only if n is even and A contains an isotropic ideal I (i.e. $I \subset I^\perp$) of dimension $n/2$. In this case: $B \cong A/I$. Note that any isotropic $n/2$-dimensional subspace I of A is an ideal of A if and only if it is abelian, i.e. $I^2 = 0$.*

Proof. "$\Rightarrow$": Since $\dim B = \dim B^*$ it is clear that $\dim T^*_w B$ is even. Moreover, it is clear from the definition of the multiplication (cf. eqn (8) and the bilinear form q_B (cf. eqn (9)) that B^* is an isotropic ideal of half the dimension of $T^*_w B$.

"$\Leftarrow$": Let I be an $n/2$-dimensional isotropic subspace of A. Since $\dim I + \dim I^\perp = n$ it follows that $I = I^\perp$. If I is an ideal of A then $I^2 = I(I^\perp) = 0$ by Prop. 2.1(i) and I is abelian. Conversely, if $I^2 = 0$ then $q(I, IA) = q(I^2, A) = 0 = q(A, I^2) = q(AI, I)$ showing that $IA + AI \subset I^\perp = I$ which implies that I is an ideal of A. Suppose that I is an ideal of A, let B denote the factor algebra A/I and $p\colon A \to B$ the canonical projection. Now, since the characteristic $\mathbb{K}$ is not equal to 2 we can choose an isotropic complementary vector subspace B_0 to I in A, i.e. $A = B_0 \oplus I$ and $B_0^\perp = B_0$ (compare [**33**, p. 386]). Denote by p_0 (resp. p_1) the projection $A \to B_0$ (resp. $A \to I$) along I (resp. along B_0). Moreover, let $f^\flat$ denote the linear map $I \to B^*\colon i \mapsto (pa \mapsto f(i, a))$. It is well-defined because of $f(I, I) = 0$. Since f is nondegenerate, $I^\perp = I$, and $\dim I = n/2 = \dim B$ it follows that $f^\flat$ is a linear isomorphism. Furthermore, $f^\flat$ has the following intertwining property: Let $a, a' \in A$ and $i \in I$ then

$$\begin{aligned} f^\flat(ai)(pa') &= f(ai, a') = f(a', ai) = f(a'a, i) = f(i, a'a) \\ &= f^\flat(i)(p(a'a)) = f^\flat(i)((pa')(pa)) = (L^*(pa)f^\flat(i))(pa') \\ &= ((pa) \cdot f^\flat(i))(pa'). \end{aligned}$$

Hence, after a completely analogous computation one has the following

$$f^\flat(ai) = (pa) \cdot f^\flat(i) \text{ and } f^\flat(ia) = f^\flat(i) \cdot (pa) \qquad \forall a \in A, i \in I.$$

We define the following bilinear map $w\colon B \times B \to B^*\colon (pb_0, pb_0') \mapsto f^\flat(p_1(b_0 b_0'))$ where b_0 and b_0' are in B_0. This is well-defined since the restriction of the projection p to B_0 is a linear isomorphism. Now, let m denote the following linear map $A \to B \oplus B^*\colon b_0 + i \mapsto pb_0 + f^\flat(i)$ where $b_0 \in B_0$ and $i \in I$. Since p restricted to B_0 and $f^\flat$ are linear isomorphisms the map m is also a linear isomorphism. Moreover, m is an isomorphism of the metrised algebra (A, f) to the T^*-extension $(T_w^* B, q_B)$: Indeed, let $b_0, b_0' \in B$ and $i, i' \in I$ then

$$\begin{aligned} m((b_0 + i)(b_0' + i')) &= m\big(p_0(b_0 b_0') + p_1(b_0 b_0') + b_0 i' + i b_0'\big) \\ &= p(p_0(b_0 b_0')) + f^\flat(p_1(b_0 b_0') + b_0 i' + i b_0') \\ &= p(b_0 b_0') + w(b_0, b_0') + (pb_0) \cdot f^\flat(i') + f^\flat(i) \cdot (pb_0') \\ &= (pb_0)(pb_0') + w(b_0, b_0') + (pb_0) \cdot f^\flat(i') + f^\flat(i) \cdot (pb_0') \\ &= (pb_0 + f^\flat(i)) \cdot (pb_0' + f^\flat(i')) \\ &= (m(b_0 + i)) \cdot (m(b_0' + i')) \end{aligned}$$

where we made use of the definition of w, the intertwining properties of $f^\flat$, the fact that p is a homomorphism, and the definition (8) of the product in $T_w^* B$. In addition we have:

$$\begin{aligned} (m^* q_B)(b_0 + i, b_0' + i') &= q_B(pb_0 + f^\flat(i), pb_0' + f^\flat(i')) \\ &= f^\flat(i)(pb_0') + f^\flat(i')(pb_0) \\ &= f(i, b_0') + f(i', b_0) \\ &= f(b_0 + i, b_0' + i') \end{aligned}$$

where the fact that B_0 could be chosen to be isotropic entered in the last equation. Hence: $m^* q_B = f$ which implies that q_B is an invariant symmetric bilinear form on $T_w^* B$ (cf. Prop. 2.3(i) and (ii)) or that w is cyclic. Therefore (A, f) and $(T_w^* B, q_B)$ are isomorphic as metrised algebras and the Theorem is proved. $\square$

The proof of this Theorem shows that the bilinear map w depends on the choice of the isotropic subspace B_0 of A complementary to the ideal I. Therefore there may be different T^*-extensions describing the "same" metrised algebra. This situation can be dealt with in the following way:

Let B_1 and B_2 be two algebras over a field $\mathbb{K}$ and let $w_1\colon B_1 \times B_1 \to B_1^*$ and $w_2\colon B_2 \times B_2 \to B_2^*$ two bilinear maps in the corresponding dual spaces. The T^*-extension $T^*_{w_1}B_1$ of B_1 is said to be **equivalent** to the T^*-extension $T^*_{w_2}B_2$ iff $B_1 = B_2 := B$ and iff there exists an isomorphism of algebras $\Phi\colon T^*_{w_1}B_1 \to T^*_{w_2}B_2$ which is the identity on the ideal B^* and which induces the identity on the factor algebra $T^*_{w_1}B_1/B^* = B = T^*_{w_2}B_2/B^*$. The two T^*-extensions $T^*_{w_1}B_1$ and $T^*_{w_2}B_2$ are said to be **isometrically equivalent** iff they are equivalent and Φ is an isometry. This situation can be depicted in the following commutative diagram:

$$
\begin{tikzcd}
 & & T^*_{w_1}B \arrow[dr, "p_1"] \arrow[dd, "\Phi"] & & \\
0 \arrow[r] & B^* \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \arrow[r] & 0 \\
 & & T^*_{w_2}B \arrow[ur, "p_2"'] & &
\end{tikzcd}
\tag{10}
$$

Here the two horizontal maps are the zero maps. i_1 and i_2 denote the canonical injections of B^* into $T^*_{w_1}B_1$ and $T^*_{w_2}B_2$, respectively, whereas p_1 and p_2 denote the canonical projections of $T^*_{w_1}B_1$ and $T^*_{w_2}B_2$ onto B, respectively.

Proposition 3.1. *Let B be an algebra over a field of characteristic not equal to 2. Furthermore, let w_1 and w_2 be two bilinear maps $B \times B \to B^*$.*

(i) *$T^*_{w_1}B_1$ is equivalent to $T^*_{w_2}B_2$ (cf. diagram (10)) if and only if there is a linear map $z\colon B \to B^*$ such that for all $b, b' \in B$*

$$
w_1(b,b') - w_2(b,b') = b \cdot (z(b')) + (z(b')) \cdot b - z(bb'). \tag{$*$}
$$

If this is the case then the symmetric part z_s of z which is defined by $z_s(b)(b') := \frac{1}{2}(z(b)(b') + z(b')(b))$ for all $b, b' \in B$ will induce a symmetric invariant bilinear form on B, i.e.: $z_s(bb')(b'') = z_s(b)(b'b'')$ for all $b, b', b'' \in B$.

(ii) *$T^*_{w_1}B_1$ is isometrically equivalent to $T^*_{w_2}B_2$ if and only if there is a linear map $z\colon B \to B^*$ such that eqn $(*)$ holds for all $b, b' \in B$ and in addition the symmetric part z_s of z vanishes.*

Proof. (i) The equivalence between $T^*_{w_1}B_1$ and $T^*_{w_2}B_2$ holds if and only if there is a homomorphism of algebras $\Phi\colon T^*_{w_1}B_1 \to T^*_{w_2}B_2$ satisfying $\Phi(b+\beta) = b + \Phi_{21}(b) + \beta$ for all $b \in B$ and $\beta \in B^*$ where Φ_{21} is the component of Φ that maps B to B^*: indeed, Φ must be the identity on B^* and we must have $b = p(b) = p(\Phi(b)) = \Phi_{11}(b)$ where Φ_{11} is the component of Φ that maps B to B.

Let z denote Φ_{21}. Clearly, Φ is a linear isomorphism for arbitrary z. Then for all $b, b' \in B$ and $\beta, \beta' \in B^*$ we have

$$\Phi((b+\beta)\cdot(b'+\beta')) = bb' + w_1(b,b') + z(bb') + b\cdot\beta' + \beta\cdot b'$$

where the multiplication was formed in $T^*_{w_1}B_1$. On the other hand:

$$\begin{aligned}(\Phi(b+\beta))\cdot(\Phi(b'+\beta')) &= (b+z(b)+\beta)\cdot(b'+z(b')+\beta') \\ &= bb' + w_2(b,b') + b\cdot(z(b')) + (z(b))\cdot b' + b\cdot\beta' + \beta\cdot b'\end{aligned}$$

where the multiplication was formed in $T^*_{w_2}B_2$. Hence Φ is a homomorphism of algebras if and only if eqn $(*)$ holds. Now, both w_1 and w_2 are cyclic maps. Hence the right hand side of eqn $(*)$ has to be cyclic. Splitting z into its antisymmetric part z_a defined by $z_a(b)(b') := \frac{1}{2}(z(b)(b') - z(b')(b))$ for all $b, b' \in B$ and its symmetric part z_s as defined above, i.e. $z = z_a + z_s$, we see that the right hand side of eqn $(*)$ evaluated on $b'' \in B$ has the following form:

$$z_a(b')(b''b) + z_a(b)(b'b'') + z_a(b'')(bb') + z_s(b')(b''b) + z_s(b)(b'b'') - z_s(bb')(b'')$$

Observing that the sum of the terms involving z_a is already cyclic and subtracting from this the same sum of terms after the cylic permutation $(b, b', b'') \mapsto (b', b'', b)$ we get the result

$$2z_s(b)(b'b'') - 2z_s(bb')(b'') = 0,$$

which proves the invariance of the symmetric bilinear form induced by z_s.

(ii) Let the isomorphism Φ be defined as in (i). Then we have for all $b, b' \in B$ and $\beta, \beta' \in B^*$

$$\begin{aligned}q_B(\Phi(b+\beta), \Phi(b'+\beta')) &= q_B(b+z(b)+\beta, b'+z(b')+\beta') \\ &= z(b)(b') + z(b')(b) + \beta(b') + \beta'(b) \\ &= z(b)(b') + z(b')(b) + q_B(b+\beta, b'+\beta')\end{aligned}$$

from which it is clear that Φ is an isometry iff $z_s = 0$. □

The apparent cohomological appeal of this Proposition is no coincidence as will become clear in the following two examples:

3.1 Lie Algebras

Let $(B, [\ ,\ ])$ be a Lie algebra over a field $\mathbb{K}$ of characteristic different from 2. Let V be a vector space over $\mathbb{K}$ such that there is a linear map ρ of B into the space of linear endomorphisms of V satisfying $\rho([b,b']) = \rho(b)\rho(b') - \rho(b')\rho(b)$ for all $b, b' \in B$. ρ is called a representation of B and V is called a B-module. For each nonnegative integer k let $C^k(B,V)$ denote the space of alternating k-linear maps $B \times \cdots \times B$ into V where $C^0(B,V)$ is defined to be equal to V. Denote by $C(B,V)$ the direct sum of all the spaces $C^k(B,V)$ $(0 \leq k < \infty)$. The coboundary operator $\delta\colon C(B,V) \to C(B,V)$ is defined for $f \in C^k(B,V)$ and $b_0, b_1, \ldots, b_k \in B$ as follows:

$$(11) \qquad (\delta f)(b_0, b_1, \ldots, b_k) := \sum_{i=0}^{k} (-1)^i \rho(b)(f(b_0, \ldots, \hat{b_i}, \ldots, b_k)) + \sum_{i<j} (-1)^{i+j} f([b_i, b_j], b_0, \ldots, \hat{b_i}, \ldots, \hat{b_j}, \ldots, b_k)$$

where the hat ˆ over a symbol means that it should be omitted. It is known that $\delta^2 = 0$. Call any k-form $f \in C^k(B,V)$ a **k-cocycle** iff $\delta f = 0$ and denote the subspace of k-cocycles by $Z^k(B,V)$. The **k-th cohomology group** $H^k(B,V)$ is defined to be the factor space $Z^k(B,V)/\delta C^{k-1}(B,V)$ for $k \geq 1$ and $Z^0(B,V)$ for $k = 0$.

In particular, the dual space B^* of B is a B-module with respect to the coadjoint action of B. Consider a bilinear map $w\colon B \times B \to B^*$ and the corresponding T^*-extension $T^*_w B$ of B. It is known that the multiplication (8) of $T^*_w B$ is anticommutative and satisfies the Jacobi identity if and only if w is antisymmetric and a 2-cocycle, i.e. $w \in Z^2(B, B^*)$ (cf. [**14**, p. 121]). Now, the additional fact that w has to be cyclic means that the trilinear form $w^\flat$ defined by $w^\flat(b, b', b'') := w(b, b')(b'')$ should be alternating. Considering the field $\mathbb{K}$ as a trivial B-module we can write $w^\flat \in C^3(B, \mathbb{K})$. Using the special case $j = n - 1$ in [**27**, Lemma 1], we can infer that $w^\flat$ is a 3-cocycle, i.e. $w^\flat \in Z^3(B, \mathbb{K})$, if and only if w is a 2-cocycle. Conversely, since every 3-cocycle in $Z^3(B, \mathbb{K})$ induces a 2-cocycle w in $Z^2(B, B^*)$ by the prescription $w(b, b') = (b'' \mapsto w^\flat(b, b', b''))$ we can conclude that **the set of all T^*-extensions of the Lie algebra B is isomorphic to the space $Z^3(B, \mathbb{K})$ of scalar 3-cocycles of B**. Next, we shall consider the notion of equivalence of T^*-extensions: We observe that the map z in Proposition 3.1 is in $C^1(B, B^*)$ and that the difference of two equivalent 2-cocycles w_1 and w_2 is nothing but δz (cf. eqn (11)). Denote by $F_s(B)$ the vector space of all symmetric invariant bilinear forms on B and for each $g \in F_s(B)$ let $\delta\colon g \mapsto \delta g\colon (b, b', b'') \mapsto g([b, b'], b'')$ be the **Cartan map** $F_s(B) \to Z^3(B, \mathbb{K})$. According to Proposition 3.1 the symmetric part of z must induce a symmetric invariant bilinear form on B whereas the antisymmetric part may be arbitrary. Identifying $C^1(B, B^*)$ canonically with the vector space of bilinear forms on B we see that z must be contained in the direct sum $C^2(B, \mathbb{K}) \oplus F_s(B)$. Therefore we have the following corollaries to Proposition 3.1:

$$(12) \qquad \{\text{equivalence classes of } T^*\text{-extensions of } B\} \cong \frac{Z^3(B, \mathbb{K})}{\delta C^2(B, \mathbb{K}) + \delta F_s(B)}$$

and

$$\{\text{isometric equivalence classes of } T^*\text{-extensions of } B\} \cong \frac{Z^3(B,\mathbb{K})}{\delta C^2(B,\mathbb{K})} = H^3(B,\mathbb{K}) \tag{13}$$

3.2 Associative Algebras

Let B be an associative algebra over a field $\mathbb{K}$ of characteristic not equal to 2. Let V be a vector space over $\mathbb{K}$ such that there are two linear maps λ and ρ of B into the space of linear endomorphisms of V satisfying $\lambda(bb') = \lambda(b)\lambda(b')$ and $\rho(bb') = \rho(b')\rho(b)$ for all $b, b' \in B$. λ and ρ are called a left and right multiplication of B in V and V is called a B-bimodule. For each nonnegative integer k let $CH^k(B,V)$ denote the space of k-linear maps $B \times \cdots \times B$ into V where $CH^0(B,V)$ is defined to be equal to V. Denote by $CH(B,V)$ the direct sum of all the spaces $CH^k(B,V)$ $(0 \leq k < \infty)$. The Hochschild coboundary operator $\delta\colon CH(B,V)$ into $CH(B,V)$ is defined as follows: Let $f \in CH^k(B,V)$ and $b_0, b_1, \ldots, b_k \in B$.

$$\begin{aligned}(\delta f)&(b_0, b_1, \ldots, b_{k-1})\\ &:= \lambda(b_0)(f(b_1, \ldots, b_{k-1})) + (-1)^{k+1}\rho(b_k)(f(b_0, \ldots, b_{k-1}))\\ &\quad + \sum_{i=0}^{k-1}(-1)^{i+1} f(b_0, \ldots, b_i b_{i+1}, \ldots, b_{k-1})\end{aligned} \tag{14}$$

It is known that $\delta^2 = 0$. Call any k-cochain $f \in CH^k(B,V)$ a **Hochschild k-cocycle** iff $\delta f = 0$ and denote the subspace of k-cocycles by $ZH^k(B,V)$. The **k-th Hochschild cohomology group** $HH^k(B,V)$ is defined to be the factor space $ZH^k(B,V)/\delta CH^{k-1}(B,V)$ for $k \geq 1$ and $ZH^0(B,V)$ for $k = 0$.

In particular, the dual space B^* of B is a B-bimodule with respect to the multiplications $\lambda(b) := L^*(b)$ and $\rho(b) := R^*(b)$ for all $b \in B$ (cf. eqs (5) and (6)). Consider a bilinear map $w\colon B \times B \to B^*$ and the corresponding T^*-extension $T^*_w B$ of B. It is known that the multiplication (8) of $T^*_w B$ is associative if and only if w is a Hochschild 2-cocycle, i.e. $w \in ZH^2(B, B^*)$ (cf. [**26**, p. 65–67]). Now, the additional fact that w has to be cyclic means that the trilinear form $w^\flat$ defined by $w^\flat(b, b', b'') := w(b, b')(b'')$ should be invariant under cyclic permutations.

Unlike in the case of a Lie algebra the cohomological transition to scalar $k+1$-linear maps requires more efforts in the associative case: Define for each integer $k \geq 0$:

$$\begin{aligned}CC^k(B) := \big\{ f\colon B \times \cdots \times B \to \mathbb{K}\ (k+1 \text{ factors }) \mid f \text{ is } k+1\text{-linear and}\\ f(b_1, b_2, \ldots, b_k, b_0) = (-1)^k f(b_0, b_1, b_2, \ldots, b_k)\ \forall b_0, \ldots, b_k \in B \big\}\end{aligned} \tag{15}$$

The vector space $CC^k(B)$ is called the **space of cyclic k-cochains** (cf. e.g. [**15**, p. 51 and p. 98, Prop. 1]). Observe that $w^\flat$ is a cyclic 2-cochain. There exists a coboundary operator δ_λ on $CC(B)$, the direct sum of all the $CC^k(B)$ ($k \geq 0$): let f be a cyclic k-cochain and $b_0, \ldots, b_k, b_{k+1} \in B$:

$$(16) \qquad (\delta_\lambda f)(b_0, \ldots, b_k, b_{k+1}) := \sum_{i=0}^{k} (-1)^i f(b_0, \ldots, b_i b_{i+1}, \ldots, b_{k+1}) + (-1)^{k+1} f(b_{k+1} b_0, \ldots, b_k).$$

It is known that $\delta_\lambda^2 = 0$. Each $f \in CC^k(B)$ satisfying $\delta_\lambda f = 0$ is called a **cyclic k-cocycle**, and the space of all cyclic k-cocycles is denoted by $ZC^k(B)$. The k-th cyclic cohomology group $HC^k(B)$ is defined to be the factor space $ZC^k(B)/\delta_\lambda CC^{k-1}(B)$ for $k \geq 1$ and $ZC^0(B)$ for $k = 0$. Now, each cyclic k-cochain f in $CC^k(B)$ can be canonically regarded as a Hochschild k-cochain $f^\sharp \in CH^k(B, B^*)$ by setting $f^\sharp(b_1, \ldots, b_k)(b_{k+1}) := f(b_1, \ldots, b_k, b_{k+1})$ for all $b_1, \ldots, b_{k+1} \in B$. An easy computation gives $((\delta f^\sharp)(b_0, b_1, \ldots, b_k))(b_{k+1}) = -(\delta_\lambda f)(b_0, \ldots, b_k, b_{k+1})$ showing that $CC(B)$ is a subcomplex of $CH(B, B^*)$ (cf. [**15**, p. 102]). In particular, $w^\flat$ is a cyclic 2-cocycle if and only if w is a Hochschild 2-cocycle. Hence we can conclude that **the set of all T^*-extensions of the associative algebra B is isomorphic to the space $ZC^2(B)$ of all cyclic 2-cocycles of B.** Next, we shall consider the notion of equivalence of T^*-extensions: We observe that the map z in Proposition 3.1 is in $CH^1(B, B^*)$ and that the difference of two equivalent 2-cocycles w_1 and w_2 is nothing but δz (cf. eqn (14)). Denote by $F_s(B)$ the vector space of all symmetric invariant bilinear forms on B and for each $g \in F_s(B)$ let δ_λ be the map $F_s(B) \to ZC^2(B)$: $g \mapsto \delta_\lambda g : (b, b', b'') \mapsto g(bb', b'')$ which is the analogon of the Cartan map in the case of a Lie algebra. This is well-defined because the invariance of g implies that $\delta_\lambda g$ is cyclic and that for $b_0, b_1, b_2, b_3 \in B$:

$$(17) \qquad \begin{aligned} &(\delta_\lambda(\delta_\lambda g))(b_0, b_1, b_2, b_3) \\ &\quad = g((b_0 b_1) b_2, b_3) - g(b_0(b_1 b_2), b_3) + g(b_0 b_1, b_2 b_3) - g((b_3 b_0) b_1, b_2) \\ &\quad = 0. \end{aligned}$$

According to Proposition 3.1 the symmetric part of z must induce a symmetric invariant bilinear form on B whereas the antisymmetric part may be arbitrary. Identifying $C^1(B, B^*)$ canonically with the vector space of bilinear forms on B we see that z must be contained in the direct sum $CC^1(B) \oplus F_s(B)$ where by definition the space of antisymmetric bilinear forms on B is equal to $CC^1(B)$. Therefore we have the following corollaries to Proposition 3.1:

$$(18) \qquad \{\text{equivalence classes of } T^*\text{-extensions of } B\} \cong \frac{ZC^2(B)}{\delta_\lambda CC^1(B) + \delta_\lambda F_s(B)}$$

and

$$(19) \qquad \{\text{isometric equivalence classes of } T^*\text{-extensions of } B\} \cong \frac{ZC^2(B)}{\delta_\lambda CC^1(B)} = HC^2(B).$$

For general nonassociative algebras a cohomology theory based on equivalence classes of abelian extensions had been formulated by M. Gerstenhaber in 1964 (cf. [**23**]).

It should be emphasized that the above cohomological formulation applies to isomorphy in the sense of diagram (10), i.e. the ideal B^* remains stable under the isomorphism in question. However, the general situation is much more difficult: for instance, it can happen that there exists an isometry between two trivial T^*-extensions $T_0^*A_1$ and $T_0^*A_2$ without A_1 being isomorphic to A_2 (see Example 4.1 of the next section).

3.3 Nilpotent Metrised Algebras are T^*-extensions

We shall now come to the main result of this paper, namely that very many finite-dimensional metrisable algebras are in fact isometric to certain T^*-extensions.

The proof requires the following little Lemma on Lie algebras:

Lemma 3.2. *Let (V, q) be a metrised vector space of finite dimension n over an algebraically closed field $\mathbb{K}$ of characteristic not equal to 2. Let L be a Lie algebra consisting of linear endomorphisms of V such that one of the following two conditions is satisfied:*

(i) *L consists of nilpotent endomorphisms, and for each $\phi \in L$ its q-transpose ϕ^+ (cf. Appendix A) is contained in L.*

(ii) *The characteristic of $\mathbb{K}$ is equal to 0, L is solvable, and each $\phi \in L$ is q-antisymmetric: $\phi^+ = -\phi$.*

Suppose W is an isotropic subspace of V (i.e. $W \subset W^\perp$) which is stable under L (i.e. $\phi W \subset W$ for all $\phi \in L$).

Then W is contained in a maximally isotropic subspace $W_{\max}$ of V which is also stable under L, and $\dim W_{\max} = [n/2]$ (i.e. the integer part of $n/2$). If n is even, then $W_{\max} = W_{\max}^\perp$. If n is odd, then $W_{\max} \subset W_{\max}^\perp$, $\dim W_{\max}^\perp - \dim W_{\max} = 1$, and $\phi W_{\max}^\perp \subset W_{\max}$ for all $\phi \in L$.

Proof. We shall use induction on n the case $n = 0$ being trivially satisfied. Hence we can assume that $n \geq 1$. If condition a) holds then by Engel's Theorem (cf. [**30**, p. 36]) L is nilpotent and there exists a nonzero vector $v \in V$ such that $\phi v = 0$ for all $\phi \in L$. If condition b) holds then by Lie's Theorem (cf. [**30**, p. 50]) there exists a nonzero L-stable vector $v \in V$ (i.e. a one-form $\lambda \in L^*$ such that $\phi v = \lambda(\phi) v$ for all $\phi \in L$). Therefore, under either condition it suffices to distinguish the following two cases:

Case 1: $W \neq 0$ or there is a nonzero L-stable vector $v \in V$ s. t. $q(v, v) = 0$.

Case 2: $W = 0$ and for all nonzero L-stable vectors $v \in V$ one has $q(v, v) \neq 0$.

In the first case the one-dimensional subspace $\mathbb{K}v$ is a nonzero isotropic L-stable subspace, hence we can restrict our attention to W. Its orthogonal space $W^\perp$ contains W by assumption and is also L-stable: Indeed, let $\phi \in L$, $w \in W$, and $w^\perp \in W^\perp$. Then $q(w, \phi w^\perp) = q(\phi^+ w, w^\perp) = 0$ because $\phi \in L$ under either condition a) or b), and because W is L-stable. Now, the factor space $V' := W^\perp/W$ is again metrised by the projection q' to V' of the restriction of q to $W^\perp \times W^\perp$ (cf. Prop. 2.3(iii) for abelian A). Denote by p the canonical projection $W^\perp \to V'$. The Lie algebra L canonically acts on V' by setting $\phi'(pw^\perp) := p(\phi w^\perp)$ since $W^\perp$ and W are L-stable. Let I denote the vector space $\{\phi \in L | \phi w^\perp \in W \text{ for all } w^\perp \in W^\perp\}$. Clearly, I is an ideal of L, and $I = \{\phi \in L | \phi' = 0\}$. The factor algebra $L' := L/I$ is clearly solvable if condition b) holds. If condition a) is satisfied then for each $\phi \in L$ there is a positive integer m such that $\phi^m = 0$. Obviously, this implies $\phi'^m = 0$, hence L' also consists of nilpotent endomorphisms of V'. Let $w^\perp$ and $x^\perp$ be two arbitrary elements in $W^\perp$. Then by definition of q' we have the following equations for arbitrary $\phi \in L$:

$$(*)\qquad \begin{aligned} q'((\phi')^+(pw^\perp), px^\perp) &= q'(pw^\perp, \phi'(px^\perp)) &&= q'(pw^\perp, p(\phi x^\perp)) \\ &= q(w^\perp, \phi x^\perp) &&= q(\phi^+ w^\perp, x^\perp) \\ &= q'(p(\phi^+ w^\perp), px^\perp) &&= q'((\phi^+)'(pw^\perp), px^\perp) \end{aligned}$$

which shows at once $(\phi')^+ = (\phi^+)'$ for all $\phi \in L$. This implies that the Lie algebra L' satisfies condition a) or b) if L satisfies a) or b), respectively. Since $\dim V' = \dim W^\perp - \dim W = \dim V - 2\dim W$ (compare eqn 26) we can use the induction hypothesis to get a maximally isotropic L'-stable subspace $W'_{\max}$ in V'. Clearly, $\dim W'_{\max} = [n/2] - \dim W$. Now, set $W_{\max} := p^{-1}W'_{\max}$ which is equal to $\{w^\perp \in W^\perp | pw^\perp \in W'_{\max}\}$, hence $W_{\max} \supset W$ and $W'_{\max} \cong W_{\max}/W$. For two arbitrary elements $w^\perp$ and $x^\perp$ of $W_{\max}$ we have $q(w^\perp, x^\perp) = q'(pw^\perp, px^\perp)$ which is equal to zero because $W'_{\max}$ is isotropic. Hence $W_{\max}$ is isotropic, and, since $\dim W_{\max} = \dim W'_{\max} + \dim W = [n/2]$, it is maximally isotropic. Moreover, for arbitrary $\phi \in L$ and $w^\perp \in W_{\max}$ we have $p(\phi w^\perp) = \phi'(pw^\perp) \in W'_{\max}$ which implies $\phi w^\perp \in W_{\max}$. It follows that $W_{\max}$ is L-stable, maximally isotropic and contains W which proves the first assertion of the Lemma in this case.

In the second case, pick a nonzero L-stable vector $v \in V$. If condition a) holds then $\phi v = 0$ for all $\phi \in L$. If condition b) holds we have $\phi v = \lambda(\phi) v$ for a certain one-form λ in L. But, since all $\phi \in L$ are q-antisymmetric we have

$$(**)\qquad \lambda(\phi) q(v, v) = q(\phi v, v) = -q(v, \phi v) = -\lambda(\phi) q(v, v)$$

which implies $\lambda = 0$ because the characteristic of $\mathbb{K}$ is not equal to 2 and $q(v, v) \neq 0$. Hence $\lambda = 0$ under either condition. Clearly, $\mathbb{K}v$ is a q-nondegenerate L-stable subspace of V, therefore $V = \mathbb{K}v \oplus (\mathbb{K}v)^\perp$, and the orthogonal space $(\mathbb{K}v)^\perp$ is also L-stable (because either $\phi^+ \in L$ for each $\phi \in L$ or $\phi^+ = -\phi$). Now, if $(\mathbb{K}v)^\perp = 0$ then V is one-dimensional, $L = 0$, and 0 is the only maximally isotropic subspace of V implying that the Lemma trivially holds. If $(\mathbb{K}v)^\perp \neq 0$ then by the Theorem of Engel (condition a)) or Lie (condition b)) there is a nonzero L-stable vector w

in $(\mathbb{K}v)^\perp$. By assumption, $q(w,w) \neq 0$, hence we get $\phi w = 0$ for all $\phi \in L$ using the same argument we had used above for v. But then it follows that L vanishes on the two-dimensional nondegenerate subspace $\mathbb{K}v \oplus \mathbb{K}w$ of V. Without loss of generality, assume that $q(v,v) = 1 = q(w,w)$ and set $\alpha := q(v,w)$. Then the nonzero vector $v + (-\alpha + \sqrt{\alpha^2 - 1})w$ is isotropic and L-stable at the same time which contradicts the assumption of case 2.

Therefore we have proved the existence of maximally isotropic L-stable subspaces $W_{\max}$ containing W in both cases. If the dimension of V is odd, say $n = 2k+1$, then $k = [n/2] = \dim W_{\max}$ and $k+1 = \dim W_{\max}^\perp$. Now both spaces are L-stable, hence there is an induced action ϕ' of each $\phi \in L$ on the one-dimensional factor space $V' := W_{\max}^\perp / W_{\max}$. By the same reasoning used in Case 1 we can conclude that V' is metrised, and using eqs. $(*)$ and $(**)$ we see that the induced action must be zero. But this means that $W_{\max}^\perp$ is mapped to $W_{\max}$ by L, and the Lemma is proved. □

The main result of this section is contained in the following corollary to the Lemma:

Corollary 3.1. *Let (A,q) be a metrised algebra of finite dimension n over an algebraically closed field $\mathbb{K}$ of characteristic not equal to 2. Suppose that A satisfies one of the following two conditions:*

a) A is nilpotent.

b) A is a solvable Lie algebra, and the characteristic of $\mathbb{K}$ is equal to 0.

Given any isotropic ideal J of A (i.e. $J \subset J^\perp$) then A contains a maximally isotropic ideal I of dimension $[n/2]$ which contains J. Moreover, if n is even then A is isometric to some T^*-extension of the factor algebra A/I. If n is odd then the ideal $I^\perp$ is abelian and A is isometric to a nondegenerate ideal of codimension one in some T^*-extension of the factor algebra A/I.

Proof. Suppose that condition a) holds. Since A is nilpotent its multiplication algebra $LR(A)$ is an associative algebra consisting of nilpotent endomorphisms (cf. Appendix A for definitions). Now the invariance of the symmetric bilinear form q implies the following equations for all $a, a', a'' \in A$:

$$q(R(a')a, a'') = q(aa', a'') = q(a, a'a'') = q(a, L(a')a'')$$

which implies $L(a)^+ = R(a)$ and, by the symmetry of q, $R(a)^+ = L(a)$ for all $a \in A$. Since $LR(A)$ is generated by all left and right multiplications $L(a)$ and $R(a')$ each element ϕ of $LR(A)$ can be written as a sum of products of the form $\phi = S(a_1)\cdots S(a_k)$ where $a_1, \dots, a_k \in A$ and S denotes left or right multiplication. It follows that the q-transpose ϕ^+ of ϕ is of the form

$$\phi^+ = (S(a_1)\cdots S(a_k))^+ = (S(a_k))^+ \cdots (S(a_1))^+.$$

But since the q-transpose of any left multiplication is a right multiplication and vice versa we can conclude that $\phi^+ \in LR(A)$ whenever $\phi \in LR(A)$. If we consider $LR(A)$ as a Lie algebra w. r. t. the natural commutator $[\phi, \phi'] = \phi\phi' - \phi'\phi$ of linear maps we can conclude that condition a) of the preceding Lemma is satisfied for $L = LR(A)$. Observing that a subspace W of A is $LR(A)$-stable if and only if it is an ideal of A we see that J is an isotropic $LR(A)$-stable subspace of A. But then Lemma 3.2 supplies us with a maximally isotropic $LR(A)$-stable subspace I of A containing J. Hence I is a maximally isotropic ideal of dimension $[n/2]$ containing J.

If condition b) holds consider the Lie algebra $ad(A) := \{ad(a) := [a, \ ]|a \in A\}$. Because of $q([a, a'], a'') = -q([a', a], a'') = -q(a', [a, a''])$ for all $a, a', a'' \in A$ we see that all linear endomorphisms $ad(a)$ are q-antisymmetric (i.e. $ad(a)^+ = -ad(a)$) for all $a \in A$. Since $ad(A)$ is isomorphic to A modulo its centre and is therefore solvable condition b) of the preceding Lemma is satisfied. Observing that a subspace $W \subset A$ is an ideal of A if and only if it is $ad(A)$-stable we can use Lemma 3.2 to conclude that every isotropic ideal J is contained in a maximally isotropic ideal of dimension $[n/2]$.

If n is even then A is isometric to some T^*-extension of A/I by Theorem 3.2.

If n is odd then $\dim I^\perp - \dim I = 1$ and $\phi I^\perp \subset I$ for all $\phi \in LR(A)$ (or $ad(A)$) according to Lemma 3.2. In particular, it follows that $A(I^\perp) + (I^\perp)A \subset I$. Hence $I^\perp \subset (A(I^\perp) + (I^\perp)A)^\perp = Z(I^\perp)$, the annihilator of $I^\perp$ in A (cf. Prop. 2.1(ii)). This proves that $I^\perp$ is abelian. Now, take any one-dimensional abelian algebra $\mathbb{K}c$ spanned by a nonzero vector c, define a nondegenerate symmetric bilinear form q_c on $\mathbb{K}c$ by $q_c(c, c) := 1$, and form the orthogonal sum $(A', q') := (A \oplus \mathbb{K}c, q \perp q_c)$. This is a metrised nilpotent algebra if condition a) holds and a metrised solvable Lie algebra if condition b) holds. Obviously, A is a nondegenerate ideal of codimension one in A'. Since $I^\perp$ is not isotropic and $\mathbb{K}$ is algebraically closed there exists a vector $d \in I^\perp$ such that $q(d, d) = -1$. Define $e := c + d$ and $I' := I \oplus \mathbb{K}e$. Then I' is an isotropic ideal of A' of dimension $(n+1)/2$: Indeed, since $q'(e, e) = q(d, d) + q_c(c, c) = -1 + 1 = 0$ and $q'(I, d+c) = q(I, d) + q'(I, c) = 0 + 0 = 0$ we have that I' is isotropic. Moreover, $cA = 0 = Ac$ by definition of the orthogonal sum, and $dA \subset I \supset Ad$ by what was proved above. This implies that I' is an ideal of A'. By Theorem 3.2 we know that A' is isometric to some T^*-extension of the factor algebra A'/I'. Observing that for each $\lambda \in \mathbb{K}, a \in A$ the linear map $\Phi \colon A' \to A/J \colon a + \lambda c \mapsto a - \lambda d + I$ is a surjective homomorphism of algebras (where again the relations $dA \subset I \supset Ad$ are used) with kernel I' we can conclude that A'/I' is isomorphic to A/J. This proves the Corollary. □

We shall now see that in every finite-dimensional nilpotent metrised algebra (A, q) there is a very natural isotropic ideal J of A. Define:

$$J := \sum_{i=0}^{\infty} C^i(A) \cap C_i(A). \tag{20}$$

Since A is finite-dimensional this sum is finite. Furthermore, we have that $C^i(A)^\perp = C_i(A)$ (cf. Prop. 2.1(iv)), hence $C^i(A) \cap C_i(A)$ is isotropic for all $0 \leq i$. For $i < j$ we have $C^i(A) \supset C^j(A) \supset C^j(A) \cap C_j(A)$ hence: $(C^j(A) \cap C_j(A))^\perp \supset (C^i(A))^\perp = C_i(A) \supset C_i(A) \cap C^i(A)$, and it follows that $q(C^i(A) \cap C_i(A), C^j(A) \cap C_j(A)) = 0$ for all $0 \leq i, j$. Therefore J is an isotropic ideal of A. Let m denote the nilindex of A and let m' be the integer part of $(m+1)/2$. Using the relations $C^i(A) \subset C_{m-i}(A) \subset C_{m-i+1}(A)$ (compare Appendix A) we can conclude that $C^{m'}(A) \subset C_{m'}(A)$. This implies that $C^{m'}(A)$ is contained in J. Now, according to the preceding Corollary there exists a maximally isotropic ideal I of A containing $J \supset C^{m'}(A)$. That means that the factor algebra A/J has nilindex at most m'. In other words, **every finite-dimensional nilpotent metrised algebra over an algebraically closed field of characteristic not two is isometric to (a nondegenerate ideal of codimension one of) a T^*-extension of a nilpotent algebra of nilindex roughly one half of the nilindex of** A.

Notes

Except for the cohomological statements eqs. (11)–(19), Lemma 3.2, and Corollary 3.1, all results of this section are already contained in my Diplomarbeit (cf. [**10**, Section 2.6]).

The dual left and right multiplications (eqs. (5) and (6)) are well-known for associative algebras (cf. [**16**, p. 413]). Trivial T^*-extensions (i.e. $w = 0$) of Lie algebras have been constructed by Medina and Revoy in [**47**] and have later been used by Kostant and Sternberg (cf. [**42**]) in their formulation of BRST quantization. They also appear as a particular case of a Manin triple (see [**17**] or the next section for a definition) where one of the isotropic subalgebras is abelian. The trivial T^*-extension of an associative algebra is used in a Proposition by H. Tachikawa (cf. [**36**, p. 55, Prop. 1.13.]) that every finite-dimensional associative algebra with unit is a homomorphic image of a finite-dimensional symmetric Frobenius algebra. In the classification of finite-dimensional two-step nilpotent metrised Lie algebras by Medina and Revoy (cf. [**48**]) a nontrivial T^*-extension of an abelian Lie algebra by an alternating three-form is constructed: thus it is a particular case of Corollary 3.1. In her theory of proto-Lie-bigebras (cf. [**41**], [**40**]) Kosmann-Schwarzbach investigates the splitting of an even-dimensional metrised Lie algebra A into the direct vector space sum of two isotropic subspaces F and F^* which are not necessarily subalgebras. Taking components of the Lie bracket in A w. r. t. F and F^* and considering F^* as the dual space of F the equivalent description consists of four objects: a bracket μ in $(F^* \wedge F^*) \otimes F$, a co-bracket γ in $(F \wedge F) \otimes F^*$, a 3-form ψ in $F^* \wedge F^* \wedge F^*$, and a 3-form ϕ in $F \wedge F \wedge F$ satisfying five compatibility conditions (cf. [**41**, p. 9], and [**40**, p. 392], Definition, and [**40**, p. 393, Prop. 3], and [**46**]). Nontrivial T^*-extensions can be viewed as a particular case of this construction, namely $\phi = 0$ and $\gamma = 0$ which forces μ to be a Lie bracket and ψ to be

a scalar 3-cocycle of this bracket. Furthermore, Keith's notion of bi-extension (see the notes for Section 2) of Lie algebras contains the T^*-extension as the special case $B = B^\perp$, $M := B/B^\perp = 0$.
The statement 3.1(iii) about decomposability properties of a trivial T^*-extension is a generalized version of a similar Theorem of Astrakhantsev (cf. [**6**, Theorem 5]): he has considered the Lie algebra TA of the tangent bundle of a Lie group admitting a metrised Lie algebra A (see also the discussion following Proposition 2.3 in Section 2 for a definition). Since coadjoint and adjoint representation are equivalent for metrised Lie algebras (cf. the discussion following Prop. 2.4 in Section 2) TA is isometric to the trivial T^*-extension T_0^*A.
I learned the notation $HH(A, V)$ for Hochschild cohomology and $HC(A)$ for cyclic cohomology in a talk by C. Kassel, Strasbourg.

4. Applications to Lie and Associative Algebras. Examples

In this Section we shall discuss the important special cases of Lie and associative algebras where we use the filiform and Heisenberg Lie algebras as illustrative (counter) examples.

Let (A, q) a metrised Lie algebra of finite dimension n over a field $\mathbb{K}$. If it contains an isotropic subalgebra B of dimension $[n/2]$ then (A, B, q) is called a **Manin pair** (cf. [**18**, p. 130]). If n is even and A contains two $n/2$-dimensional isotropic subalgebras B_1 and B_2 such that $A = B_1 \oplus B_2$ then (A, B_1, B_2, q) is called a **Manin triple** (cf. [**17**, p. 902]).

Theorem 4.1. *Let (A, q) be a metrised Lie algebra of finite dimension n over an algebraically closed field $\mathbb{K}$ of characteristic zero.*

- (i) *Then there exists a solvable subalgebra B of A such that (A, B, q) is a Manin pair.*
- (ii) *If the maximal semisimple ideal of A is zero then the algebra B in (i) can be chosen to be a nilpotent subideal contained in the radical R of A.*
- (iii) *If A is solvable then the algebra B in (i) can be chosen to be an abelian ideal of A.*
- (iv) *If A is semisimple and of even dimension there are two solvable isotropic subalgebras B_1 and B_2 of A such that (A, B_1, B_2, q) is a Manin triple.*

Proof. Part (iii) had already been shown in the preceding Section (cf. Cor. 3.1b).

(ii) Let L be a Levi subalgebra of A. By the Double Extension Theorem (see Thm. 2.2(iii)) we know that the orthogonal space $R^\perp$ of R is contained in R. Take a maximally isotropic abelian ideal I in the solvable metrised Lie algebra $R/R^\perp$ (cf. (i)) and define B to be the inverse image of I in R under the canonical projection $R \to R/R^\perp$. Since $R^\perp$ is isotropic it follows that B is isotropic. Moreover, we can conclude that $\dim B = \dim I + \dim R^\perp = [(\dim(R/R^\perp))/2] + \dim R^\perp = [n/2]$

because $A = L \oplus R$ and $\dim L = \dim R^\perp$. Since $B/R^\perp$ is abelian and $R^\perp$ is central we can conclude that B is nilpotent.

(i) Again using the Double Extension Theorem we know that A decomposes into an orthogonal direct sum $A = A_0 \oplus A_1$ where A_0 is the maximal semisimple ideal of A and A_1 contains no nonzero semisimple ideals. Let n_0 and n_1 be the dimensions of A_0 and A_1, respectively. Let B_0' be a maximal solvable subalgebra (a so-called Borel subalgebra) of A_0. B_0' decomposes uniquely into $H \oplus N$ where H is a Cartan subalgebra of A_0 and N is the nilradical of B_0' which may be visualized as the nilpotent algebra spanned by all root vectors corresponding to positive roots (cf. [**28**, p. 84]). By Thm. 2.1(ii) we can conclude that all the simple ideals of A_0 are mutually orthogonal under the restriction q_0 of q to A_0. Writing $q(x,y) = \mathrm{Kil}\,(\phi x, y)$ where $x, y \in A$, Kil denotes the Killing form of A_0 and ϕ is a suitable linear endomorphism of A_0 we can see that the invariance of q entails that ϕ commutes with all linear maps $ad(x)$. By Schur's Lemma we can conclude that $q = \sum \lambda_i \mathrm{Kil}_{\,i}$ where the sum goes over the simple ideals of A_0, λ_i are nonzero elements of $\mathbb{K}$, and $\mathrm{Kil}_{\,i}$ denotes the Killing form restricted to the ith simple ideal. Hence it follows that $q(N,N) = 0 = q(H,N)$ and the restriction of q to $H \times H$ is nondegenerate because the same is true for Kil replacing q (cf. [**28**, p. 36]). If r_0 notes the dimension of H (the so-called rank of A_0) take any isotropic $[r_0/2]$-dimensional subspace V of H. Then $B_0 := V \oplus N$ is a solvable isotropic subalgebra of A_0 (because $[H,H] = 0$ and $[H,N] \subset N$) whose dimension is $[n_0/2]$ since $n_0 = 2\dim N + r_0$. Now, if not both A_0 and A_1 are odd-dimensional we can simply take the direct sum of ideals $B_0 \oplus B_1$ where B_1 is the nilpotent isotropic subalgebra of A_1 constructed in (ii) to get a solvable isotropic subalgebra of A of dimension $[n/2]$. On the other hand, assume that both A_0 and A_1 are odd-dimensional. Consider the orthogonal space $B_0^\perp$ to B_0 in A_0 and the orthogonal space $B_1^\perp$ to B_1 in A_1. Then there is a nondegenerate vector b_0 in $H \cap B_0^\perp$ and a nondegenerate vector b_1 in $R \cap B_1$. Since A_0 and A_1 are mutually orthogonal the restriction of q to the two-dimensional vector space $\mathbb{K}b_0 \oplus \mathbb{K}b_1$ is nondegenerate. Now take a nonzero isotropic vector b in this two-dimensional vector space. Being orthogonal to both B_0 and B_1 the vector subspace $B := \mathbb{K}b \oplus B_0 \oplus B_1$ of A is isotropic. But $[b, B_0] = [b_0, B_0] \subset [H, B_0] \subset B_0$ and $[b, B_1] = [b_1, B_1] \subset B_1$ (because $B_1^\perp = \mathbb{K}b_1 \oplus B_1$, cf. Cor. 3.1b) and its proof). Hence B is a subalgebra of A which is solvable since it is contained in the solvable subalgebra $B_0' \oplus R$ of A.

(iv) As in (iii) pick a Cartan subalgebra H of A. Since A is even-dimensional H is also even-dimensional. Choose an ordering on the set of all roots corresponding to H. Again, let N^+ and N^- denote the nilpotent subalgebras spanned by all the root vectors corresponding to positive roots and negative roots, respectively. Because $q = \sum \lambda_i \mathrm{Kil}_{\,i}$ (compare (iii)) N^+ and N^- are both isotropic and the restriction of q to $H \times H$ is nondegenerate. It follows that there exist two isotropic subspaces H^+ and H^- of H such that $H = H^+ \oplus H^-$. Since H is abelian H^+ and H^- are both abelian subalgebras of A. Because $[H, N^+] \subset N^+$ and $[H, N^-] \subset N^-]$ the vector spaces $B_1 := H^+ \oplus N^+$ and $B_2 := H^- \oplus N^-$ are solvable subalgebras of A. Since $q(H, N^+) = 0 = q(H, N^-)$ they are both isotropic and span A because $A = N^- \oplus H \oplus N^+$. □

The next Theorem characterizes metrised associative algebras in a manner very similar to the Double Extension Theorem (cf. Thm. 2.2):

Theorem 4.2. *Let (A, f) be a finite-dimensional metrised associative algebra over an algebraically closed field $\mathbb{K}$ of characteristic not equal to 2. Let A_0 denote the largest semisimple ideal of A and A_1 its orthogonal space.*

Then (A, f) is given by the orthogonal direct sum $(A_0 \oplus A_1, f_0 \perp f_1)$ where f_0 (f_1) denotes the restriction of f to $A_0 \times A_1$ ($A_1 \times A_1$). Moreover, A_1 does not contain any nonzero semisimple ideal, and the radical R of A is contained in A_1. Let L denote a Levi subalgebra of A_1 (cf. [**16**, p. 491, Thm. 72.19]*). Then the orthogonal space $R^\perp$ (w. r. t. f_1) of R is contained in R, and the subalgebra $L \oplus R^\perp$ is nondegenerate w. r. t. f_1. Let f_L be the restriction of f to $L \times L$ and p_L the canonical projection $L \oplus R \to L$. Then $L \oplus R^\perp$ is isometric to the T^*-extension $(T_0^* L, q_L + p_L^* f_L)$. Denote by A' the factor algebra $R/R^\perp$ and by f' the projection to A' of the restriction of f_1 to $R \times R$. Then (A', f') is a nilpotent metrised associative algebra and therefore isometric to a suitable (nondegenerate ideal of codimension 1 of a) T^*-extension (cf. Cor. 3.1).*

Proof. The well-known Wedderburn-Artin-Theorem implies that every finite-dimensional semisimple associative algebra B over an algebraically closed field $\mathbb{K}$ decomposes -up to permutations- uniquely into a finite direct sum $\oplus B_i$ of simple ideals B_i each of which is isomorphic to the algebra of all linear endomorphisms of some finite-dimensional vector space over $\mathbb{K}$ (cf. e. g. [**29**, p. 40–43, paragraphs 3–4], and the fact that any finite-dimensional division ring over an algebraically closed field $\mathbb{K}$ is identical with $\mathbb{K}$). Moreover, any ideal I of B is a direct sum of some of the B_i because $I = IB$ (since B has a unit element) $= \oplus IB_i \subset \oplus I \cap B_i \subset I$ and $I \cap B_i$ is either zero or equal to B_i. Hence I is semisimple and there is a unique semisimple ideal J of B such that $B = I \oplus J$. So if B and B' are semisimple ideals of A intersecting in I then $B + B' = J \oplus B'$ is semisimple whence A_0 is well-defined. The intersection of A_0 with its orthogonal space A_1 is an abelian ideal of the semisimple algebra A_0 (cf. Prop. 2.1(i)) and therefore has to vanish which implies that A is an orthogonal direct sum of A_0 and A_1. Since any nilpotent ideal of A_0 vanishes it follows that $A_0 R + R A_0 \subset A_0 \cap R = 0$ and therefore we have (since A_0 has a unit element) $f(A_0, R) = f(A_0 A_0, R) \subset f(A_0, A_0 R) \subset 0$ whence $R \subset A_1$. Consider now the factor algebra $(R^\perp + R)/R \cong R^\perp/(R \cap R^\perp)$: This is an ideal in the semisimple factor algebra A_1/R which is semisimple by the above reasoning. Hence $R \cap R^\perp$ is the radical of $R^\perp$. By the Wedderburn-Mal'cev-Theorem (cf. [**16**, p. 491, Thm. 72.19]) there is a semisimple Levi subalgebra M' of $R^\perp$ complementary to $R^\perp \cap R$ and a unique semisimple ideal M of the Levi subalgebra L both projecting one-one onto $(R + R^\perp)/R$. Hence M' is contained in the subalgebra $M \oplus R$ of A_1 and both M and M' are Levi subalgebras of $M \oplus R$. By the above-mentioned Wedderburn-Mal'cev-Theorem there is an element n in R such that M' is the image of M under the automorphism $(id - L(n))(id - R(n))^{-1}$ of $M \oplus R$ and A_1 (where id, $L(n)$, $R(n)$ denote the identity map in A_1, left multiplication with n, and right multiplication with n, respectively). Hence $L' := (id - L(n))(id - R(n))^{-1}L$ is a Levi subalgebra of A_1 containing M' as a semisimple ideal. On the other hand, M' is contained in $R^\perp$ whence $M'R + RM' \subset (R^\perp)R + R(R^\perp) = 0$. But this means that M' is a semisimple ideal of A_1 and since $A_0A_1 + A_1A_0 = 0$ we have that M' is a semisimple ideal of A and should be contained in A_0. It follows that M' vanishes implying $R^\perp \subset R$. Now the restriction of f_1 to $L \oplus R^\perp \times L \oplus R^\perp$ has to be nondegenerate because $f_1(R, R^\perp) = 0$ and f_1 is nondegenerate on A_1. Moreover, since L is a subalgebra and $R^\perp$ is an ideal of A_1 the vector space $L \oplus R^\perp$ is a subalgebra of A_1. Because of the dimension formula for the orthogonal space (cf. eqn (26) in Appendix A) L and $R^\perp$ have the same dimension. Since $R^\perp$ is an isotropic ideal we can use the proof of Thm. 3.2 for the particular case $w = 0$ to show that $L \oplus R^\perp$ is isomorphic to T_0^*L via the map m defined in that proof. That m is also an isometry, i.e. $m^*(q_L + p_L^* f_L) = f_1$ restricted to $L \oplus R^\perp \times L \oplus R^\perp$, can be seen by a straight forward modification of the last four lines of the proof of Thm. 3.2 which we leave to the reader. The rest of the Theorem is a consequence of Prop. 2.3(iii) and Cor. 3.1. □

We see that A_1 is a (semi)direct sum of a subalgebra (L) and an ideal (R) which was a characteristic feature of the Double Extension construction 2.2 for Lie algebras. However, in contrast to the Lie case this will not in general be true for nilpotent metrised associative algebras: the two-dimensional algebra $K(3)$ (cf. Section 2) spanned by x and x^2 over the field $\mathbb{K}$ with $x^3 = 0$ is metrised by declaring x and x^2 to be isotropic and $f(x, x^2) := 1$. But here every one-dimensional subalgebra of $K(3)$ is identical with $\mathbb{K}x^2$, hence no (semi)direct sum of a nontrivial subalgebra and a nontrivial ideal is possible.

The following Proposition shows that left symmetric algebras (cf. Appendix A for a definition) are automatically associative when supposed to be metrised:

Proposition 4.1. *Let (A, f) be a left-symmetric metrised algebra over a field of characteristic different from 2.*

Then (A, f) is a metrised associative algebra.

Proof. Let $a, b, c, d \in A$. Then we have the following identity for the associator $(a, b, c) := (ab)c - a(bc)$:

$$f((a, b, c), d) = f((ab)c - a(bc), d) = f(ab, cd) - f(a, (bc)d) = f(a, b(cd) - (bc)d) \ ,$$

whence by the symmetry of f

$$(*) \qquad f((a,b,c),d) = -f((b,c,d),a) \ \forall \ a,b,c,d \in A.$$

It follows that

$$\begin{aligned} f((a,b,c),d) &= -f((b,c,d),a) = -f((c,b,d),a) \\ &= +f((b,d,a),c) = +f((d,b,a),c) \\ &= -f((b,a,c),d) = -f((a,b,c),d) \end{aligned}$$

where we have used $(*)$ and the symmetry of the associator in the first two arguments in interchanging order. This clearly implies $2f((a,b,c),d) = 0$ for all $a,b,c,d \in A$, hence $(a,b,c) = 0$ by the nondegeneracy of f. □

The following example shows that **two nonisomorphic Lie algebras may have isometric T^*-extensions**:

Example 4.1. Let (A,q) be the metrised Lie algebra spanned by eight basis elements $g,d,G,D,\Gamma,\Delta,\gamma,\delta$ over an arbitrary field $\mathbb{K}$ where the only nonvanishing Lie brackets are given by $[g,d] = d$, $[g,D] = D$, $[d,G] = -D$, $[g,\Delta] = -\Delta$, $[d,\Delta] = \Gamma$, $[g,\delta] = -\delta$, $[d,\delta] = \gamma$ and the only nonvanishing scalar products are defined by $1 = q(g,\gamma) = q(d,\delta) = q(G,\Gamma) = q(D,\Delta)$. That (A,q) is well-defined is easily calculated; note that the $\mathbb{K}$-span of $\{G,D,\Gamma,\Delta,\gamma,\delta\}$ is an abelian ideal on which the two-dimensional nonabelian Lie algebra spanned by g and d acts via its adjoint representation (on $\mathbb{K}$-span$\{G,D\}$) and its coadjoint representation (on $\mathbb{K}$-span$\{\gamma,\delta\}$ and on $\mathbb{K}$-span$\{\Gamma,\Delta\}$). Now, it is not difficult to see that $B_1 := \mathbb{K}$-span$\{g,d,G,D\}$ and $B_2 := \mathbb{K}$-span$\{g,d,\Gamma,\Delta\}$ are maximally isotropic subalgebras of A and $I_1 := \mathbb{K}$-span$\{\Gamma,\Delta,\gamma,\delta\}$ and $I_2 := \mathbb{K}$-span$\{G,D,\gamma,\delta\}$ are maximally isotropic ideals of A. Furthermore, $A = B_1 \oplus I_1 = B_2 \oplus I_2$. By Thm. 3.2 and its proof we can conclude that (A,q) is both isometric to the trivial T^*-extension $(T_0^*B_1, q_{B_1})$ of B_1 and to the trivial T^*-extension $(T_0^*B_2, q_{B_2})$ of B_2 (we have chosen the subalgebras B_1 and B_2 as complementary isotropic subspaces to the ideals I_1 and I_2 whence the cocycles w_1 and w_2 vanish). However, as can immediately be seen by the above Lie brackets, the dimension of $[B_1,B_1]$ is equal to two whereas the dimension of $[B_2,B_2]$ equals three. Therefore, B_1 and B_2 are not isomorphic.

For a given positive integer n and an arbitrary field $\mathbb{K}$ let $\mathcal{H}_n$ denote the Lie algebra spanned by the $2n+1$ elements $\{q_1,\ldots,q_n,p_1,\ldots,p_n,e\}$ over $\mathbb{K}$ where the only nonvanishing Lie brackets are given by $e = [q_1,p_1] = [q_2,p_2] = \cdots = [q_n,p_n]$. $\mathcal{H}_n$ is called the **(nth) Heisenberg algebra** (over $\mathbb{K}$). It is obviously a nilpotent Lie algebra of nilindex 2 where $\mathbb{K}\,e$ equals both the centre and the derived algebra $[\mathcal{H}_n,\mathcal{H}_n]$. It is easy to calculate that each nonzero ideal of $\mathcal{H}_n$ contains e whence it follows that $\mathcal{H}_n$ is indecomposable for all $n \geq 1$. Let $\{q^1, \ldots, q^n, p^1, \ldots, p^n, \epsilon\}$ be the basis of the dual space $\mathcal{H}_n^*$ of $\mathcal{H}_n$ that is (in obvious notation) dual to the above defining basis. The coadjoint representation of $\mathcal{H}_n$ is quickly computed: the only nonvanishing parts are the following: $ad^*(q_i)(\epsilon) = -p^i$, $ad^*(p_i)(\epsilon) = q^i$ (for all $1 \leq i \leq n$). The scalar cocycles, coboundaries and cohomology groups $H(\mathcal{H}_n, \mathbb{K})$ (cf. Section 3 for definitions) have been characterized and computed by L. J. Santharoubane (cf. [**51**]): for instance, he has found the dimension formula

$$\begin{aligned} \dim\ H^k(\mathcal{H}_n, \mathbb{K}) &= \dim\ H^{2n+1-k}(\mathcal{H}_n, \mathbb{K}) \\ &= \binom{2n}{k} - \binom{2n}{k-2} \qquad (0 \leq k \leq n) \end{aligned} \tag{21}$$

where the occurring binomial coefficients are defined to be zero for negative lower entries. In order to construct T^*-extensions we note that for the Heisenberg algebras the set of equivalence classes of T^*-extensions in the sense of eqn (12) is already isomorphic to $H^3(\mathcal{H}_n, \mathbb{K})$ because the Cartan map vanishes (cf. Section 3): indeed, let f be a symmetric invariant bilinear form on $\mathcal{H}_n$. Then we have that $f(e,e) = f([q_1,p_1],e) = f(q_1,[p_1,e]) = 0$; $f(q_i,e) = f(q_i,[q_1,p_1]) = f([q_i,q_1],p_1) = 0$ and likewise $f(p_j,e) = 0$ for all $1 \leq i,j \leq n$. But since $[\mathcal{H}_n, \mathcal{H}_n] = \mathbb{K}e$ the trilinear form $(\delta f)(a,b,c) = f([a,b],c)$ has to vanish for all $a,b,c \in \mathcal{H}_n$ whence eqs. (12) and (13) are in fact equivalent in this case.

Example 4.2. Let $\mathbb{K}$ be a field of characteristic not two and consider the first Heisenberg algebra $\mathcal{H}_1$ over $\mathbb{K}$. Clearly, the third scalar cohomology group $H^3(\mathcal{H}_1, \mathbb{K})$ is spanned by the volume form $w^\flat := q^1 \wedge p^1 \wedge \epsilon$. Consider the T^*-extension $(A,q) := (T_w^*\mathcal{H}_1, q_{\mathcal{H}_1})$ where we have set $w(a,b)(c) := w^\flat(a,b,c)$ for all $a,b,c \in \mathcal{H}_1$. The only nonvanishing Lie brackets for the basis elements $q_1, p_1, e, q^1, p^1, \epsilon$ are easily computed (cf. eqn (8)): $[q_1,p_1] = e + \epsilon$, $[q_1,e] = -p^1$, $[p_1,e] = q^1$, $[q_1,\epsilon] = -p^1$, and $[p_1,\epsilon] = q^1$. It follows that the nilindex of A is three (in contrast to the trivial T^*-extension $T_0^*\mathcal{H}_1$ which has nilindex two, cf. Thm. 3.1(ii)). Moreover, (A,q) clearly decomposes into the orthogonal direct sum of the five-dimensional ideal I spanned by $q_1, p_1, e+\epsilon, q^1, p^1$ and the one-dimensional ideal J spanned by $e - \epsilon$. Hence (A,q) is an example of a **decomposable** metrised Lie algebra which is a **T^*-extension of an indecomposable Lie algebra** which shows that statement (iii) of Thm. 3.1 cannot be extended in general to nontrivial T^*-extensions. The metrised five-dimensional Lie algebra I is isometric to the Lie algebra W_3 of Favre and Santharoubane (cf. [**20,** Sections 4 and 5]).

Now suppose that (A,q) was a Manin triple. Then there would exist two isotropic three-dimensional complementary subalgebras B and C of A. Using the well-known fact that every three-dimensional nilpotent Lie algebra is either abelian or isomorphic to $\mathcal{H}_1$ we can conclude that both B and C would have to be isomorphic to $\mathcal{H}_1$: if one of the algebras, say C, was abelian it would have to be an isotropic ideal of A (cf. Thm. 3.2) which would imply that A was isometric to the trivial T^*-extension of B. But this would be impossible because A would be of nilindex at most two in contrast to the above. Now, for $b \in B$ and $c \in C$ let $\rho(b)(c)$ and $-\lambda(c)(b)$ denote the C-component and B-component of the bracket $[b, c]$ w. r. t. the decomposition $A = B \oplus C$, respectively. Using the Jacobi identity it is clear that ρ is a representation of B on C which is equivalent to the coadjoint representation of B since B and C are dual to each other by the invariant symmetric bilinear form q. An analogous statement is true for λ as a representation of C on B. Let β and γ span the centres of B and C, respectively. It follows by the Lie brackets of $\mathcal{H}_1$ that the dimension of the vector space $\rho(B)(\gamma)$ would be two whereas $\lambda(\gamma)(B) = 0$. Hence $[B, \gamma] = \rho(B)(\gamma) \subset C$, and in a completely analogous manner we would have $[C, \beta] \subset B$. But this would mean that the derived ideal $[A, A]$ was at least four-dimensional which would directly contradict the above Lie brackets. Hence (A, q) is also an **example of an even-dimensional metrised Lie algebra which is no Manin triple**.

Another family of finite-dimensional nilpotent Lie algebras provides us with examples having arbitrarily large nilindex: for an integer $n \geq 2$ let $\mathcal{L}_n$ denote the Lie algebra spanned by $n+1$ elements $e_0, e_1, \ldots, e_n$ over an arbitrary field $\mathbb{K}$ where the only nonvanishing Lie brackets are given by $[e_0, e_i] = e_{i+1}$ for all $1 \leq i \leq n-1$. $\mathcal{L}_n$ is called the (nth) **filiform algebra** (over $\mathbb{K}$) and has been defined and investigated in the Thesis of M. Vergne (cf. [**55**]). Clearly, $\mathcal{L}_n$ is a nilpotent Lie algebra of nilindex n with centre spanned by e_n. It is easy to calculate that each nonzero ideal of $\mathcal{L}_n$ contains e_n whence it follows that $\mathcal{L}_n$ is indecomposable for all $n \geq 2$. Let $e^0, e^1, \ldots, e^n$ be the basis of the dual space $\mathcal{L}_n^*$ of $\mathcal{L}_n$ that is in (obvious notation) dual to the above defining basis. The coadjoint representation of $\mathcal{L}_n$ is quickly computed: the only nonvanishing parts are $ad^*(e_0)e^i = -e^{i-1}$ and $ad^*(e_{i-1})e^i = e^0$ for all $2 \leq i \leq n$. Some remarks concerning the scalar cohomology groups $H^k(\mathcal{L}_n, \mathbb{K})$ of $\mathcal{L}_n$ for $0 \leq k \leq 3$ are contained in Appendix B. In order to construct T^*-extensions we again note that for the filiform algebras the set of equivalence classes of T^*-extensions in the sense of eqn (12) is already isomorphic to $H^3(\mathcal{L}_n, \mathbb{K})$ because the Cartan map vanishes (cf. Section 3): indeed, let f be a symmetric invariant bilinear form on $\mathcal{L}_n$. Then for $2 \leq j \leq n$ we have that $f(e_0, e_j) = f(e_0, [e_0, e_{j-1}]) = f([e_0, e_0], e_{j-1}) = 0$ and for all $1 \leq i \leq n$ $f(e_i, e_j) = f(e_i, [e_0, e_{j-1}]) = f([e_{j-1}, e_i], e_0) = 0$. But since $[\mathcal{L}_n, \mathcal{L}_n]$ is spanned by all the e_j with $2 \leq j \leq n$ it follows that the trilinear form $(\delta f)(a, b, c) = f([a, b], c)$ has to vanish for all $a, b, c \in \mathcal{L}_n$ whence eqs (12) and (13) are in fact also equivalent in this case.

Example 4.3. Let $\mathbb{K}$ be a field of characteristic not two and consider the nth filiform algebra $\mathcal{L}_n$ over $\mathbb{K}$ for some integer $n \geq 2$. Then the 3-form $w^\flat := e^0 \wedge e^{n-1} \wedge e^n$ is a scalar 3-cocycle of $\mathcal{L}_n$ (compare Appendix B). Forming the map $w\colon \mathcal{L}_n \times \mathcal{L}_n \to \mathcal{L}_n^*$ by $w(a, b)(c) := w^\flat(a, b, c)$ for all $a, b, c \in \mathcal{L}_n$ we can construct the T^*-extension $(A, q) = (T_w^*\mathcal{L}_n, q_{\mathcal{L}_n})$ of $\mathcal{L}_n$. Using the abovementioned basis $e_0, e_1, \dots, e_n, e^0, e^1, \dots, e^n$ the only nonvanishing Lie brackets of A are easily computed (cf. eqn (8)): $[e_0, e_i] = e_{i+1}$ (for all $1 \leq i < n-1$), $[e_0, e_{n-1}] = e_n + e^n$, $[e_0, e_n] = -e^{n-1}$, $[e_{n-1}, e_n] = e^0$, $[e_0, e^i] = -e^{i-1}$, $[e_{i-1}, e^i] = e^0$ (for all $2 \leq i \leq n$). Denoting by ad_w the adjoint representation in A we deduce from these Lie brackets that $ad_w(e_0)^{2n-2}e_1 = 2(-1)^{n-1}e^1 \neq 0$. Using Thm. 3.1(i) we can conclude that $T_w^*\mathcal{L}_n$ has maximal nilindex $2n-1$.

Appendix A: Elementary Notions of Non-Associative Algebras and Bilinear Forms

In this Appendix we shall recall some definitions and concepts from the theory of nonassociative algebras and bilinear forms. Most of this material can be found in the books of R. Schafer (cf. [**52**]) and N. Jacobson (cf. [**30**], [**31**], [**32**], and [**33**]) and the articles of A. A. Albert (cf. [**1**], [**2**], and [**3**]).

A **non-associative algebra** (or shorter: **algebra**) is a vector space A over some field $\mathbb{K}$ together with a **multiplication**, i.e. a bilinear map $A \times A \to A$ denoted by $(a, b) \mapsto ab$. For each $a \in A$ let $L(a)$ (resp. $R(a)$) denote the **left multiplication** (resp. **right multiplication**) map $A \to A$: $b \mapsto ab$ (resp. $b \mapsto ba$). For two vector subspaces V and W of A let VW denote the $\mathbb{K}$-linear span of all multiplications vw with $v \in V$ and $w \in W$. A vector subspace I is called an **ideal** (resp. a **subalgebra**) iff $AI + IA \subset I$ (resp. $II \subset I$). A **homomorphism** m from an algebra A to an algebra B is a linear map $A \to B$ satisfying $m(aa') = (ma)(ma')\ \forall a, a' \in A$. Clearly, the image of any homomorphism is a subalgebra of B whereas the kernel is an ideal of A. Conversely, given an arbitrary ideal I in an algebra A the factor space A/I carries a well-defined canonical multiplication $(a + I)(a' + I) := aa' + I$ such that the canonical projection is an algebra homomorphism $A \to A/I$. A/I is called the **factor algebra** A **mod** I. Furthermore, a linear map d mapping an algebra A into A (resp. another algebra B) and satisfying the identity $d(aa') = (da)a' + a(da')$ is called a **derivation of** A (resp. a **derivation of** A **in** B).

For any integer $n \geq 1$ let A^n denote the $\mathbb{K}$-linear span of all n-fold multiplications of elements in A no matter how associated, i.e. $A^1 := A$, $A^2 := AA$, $A^3 := (A^2)A + A(A^2)$, $A^4 := A(A^3) + (A^2)(A^2) + (A^3)A$ etc. $(A^n)_{n\geq 1}$ is called the **series of powers of** A and clearly consists of ideals. An algebra is called **abelian** (resp. **perfect**) iff $A^2 = 0$ (resp. $A^2 = A$). A is called **nilpotent** (**of length** k) iff there is a (smallest) integer k such that $A^k = 0$. Subalgebras and factor algebras of nilpotent algebras are again nilpotent. Since for each ideal I of an algebra A the vector space $AI + IA$ is obviously an ideal it is clear that the **central descending series** $(C^n(A))_{n\geq 0}$ which is inductively defined by $C^0(A) := A$, $C^{n+1}(A) := AC^n(A) + C^n(A)A$ consists of ideals. It can be shown by induction that $A^{2^n} \subset C^n(A) \subset A^{n+1}$ (cf. [**52**, p. 19]). For each vector subspace V of an algebra A define the vector space $C(V) := \{a \in A \mid Aa \subset V \text{ and } aA \subset V\}$. It is easy to see that $C(I)$ is an ideal of A if I is an ideal of A. Hence the **central ascending series** $(C_n(A))_{n\geq 0}$ which is inductively defined by $C_0(A) := 0$, $C_{n+1}(A) := C(C_n(A))$ consists of ideals. In particular, the ideal $C_1(A)$ is equal to the subspace $\{a \in A \mid Aa = 0 \text{ and } aA = 0\}$ which is called the **annihilator** $Z(A)$ **of** A. More generally, for any vector subspace V of A the **annihilator** $Z(V)$ **of** V **in** A is defined as the subspace $\{a \in A \mid Va = 0 \text{ and } aV = 0\}$. In general, $Z(V)$ is not a subalgebra even if V is an ideal. For Lie algebras $Z(A)$ is called the **centre of** A. In a similar manner as in the case of Lie algebras (cf. [**30**, p. 29]) and using the above mutual inclusion relation between $(C^n(A))_{n\geq 0}$ and $(A^m)_{m\geq 1}$ it can easily be shown that an algebra A is nilpotent iff there is an integer m such that $C^m(A) = 0$ iff there is an integer m' such that $C_{m'}(A) = 0$. In that case, if m and m' are the smallest such integers they coincide, and for nonzero A the number m is called the **nilindex of** A. For any nilpotent algebra of nilindex m it is easily shown by induction that $C^i(A)$ is contained in $C_{m-i}(A)$ for all $0 \leq i \leq m$. The **derived series** $(D^n(A))_{n\geq 0}$ which is inductively defined by $D^0(A) := A$, $D^{n+1}(A) := (D^n(A))(D^n(A))$ is obviously a series of subalgebras of A with $D^{n+1}(A)$ being an ideal of $D^n(A)$. An algebra A is called **solvable** (**of length** k) iff there is a (smallest) integer k such that $D^k(A) = 0$. Since $D^k(A) \subset A^{2^k}$ every nilpotent algebra is solvable. Subalgebras and factor algebras of solvable algebras are again solvable, moreover, if an ideal I of A is solvable and the factor algebra A/I is solvable then A will be solvable which is shown as in the case of Lie algebras (cf. [**30**, p. 24] or [**52**, p. 18]). Hence the sum of two solvable ideals I and J, $I + J$, will again be solvable. It follows that there is a unique maximal solvable ideal R in each finite-dimensional algebra, the so-called **radical**. An algebra with vanishing radical is called **semisimple**. In particular, the factor algebra A/R is always semisimple. Any nonabelian algebra A all of whose nonzero ideals are equal to A is called **simple**. One could call A **strongly semisimple** (semisimple in the sense of Albert, cf. [**3**]) iff it is isomorphic to a finite direct sum (see the definition further down) of simple algebras. This notion leads to a definition of a different radical, see [**3**].

We shall now mention some well-known classes of algebras: an algebra A is called **commutative** (resp. **anticommutative**) iff for any two elements a and b in A the identity $ab = ba$ (resp. $aa = 0$) holds. We shall speak of an **(anti)commutative** algebra iff it is either commutative or anticommutative. For three elements a, b, c in an algebra the **associator** (a, b, c) is defined by the term $(ab)c - a(bc)$. An algebra A is called **associative** (resp. **alternative** resp. **left symmetric**) iff for all $a, b, c \in A$ the associator (a, b, c) vanishes (resp. iff $(a, a, b) = 0 = (b, a, a)$ resp. iff $(a, b, c) = (b, a, c)$)). Any anticommutative algebra A satisfying the **Jacobi identity** $(ab)c + (ca)b + (bc)a = 0\ \forall a, b, c \in A$ is called a **Lie algebra**. Any commutative algebra over a field $\mathbb{K}$ of characteristic not two satisfying the **Jordan identity** $(a, b, aa) = 0\ \forall a, b \in A$ is called a **Jordan algebra**. Note that for associative, alternative and Lie algebras the multiplication of two ideals I and J, IJ, is again an ideal. Hence the derived series $(D^n(A))_{n\geq 0}$ consists of ideals for these classes. Also, the annihilator of a vector subspace V (resp. an ideal I), $Z(V)$ (resp. $Z(I)$) is a subalgebra (resp. an ideal) of A if A is associative or Lie. Moreover, it is known that any solvable subalgebra of an associative or alternative or Jordan algebra is nilpotent (cf. [**52**, p. 30–32 and p. 95–96]) hence solvability is equivalent to nilpotency in these classes (which however is not the case for Lie algebras). The set of all bijective homomorphisms of an algebra A onto itself is called its **automorphism group** $Aut(A)$. Furthermore, the set of all derivations of A, $\mathrm{Der}(A)$, forms a Lie algebra w. r. t. the commutator $[d, d'] := dd' - d'd\ \forall d, d' \in \mathrm{Der}(A)$ of linear maps. Let $LR(A)$ (resp. $LR(A, 1)$) denote the associative subalgebra (resp. with unity) of the space of all $\mathbb{K}$-linear maps $A \to A$ generated by all left and all right multiplications. It follows that A is nilpotent iff $LR(A)$ is nilpotent. Moreover, a vector subspace I of A is an ideal iff it is invariant under all maps in $LR(A)$ (or $LR(A, 1)$). Let $K(A)$ denote the so-called **commutant of** A, i.e. the set of all $\mathbb{K}$-linear maps $A \to A$ that commute with all left and all right multiplications. $K(A)$ is an associative algebra with unity. Moreover, the intersection of $\mathrm{Der}(A)$ and $K(A)$ consists of the space of all those $\mathbb{K}$-linear maps that map A to $Z(A)$ and A to 0. Also, $\mathrm{Der}(A) \cap K(A)$ contains $[K(A), K(A)]$ (cf. [**30**, p. 290]). Hence, $K(A)$ is commutative in case A is perfect or has vanishing annihilator.

If A and B are two algebras then there is a canonical algebra structure on the **direct** vector space **sum** $A \oplus B$ given by $(a + b)(a' + b') := aa' + bb'\ \forall a, a' \in A;\ \forall b, b' \in B$ such that A and B are ideals of $A \oplus B$. Conversely, call an algebra A **decomposable** iff it is zero or equal to the direct sum of two non-zero ideals, $A = I \oplus J$ and **indecomposable** otherwise. For finite-dimensional algebras there is the

Theorem 4.3 (Decomposition Theorem). *Let A be a finite-dimensional algebra over a field $\mathbb{K}$. Assume that there are two decompositions $A = I_1 \oplus \cdots \oplus I_k \oplus \cdots \oplus I_K$ and $A = J_1 \oplus \cdots \oplus J_m \oplus \cdots \oplus J_M$ of A into direct sums of indecomposable ideals where k, K, m, M are integers s. t. $0 \leq k \leq K$ and $0 \leq m \leq M$ and the ideals I_i (resp. J_j) are non-abelian for $1 \leq i \leq k$ (resp. $1 \leq j \leq m$) and abelian otherwise.*
Then $K = M$ and $k = m$ and there is a permutation $'$ of the set $\{1, 2, \ldots, M\}$ leaving invariant the set $\{1, 2, \ldots, m\}$ such that the canonical projection $p_{j'} : A \to I_{j'}$ restricted to the ideal J_j is an isomorphism of algebras. The induced permutation of $\{1, 2, \ldots, m\}$ is uniquely defined by the condition $J_j \cap I_{j'} \neq 0$. Moreover, all the indecomposable abelian ideals in the above decomposition are one-dimensional and belong to the annihilator Z of A.
Furthermore, one has $J_j + Z = I_{j'} + Z$ and $J_j^2 = I_{j'}^2$ for all $1 \leq j \leq M$. In

particular, if A is perfect or has vanishing annihilator it follows that $m = M$ and the above decomposition is unique up to permutations.

Proof. Sketch of proof (see also [**10**, p. 43–48] for details): Since A is finite-dimensional it is Artinian and Noetherian as a module over the ring $LR(A,1)$. The Wedderburn-Remak-Krull-Schmidt Theorem (cf. [**32**, p. 110–115]) now states that any decomposition of A into indecomposable $LR(A,1)$-submodules (= indecomposable ideals) is unique up to module isomorphisms. More precisely, it is shown that $K = M$ and there is a permutation $'$ of $\{1,2,\ldots,M\}$ such that $A = J_j \oplus I^{(j')}$ where $I^{(j')}$ denotes the ideal $I_1 \oplus \cdots \oplus I_{j'-1} \oplus I_{j'+1} \oplus \cdots \oplus I_M$. Hence J_j is isomorphic to the factor algebra $A/I^{(j')}$ which in turn is obviously isomorphic to $I_{j'}$. Also, the annihilator of $I^{(j')}$, $Z(I^{(j')})$, clearly contains the ideals $I_{j'}$,J_j and Z, hence $I_{j'} \oplus (Z(I^{(j')}) \cap I^{(j')}) = Z(I^{(j')}) = J_j \oplus (Z(I^{(j')}) \cap I^{(j')})$. But on the other hand, the ideal $Z(I^{(j')}) \cap I^{(j')}$ is contained in Z, hence $I_{j'} + Z = J_j + Z$ and by squaring both sides of this equation: $I_{j'}^2 = J_j^2$. Clearly, $I_{j'}$ is nonabelian iff J_j is nonabelian, and in that case: $I_{j'} \cap J_j \neq 0$ which fixes the permutation on the nonabelian ideals. The rest of the theorem now follows easily. $\square$

If A and B are two algebras then there is a canonical algebra structure on the **tensor product** $A \otimes B$ given by $(a \otimes b)(a' \otimes b') := aa' \otimes bb'$ $\forall a, a' \in A$; $\forall b, b' \in B$. If A and B are associative then $A \otimes B$ will again be associative. If A is (anti)commutative, nilpotent, solvable, associative, alternative, Lie or Jordan and B is commutative and associative then $A \otimes B$ will have the same property as A.

Let A be an arbitrary vector space over a field $\mathbb{K}$. A **bilinear form** f on A is defined to be a bilinear map $f\colon A \times A \to \mathbb{K}$. f is called **symmetric** (resp. **antisymmetric**) iff $f(a,b) = f(b,a)$ (resp. $f(a,a) = 0$) $\forall a, b \in A$. For any subspace V of A let $V^\perp$ (resp. ${}^\perp V$) denote the **right orthogonal space** (resp. **left orthogonal space**) of V, i.e. $V^\perp := \{a \in A \mid f(v,a) = 0\ \forall v \in V\}$ (resp. ${}^\perp V := \{a \in A \mid f(a,v) = 0\ \forall v \in V\}$). f is called **non-degenerate** iff $A^\perp = 0 = {}^\perp A$. For two subspaces V and W of A one has the following basic duality relations:

$$V \subset W \quad \text{implies} \quad V^\perp \supset W^\perp \quad \text{and} \quad {}^\perp V \supset {}^\perp W \tag{22}$$

$$(V+W)^\perp = V^\perp \cap W^\perp \quad \text{and} \quad {}^\perp(V+W) = {}^\perp V \cap {}^\perp W \tag{23}$$

$$(V \cap W)^\perp \supset V^\perp + W^\perp \quad \text{and} \quad {}^\perp(V \cap W) \supset {}^\perp V + {}^\perp W \tag{24}$$

which are immediate consequences of the definition. Moreover, if A is **finite-dimensional** the following inversion and dimension formulae hold:

$${}^\perp(V^\perp) = V + {}^\perp A \quad \text{and} \quad ({}^\perp V)^\perp = V + A^\perp \tag{25}$$

$$\dim V^\perp = \dim A - \dim V + \dim(V \cap {}^\perp A) \tag{26}$$

$$\dim {}^\perp V = \dim A - \dim V + \dim(V \cap A^\perp) \tag{27}$$

$$\dim(V \cap V^\perp) = \dim(V \cap {}^\perp V) \tag{28}$$

For the special case of a symmetric nondegenerate bilinear form q these relations are well-known since all left and right orthogonal spaces coincide and $A^\perp = 0 = {}^\perp A$. The general case is proved by expressing the bilinear form f as $f(a,b) = q(Fa,b)$ where $a,b \in A$ and F is a uniquely determined linear endomorphism of A and using the familiar kernel-image-dimension formulae (see [**10**, p. 12–13] for details). For infinite-dimensional A relations (25) do no longer hold in general but have to be replaced by the weaker inclusions ${}^\perp(V^\perp) \supset V + {}^\perp A$ and $({}^\perp V)^\perp \supset V + A^\perp$. It may e.g. happen that proper subspaces can have zero orthogonal spaces.

Let f (resp. g) be a bilinear form on a vector space A (resp. B) over a field $\mathbb{K}$. Then there is a canonical bilinear form $f \perp g$ (resp. $f \otimes g$) on the direct sum $A \oplus B$ (resp. on the tensor product $A \otimes B$) of A and B defined by $f \perp g(a+b, a'+b') := f(a,a') + g(b,b')$ (resp. $f \otimes g(a \otimes b, a' \otimes b') := f(a,a')g(b,b')$) for all $a, a' \in A$ and $b, b' \in B$. Moreover, $f \perp g$ (resp. $f \otimes g$) is **nondegenerate if and only if f and g are nondegenerate**: This is easily checked for $f \perp g$. For $f \otimes g$ note that e.g. $a \in A^\perp$ implies that $a \otimes b \in (A \otimes B)^\perp$ for any $b \in B$ proving sufficiency. Conversely, choose a basis (a_i) in A and (b_j) in B: Then $\sum \alpha^{ij} a_i \otimes b_j \in {}^\perp(A \otimes B)$ with $\alpha^{ij} \in \mathbb{K}$ implies for all $a \in A$ and for all $b \in B$: $0 = f \otimes g(\sum \alpha^{ij} a_i \otimes b_j, a \otimes b) = \sum \alpha^{ij} f(a_i, a) g(b_j, b) = f(\sum \alpha^{ij} g(b_j, b) a_i, a)$ which implies by nondegeneracy of f for all $b \in B$: $0 = \sum \alpha^{ij} g(b_j, b) = g(\sum \alpha^{ij} b_j, b)$ which implies $\alpha^{ij} = 0$ by nondegeneracy of g. The case $\sum \alpha^{ij} a_i \otimes b_j \in (A \otimes B)^\perp$ is treated in an analogous manner.

A subspace V of A is called **nondegenerate (with respect to f)** iff $V \cap V^\perp = 0$ and $V \cap {}^\perp V = 0$. For finite-dimensional A these two conditions are equivalent because of equation (28). Moreover, the dimension formulae (26) and (27) imply that $A = V \oplus V^\perp$ and $A = V \oplus {}^\perp V$ for each nondegenerate subspace V (which is false in general if A is infinite-dimensional). Furthermore, it is easy to check that f is nondegenerate if and only if both the restriction of f to $V \times V$ and $V^\perp \times V^\perp$ are nondegenerate.

The intersection of ${}^\perp A$ and $A^\perp$ is called the **kernel N_f of f**. If A' is another vector space over the field $\mathbb{K}$ and $m \colon A' \to A$ is a linear map then the **pull-back** $m^* f$ is defined to be the bilinear form $(a'_1, a'_2) \mapsto f(ma'_1, ma'_1)$ for all $a'_1, a'_2 \in A'$. The kernel N_{m^*f} of m^*f contains the kernel of m. For surjective m it follows that $N_{m^*f} = m^{-1} N_f$. Now suppose that g is a bilinear form on A', m is surjective, and $\operatorname{Ker} m \subset N_f$. Then the **projection g^m of g** given by $g^m(ma'_1, ma'_2) := g(a'_1, a'_2)$ is a well-defined bilinear form on A whose kernel N_{g^m} equals $mN_g \cong N_g/\operatorname{Ker} m$. Let q be a nondegenerate symmetric bilinear form on a vector space A. A subspace V is called **isotropic** iff $V \subset V^\perp$. If A has finite dimension n the dimension of a maximally isotropic subspace is an invariant of (A,q) known as the **Witt index of (A,q)**. If the field $\mathbb{K}$ has characteristic not equal to 2 and is algebraically closed the Witt index of any (A,q) is equal to $[n/2]$, i.e. the integer part of $n/2$. Let A

be finite-dimensional and ϕ be a linear endomorphism of A. The **q-transpose of** ϕ, ϕ^+, is uniquely defined by $q(\phi^+ a, b) := q(a, \phi b)$ $\forall a, b \in A$. Clearly, $(\phi^+)^+ = \phi$.

Appendix B: A Note on the Third Scalar Cohomology of the Filiform Lie Algebras

The family of filiform Lie algebras $\mathcal{L}_n$, $n \geq 2$, has been defined at the end of Section 4, and we use the definitions and notations for Lie algebra cohomology as has been sketched in Section 3. Let V denote the vector subspace of $\mathcal{L}_n$ spanned by $e_1, \ldots, e_n$ and let ϕ be the linear endomorphism of V defined by $\phi e_i := [e_0, e_i]$ $(1 \leq i \leq n)$. Clearly, V is an abelian ideal of $\mathcal{L}_n$ and ϕ is nilpotent. Let π_2 denote the canonical projection of $\mathcal{L}_n$ onto V along e_0. Then every alternating k-form $f \in C^k(\mathcal{L}_n, \mathbb{K})$, $k \geq 1$, allows for the decomposition $f = e_0 \wedge \pi_2^* g + \pi_2^* f'$ where g is in $C^{k-1}(V, \mathbb{K})$ and f' is equal to the restriction of f to V in each argument, i.e. $f' \in C^k(V, \mathbb{K})$: this can easily be checked by evaluating f on k vectors in $\mathcal{L}_n$ each having the form $\lambda e_0 + v$ where $\lambda \in \mathbb{K}$ and $v \in V$. Likewise, we have the decomposition $\delta f = e^0 \wedge \pi_2^* h + \pi_2^* (\delta f)'$ where h is contained in $C^k(V, \mathbb{K})$. Since the coboundary operator δ involves Lie brackets and V is abelian we can conclude that $\pi_2^*(\delta f)'$ vanishes. Moreover for k elements $v_1, \ldots, v_k$ in V we have:

$$
\begin{aligned}
h(v_1, \ldots, v_k) &= e^0(e_0) h(v_1, \ldots, v_k) = (\delta f)(e_0, v_1, \ldots, v_k) \\
&= \sum_{i=1}^{k} (-1)^i f(\phi v_i, v_1, \ldots, \hat{v_i}, \ldots, v_k) \\
&= -\sum_{i=1}^{k} f(v_1, \ldots, \phi v_i, \ldots, v_k) \\
&= (\phi f')(v_1, \ldots, v_k)
\end{aligned}
\tag{29}
$$

where ϕ also denotes the natural extension of a linear endomorphism to the tensor algebra of V as a derivation on tensor products. Hence it follows that a k-cochain f is a cocycle iff its restriction to V is in $C^k(V, \mathbb{K})^\phi$ by which we denote the kernel of ϕ in $C^k(V, \mathbb{K})$. Therefore we get the following characterization of cocycles and coboundaries of $\mathcal{L}_n$ (where $C^0(V, \mathbb{K}) = \mathbb{K}$ and $C^k(V, \mathbb{K}) = 0$ for $k < 0$):

$$
Z^k(\mathcal{L}_n, \mathbb{K}) = e^0 \wedge \pi_2^*(C^{k-1}(V, \mathbb{K})) \oplus \pi_2^*(C^k(V, \mathbb{K})^\phi), \tag{30}
$$

$$
B^k(\mathcal{L}_n, \mathbb{K}) = e^0 \wedge \pi_2^*(\phi C^{k-1}(V, \mathbb{K})). \tag{31}
$$

Using the kernel-image-dimension-formula for the map ϕ we have the following formula for the dimension of the cohomology groups $H^k(\mathcal{L}_n, \mathbb{K})$:

$$
\dim H^k(\mathcal{L}_n, \mathbb{K}) = \dim(C^k(V, \mathbb{K})^\phi) + \dim(C^{k-1}(V, \mathbb{K})^\phi). \tag{32}
$$

The direct computation of (the dimension of) the kernel $C^k(V,\mathbb{K})^\phi$ turns out to be quite difficult. However, if we assume $\mathbb{K}$ to be algebraically closed and of characteristic zero we can make use of the following trick:
Recall that the induced action of ϕ on the dual space V is given in the basis $e^0, e^1, \ldots, e^n$ dual to the one given above reads $\phi e^i = -e^{i-1}$ $(2 \leq i \leq n)$ and $\phi e^1 = 0$. Now rescale ϕ and this basis in the following way: set $x := -\phi$ and $f_i := ((n-i)!)^{-1} e^i$ $(1 \leq i \leq n)$ and define two other matrices h and y in the following manner:

$$
\begin{aligned}
hf_i &:= (n+1-2i)f_i \qquad (1 \leq i \leq n),\\
yf_i &:= if_{i+1} \qquad (1 \leq i \leq n-1),\\
yf_n &:= 0
\end{aligned}
\tag{33}
$$

and clearly:

$$
\begin{aligned}
xf_i &= (n+1-i)f_{i-1} \qquad (2 \leq i \leq n),\\
xf_1 &= 0.
\end{aligned}
$$

The three matrices h, x, y span a $sl(2,\mathbb{K})$-Lie algebra which is irreducibly represented in V (cf. e.g. [**28**, p. 32], the highest weight λ being $n-1$ and his vectors v_i equalling our f_{i+1}). Moreover, every irreducible representation of $sl(2,\mathbb{K})$ in an n-dimensional vector space allows for a basis $\{f_1, \ldots, f_n\}$ such that the above equations hold (cf. e.g. [**28**, p. 33]). In particular, the kernel of x in each irreducible representation of $sl(2,\mathbb{K})$ is one-dimensional. In order to compute the dimension of the kernel of $\phi = -x$ acting on $C^k(V,\mathbb{K}) = V^* \wedge V^* \wedge \ldots \wedge V^*$ (k factors, $0 \leq k \leq n$) we can use Weyl's Theorem stating that the induced $sl(2,\mathbb{K})$-representation on $C^k(V,\mathbb{K})$ is completely reducible (cf. e.g. [**28**, p. 28]). It follows that the kernel of ϕ decomposes into the direct sum of the kernels of ϕ in the irreducible constituents. Consequently:

$$
\begin{aligned}
\dim(C^k(V,\mathbb{K}))^\phi = {} & \text{number of irreducible } sl(2,\mathbb{K})\\
& - \text{submodules in } C^k(V,\mathbb{K}).
\end{aligned}
\tag{34}
$$

This number of irreducible submodules is in turn equal to the dimension of the zero-eigenspace plus the dimension of the one-eigenspace of the map h (cf. e.g [**28**, p. 33]). The sum of these dimensions can much simpler be calculated than the kernel of ϕ directly since the basis k-forms $f_{i_1} \wedge \cdots \wedge f_{i_k}$ $(1 \leq i_1 < \cdots < i_k \leq n)$ are eigenvectors of h: $h1 = 0$ and $hf_{i_1} \wedge \cdots \wedge f_{i_k} = (k(n+1) - 2(i_1 + \cdots + i_k))f_{i_1} \wedge \cdots \wedge f_{i_k}$ for $1 \leq k \leq n$. Denoting by $[\alpha]$ the integer part of a real number α we find that

$$
\begin{aligned}
\dim(C^k(V,\mathbb{K}))^\phi = \#\Big\{ & \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} \mid i_1 + \cdots + i_k\\
& = [(k(n+1)/2]\Big\}.
\end{aligned}
\tag{35}
$$

For small k this combinatorial problem can be solved without much effort to give the first four cohomology groups of $\mathcal{L}_n$:

$$\dim H^0(\mathcal{L}_n,\mathbb{K}) = 1,\ \dim H^1(\mathcal{L}_n,\mathbb{K}) = 2,\ \dim H^2(\mathcal{L}_n,\mathbb{K}) = [n/2]+1 \tag{36}$$

and

$$\dim H^3(\mathcal{L}_n,\mathbb{K}) = \binom{[n/2]+1}{2} \qquad \text{if } n \text{ is even,} \tag{37}$$

$$\dim H^3(\mathcal{L}_n,\mathbb{K}) = \binom{[n/2]+1}{2} + \left[\frac{[n/2]+1}{2}\right] \qquad \text{if } n \text{ is odd.} \tag{38}$$

Proof. Sketch of proof: the cases $k = 0, 1, 2$ are straight forward using eqs. (35) and (32). For the case $k = 3$ note that any triple $\{j_1, j_2, j_3\} \subset \{1, \dots, n-2\}$ satisfying $j_1+j_2+j_3 = [3(n-1)/2]$ can uniquely be shifted to a triple $\{j_1+1, j_2+1, j_3+1\} \subset \{2, \dots, n-1\}$ satisfying $j_1+1+j_2+1+j_3+1 = [3(n-1)/2]+3 = [3(n+1)/2]$. Hence it follows by eqn (35) that $\dim(C^3(V,\mathbb{K}))^\phi$ for $\mathcal{L}_n$ is equal to the sum of $\dim(C^3(V,\mathbb{K}))^\phi$ for $\mathcal{L}_{n-2}$ plus the number of "extreme" triples of the form $\{1, i_1, i_2 < n\}$, $\{1 < i_1, i_2, n\}$, and $\{1, i_2, n\}$. Since there are at most two varying indices in the latter triples their number can quickly be calculated leaving a recursion formula for the dimension $\dim(C^3(V,\mathbb{K}))^\phi$ which can be solved. □

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