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FURTHER RESULTS ON VERTEX COVERING OF POWERS OF COMPLETE GRAPHS

S. Y. ALSARDARY

Abstract. A snake in a graph G is defined to be a closed path in G without proper chords. Let K_n^d be the product of d copies of the complete graph K_n. Wojciechowski [**13**] proved that for any $d \geq 2$ the hypercube K_2^d can be vertex covered with at most 16 disjoint snakes. Alsardary [**6**] proved that for any odd integer $n \geq 3, d \geq 2$ the graph K_n^d can be vertex covered with $2n^3$ snakes. We show that for any even integer $n \geq 4$, $d \geq 2$ the graph K_n^d, can be vertex covered with n^3 snakes.

1. Introduction

Throughout this paper we consider only finite, undirected, simple graphs. We define a **path** in a graph G to be a sequence of distinct vertices of G with every pair of consecutive vertices being adjacent. A **closed** path is a path whose first vertex is adjacent to the last one. A **chord** of a path P in a graph G is an edge of G joining two nonconsecutive vertices of P. If e is a chord in a closed path P, then e is called **proper** if it is not the edge joining the first vertex of P to its last vertex. Note that a proper chord of a closed path corresponds to the standard notion of a chord in a cycle. A **snake** in a graph G is a closed path in G without proper chords, and an **open** snake is a path without chords. For people interested in trees, the term **snake** means a tree with no vertex of degree more than two. In graph theory, objects which the author calls **snakes** are known as **induced cycles**.

The (cartesian) **product** of two graphs G and H is the graph $G \times H$ with the vertex set $V(G) \times V(H)$ and the edge set defined in the following way: (g_1, h_1) is adjacent to (g_2, h_2) if either $g_1g_2 \in E(G)$ and $h_1 = h_2$, or else $g_1 = g_2$ and $h_1h_2 \in E(H)$. Let K_n^d be the product of d copies of the complete graph K_n, $n \geq 2, d \geq 1$. It is convenient to think of the vertices of K_n^d as d-tuples of n-ary digits, *i.e.*, the elements of the set $\{0, 1, \ldots, n-1\}$, with edges between any two d-tuples differing at exactly one coordinate.

Let $S(K_n^d)$ be the length of the longest snake in K_n^d. The problem of estimating the value of $S(K_n^d)$ has a long history. It was first met by Kautz [**9**] in 1958 in the case $n = 2$ (known in the literature as the snake-in-the-box problem) in

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constructing a type of error-checking code for a certain analog-to-digital conversion systems. As a consequence several authors became interested in estimating the value of $S(K_2^d)$.

Computing the exact values of $S(K_2^d)$ for all d, seems hopeless. In fact, at the present time only the first five values are known. They are $4, 6, 8, 14$ and 26 (for $d = 2, 3, 4, 5$ and 6, respectively); the value of K_2^7 is unknown.

Evdokimov [**8**] in 1969 was the first to prove that

$$S(K_2^d) \geq c2^d, \tag{1}$$

for some constant $c > 0$. Other shorter proofs were given by Abbott and Katchalski [**3**] and Wojciechowski [**12**]. The best value of c in (1) was given by Abbott and Katchalski [**5**] who proved the following result.

Theorem 1.1. *The following inequality*

$$S(K_2^d) \geq \left(\frac{77}{256}\right)2^d,$$

holds for all d.

Diemer [**7**] established the following upper bound on $S(K_2^d)$:

$$S(K_2^d) \leq 2^{d-1} - \frac{2^{d-1}}{d(d-5)+7},$$

for all $d \geq 7$, and Solov'jeva [**10**] proved that

$$S(K_2^d) \leq 2^{d-1} - \frac{2^d}{d^2 - d + 2},$$

for all $d \geq 7$. Snevily [**11**] improved these upper bounds by showing the following result which is the best upper bound so far.

Theorem 1.2. *The following inequality*

$$S(K_2^d) \leq 2^{d-1} - \frac{2^{d-1}}{20d - 41},$$

holds for all $d \geq 12$.

The general case of the problem (i.e., estimating $S(K_n^d)$ with an arbitrary value of n) was introduced by Abbott and Dierker [**1**] in 1977, and developed further by Abbott and Katchalski [**2**], [**4**] and Wojciechowski [**14**]. The following generalization of (1) is a result of these investigations.

Theorem 1.3. *For any integer $n \geq 2$, there is a constant $c_n > 0$ such that*

$$S(K_n^d) \geq c_n n^d,$$

for any $d \geq 2$.

In the case of n being even, the above theorem has been proved by Abbott and Katchalski [**4**]. They proved the following result.

Theorem 1.4. *There is a constant $\alpha > 0$ such that if $n \geq 2$ is an even integer, then*

$$S(K_n^d) \geq \alpha(n/2)^{d-1} S(K_2^{d-1}),$$

for any $d \geq 2$.

Theorem 1.4 together with Theorem 1.1 imply that Theorem 1.3 holds for even integer $n \geq 2$.

In the case of n being odd, Theorem 1.3 was proved by Wojciechowski [**14**]. Actually he showed the following stronger result.

Theorem 1.5. *If $n \geq 3$ is an odd integer, then*

$$S(K_n^d) \geq 2(n-1)n^{d-4},$$

for any $d \geq 5$.

Note that in Theorem 1.3 the constant c_n depends on n. The following result of Abbott and Katchalski [**4**] shows that it cannot be made independent of n.

Theorem 1.6. *For any $n \geq 2$ and $d \geq 2$*

$$S(K_n^d) \leq \left(1 + \frac{1}{d-1}\right) n^{d-1}.$$

However, the following conjecture has been proposed by Wojciechowski [**15**].

Conjecture 1.7. *There is a constant $c > 0$ such that*

$$S(K_n^d) \geq c n^{d-1},$$

for any $n \geq 2$, $d \geq 1$.

During the XXIII Southeastern International Conference, Boca Raton 1992, Erds posed the problem of deciding whether there is a number k such that for every $d \geq 2$ the vertices of K_2^d can be covered using at most k snakes, and if the answer to the above question is positive, then whether it can be done in such a way that the snakes are pairwise vertex-disjoint. Wojciechowski [**13**] proved the following stronger result.

Theorem 1.8. *For every $d \geq 2$, there is a subgroup $\mathcal{H}_d \subset K_2^d$ and a snake $C_d \subset K_2^d$ such that $|\mathcal{H}_d| \leq 16$ and C_d uses exactly one element of every coset of $\mathcal{H}_d$, where the group structure of K_2^d is of the product $(\mathbf{Z}_2)^d$.*

Theorem 1.8 implies that both questions posed by Erds have positive answers.

Conjecture 1.9. *For any $d \geq 2$ the vertices of K_2^d can be covered with at most 16 vertex-disjoint snakes.*

Wojciechowski [**15**] conjectured the following generalization of Corollary 1.9.

Conjecture 1.10. *For any integer $n \geq 2$, there is an integer r_n such that the graph K_n^d can be vertex-covered with at most r_n vertex-disjoint snakes for any $d \geq 2$.*

Alsardary [**6**] proved the following result.

Theorem 1.11. *Let $n \geq 3$ be an odd integer and $r_n = 2n^3$. For any $d \geq 2$ the vertices of K_n^d can be covered with r_n snakes.*

In this paper we prove the following result that is a weaker version of the Conjecture 1.10. It settle the part of Conjecture 1.10 that corresponds to the first question posed by Erds.

Theorem 1.12. *Let $n \geq 4$ be an even integer and $r_n = n^3$. For any $d \geq 2$ the vertices of K_n^d can be covered with r_n snakes.*

Finally we outline the organization of this paper. In Sections 2 and 3 we present the preliminary definitions and results. In Section 4 we prove Theorem 1.12. In Section 5 we give a conclusion.

2. Preliminaries

In this section we give our basic definitions and prove some preliminary results.

We define an m-**path** in a graph G to be a path containing m vertices, *i.e.*, a path of length $m-1$. If P is an m-path, then we will write $m = |P|$. A **chain** $\mathcal{P}$ of paths in a graph G is a sequence $(P_1, P_2, \ldots, P_m)$ of paths in G such that each path in $\mathcal{P}$ has at least two vertices, and the last vertex of P_i is equal to the first vertex of P_{i+1}, where $1 \leq i \leq m-1$. When we need to specify the number m of paths in a chain, we refer to it as an m-**chain** of paths. An m-chain $\mathcal{P} = (P_i)_{i=1}^m$ of paths will be called **closed** if the first vertex of P_1 is equal to the last vertex of P_m.

Given an m-path $P = (a_i)_{i=1}^m$ in a graph G and an m-chain of paths $\mathcal{L} = (P_i)_{i=1}^m$ in a graph H, let $P \otimes \mathcal{L}$ be the $(\sum_{i=1}^m |P_i|)$-path in the graph $G \times H$ constructed in the following way. For any path

$$P_i = (b_{i1}, b_{i2}, \ldots, b_{i_{k_i}})$$

in $\mathcal{L}$, let P_i' be the path

$$((a_i, b_{i1}), (a_i, b_{i2}), \dots, (a_i, b_{ik_i}))$$

in $G \times H$. Note that for any $1 \leq i \leq m-1$, the last vertex of the path P_i' is adjacent to the first vertex of the path P_{i+1}'. Let $P \otimes \mathcal{L}$ be the path obtained by joining together (juxtaposing) the paths $P_1', P_2', \cdots, P_m'$. We will say that $P \otimes \mathcal{L}$ is the path **generated** by P and $\mathcal{L}$. Note that the path generated by a closed path and a closed chain of paths is a closed path.

If $\mathcal{R}$ is an sm-chain of paths in a graph H, then the m-**splitting** of $\mathcal{R}$ is the sequence $(\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_m)$ of s-chains of paths in H which joined together (juxtaposed) give $\mathcal{R}$. The above definition of the operation $\otimes$ can be generalized in the following way. Let $\mathcal{L} = (P_i)_{i=1}^m$ be an m-chain of s-paths in a graph G, let $\mathcal{R}$ be an sm-chain of paths in H, and let $(\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_m)$ be the m-splitting of $\mathcal{R}$. Note that for any $1 \leq i \leq m-1$, the last vertex of the path $P_i \otimes \mathcal{R}_i$ in the graph $G \times H$ is equal to the first vertex of the path $P_{i+1} \otimes \mathcal{R}_{i+1}$. Define $\mathcal{L} \otimes \mathcal{R}$ to be the chain of paths given by

$$\mathcal{L} \otimes \mathcal{R} = (P_1 \otimes \mathcal{R}_1, P_2 \otimes \mathcal{R}_2, \dots, P_m \otimes \mathcal{R}_m).$$

We will say that $\mathcal{L} \otimes \mathcal{R}$ is the chain of paths **generated** by $\mathcal{L}$ and $\mathcal{R}$. Note that the chain of paths generated by two closed chains of paths is also a closed chain of paths.

Let $\mathcal{L} = (P_i)_{i=1}^m$ be a chain of paths in a graph G. We say that $\mathcal{L}$ is **openly separated** if for $i \leq m-1$ and $j = i+1$, the paths P_i and P_j have exactly one vertex in common, and otherwise P_i and P_j are vertex disjoint. We say that $\mathcal{L}$ is **closely separated** if $\mathcal{L}$ is closed, the paths P_i and P_j have exactly one vertex in common when either $i \leq m-1$ and $j = i+1$, or $i = 1$ and $j = m$ and otherwise P_i and P_j are vertex disjoint.

If P is a path, then let $-P$ be the path obtained from P by reversing the order of vertices, and if $\mathcal{L} = (P_i)_{i=1}^m$ is a chain of paths, then let

$$-\mathcal{L} = (-P_m, -P_{m-1}, \dots, -P_1)$$

be the chain of paths obtained from $\mathcal{L}$ by reversing the order of paths and reversing every path. The expression $(-1)^i X$, where X is a path or a chain of paths, will mean X for i even and $-X$ for i odd.

Let $\mathcal{L}$ be an sm-chain of paths, and let

$$\mathcal{R} = (\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_m)$$

be the m-splitting of $\mathcal{L}$. The **alternate matrix** of the splitting $\mathcal{R}$ is the following $(m \times s)$-matrix $\mathcal{A}$ of paths:

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_1 \\ -\mathcal{L}_2 \\ \vdots \\ (-1)^{m-1}\mathcal{L}_m \end{pmatrix} = \begin{pmatrix} Q_1^1 & Q_1^2 & \cdots & Q_1^s \\ Q_2^1 & Q_2^2 & \cdots & Q_2^s \\ \vdots & \vdots & & \vdots \\ Q_m^1 & Q_m^2 & \dots & Q_m^s \end{pmatrix}$$

where $(Q_i^1, Q_i^2, \dots, Q_i^s)$ is the sequence of paths forming the s-chain $(-1)^{i-1}\mathcal{L}_i$. The splitting $\mathcal{R}$ will be called **openly alternating** if the paths Q_j^s and Q_{j+1}^s have exactly one vertex in common for every odd j, $1 \leq j \leq m-1$, the paths Q_j^1 and Q_{j+1}^1 have exactly one vertex in common for every even j, $2 \leq j \leq m-1$, and otherwise the paths Q_j^i and Q_l^i are vertex disjoint, $1 \leq i \leq s$, $1 \leq j, l \leq m$, $j \neq l$. Note that the splitting $\mathcal{R}$ is openly alternating if for every column of its alternate matrix $\mathcal{A}$ the paths in the column are mutually vertex disjoint except for the shared vertices which are necessary for $\mathcal{L}$ to be a chain of paths, *i.e.* Q_1^s and Q_2^s have exactly one vertex in common, Q_2^1 and Q_3^1 have exactly one vertex in common, and so on.

Assume that the sm-chain $\mathcal{L}$ is a closed chain of paths and m is even. Then, the splitting $\mathcal{R}$ is **closely alternating** if the paths Q_j^s and Q_{j+1}^s have exactly one vertex in common for each odd j, $1 \leq j \leq m-1$, the paths Q_j^1 and Q_{j+1}^1 have exactly one vertex in common for each even j, $2 \leq j \leq m-1$, the paths Q_1^1 and Q_m^1 have exactly one vertex in common, and otherwise the paths Q_j^i and Q_l^i are vertex disjoint, $1 \leq i \leq s$, $1 \leq j, l \leq m$, $j \neq l$. Note that the splitting $\mathcal{R}$ is closely alternating if for every column of its alternate matrix $\mathcal{A}$ the paths in the column are mutually vertex disjoint except for the shared vertices which are necessary for $\mathcal{L}$ to be a closed chain of paths.

Let H be a graph, $d \geq 1$ be an integer, $\mathcal{L}$ be an n^d-chain of paths in H. We define that $\mathcal{L}$ is **openly well distributed** if either $d = 1$ and $\mathcal{L}$ is an openly separated chain of open snakes, or $d \geq 2$, every chain $\mathcal{L}_i$ in the n-splitting

$$S = (\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n)$$

of $\mathcal{L}$ is openly well distributed and S is openly alternating.

We are going now to define the notion of a closely well distributed chain of paths. We need to consider the two cases of n being even and n being odd separately. Assume first that n is even and $\mathcal{D}$ is an n^{d+1}-chain of paths in H. We say that $\mathcal{D}$ is **closely well distributed** if every chain $\mathcal{D}_i$ in the n-splitting

$$S = (\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n)$$

of $\mathcal{D}$ is openly well distributed and S is closely alternating. In the case when n is odd we assume that $\mathcal{D}$ is an $(n-1)n^d$-chain of paths in H and say that $\mathcal{D}$ is **closely well distributed** if every chain $\mathcal{D}_i$ in the $(n-1)$-splitting

$$S = (\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_{n-1})$$

of $\mathcal{D}$ is openly well distributed and S is closely alternating.

Let $\mathcal{L}$ be a chain of paths in a graph G. We define that $\mathcal{L}$ **joins** u_1 to u_2 if u_1 is the first vertex of the first path of $\mathcal{L}$ and u_2 is the last vertex of the last path. If

v is the first or is the last vertex of a path P, then let $P - v$ be the path obtained from P by removing v. Let $\mathcal{L} = (P_i)_{i=1}^m$ be a chain of paths joining u_1 to u_2, and $\mathcal{H} = (Q_i)_{i=1}^m$ be a chain of paths joining v_1 to v_2. We define that $\mathcal{L}$ and $\mathcal{H}$ are **parallel** if for each $i, 1 \leq i \leq m$, the paths P_i and Q_i are vertex disjoint. We also say that $\mathcal{L}$ and $\mathcal{H}$ are **internally parallel** if the following conditions are satisfied:

(i) the paths $P_1 - u_1$ and Q_1 are vertex disjoint,
(ii) the paths P_1 and $Q_1 - v_1$ are vertex disjoint,
(iii) the paths $P_m - u_2$ and Q_m are vertex disjoint,
(iv) the paths P_m and $Q_m - v_2$ are vertex disjoint, and
(v) for each $i, 1 < i < m$, the paths P_i and Q_i are vertex disjoint.

For any integers $n \geq 2$ and $d \geq 1$, we define, by induction on d, the n^d-path π_n^d in K_n^d. Let π_n^1 be the n-path $(0, 1, \ldots, n-1)$ in K_n. If $d \geq 1$ and the path π_n^d in K_n^d is defined, then let

$$\pi_n^{d+1} = \pi_n^1 \otimes (\pi_n^d, -\pi_n^d, \pi_n^d, -\pi_n^d, \ldots, (-1)^{n-1}\pi_n^d).$$

In the following lemma we show that π_n^d is either a Hamiltonian path or a Hamiltonian cycle in K_n^d.

Lemma 2.1. *If v is a vertex of K_n^d, then v is a vertex of the path π_n^d.*

Proof. We are going to use induction with respect to d. For $d = 1$, the lemma is true since π_n^1 is the n-path $(0, 1, \ldots, n-1)$ in K_n. Assume that $d \geq 1$ and that the lemma is true for d, we show that it is true for $d+1$. Let $v = (a_1, a_2, \ldots, a_d, a_{d+1})$ be a vertex of K_n^{d+1}. By the inductive hypothesis, $(a_2, a_3, \ldots, a_{d+1})$ is a vertex of π_n^d. Since

$$\pi_n^{d+1} = \pi_n^1 \otimes (\pi_n^d, -\pi_n^d, \pi_n^d, -\pi_n^d, \ldots, (-1)^{n-1}\pi_n^d),$$

is a path in $K_n \times K_n^d = K_n^{d+1}$ and since a_1 is a vertex of π_n^1, it follows that $v = (a_1, a_2, \ldots, a_d, a_{d+1})$ is a vertex of π_n^{d+1}. $\square$

3. Permuting the Vertices of K_n^3

To prove Theorems 1.12 we need to construct many snakes in K_n^d. The idea of the construction is to build one snake and then get the rest of them by suitable permutations of the vertices of K_n^d. To construct the first snake, following the technique of Wojciechowski [**14**], we will combine a closed path in K_n^{d-3} with a closely well-distributed chain of paths in K_n^3. The permutations that are necessary to get the remaining snakes will be actually performed on the vertices of the chain of paths in K_n^3. In this section we will define a suitable class of permutations of the vertices of K_n^3 and present the results that will be needed later.

Let

$$\alpha : \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, n-1\}$$

be a function defined by

$$\alpha(i) = i + 1 \bmod n.$$

Let $x = (a_1, a_2, a_3) \in V(K_n^3)$, where $a_1, a_2, a_3 \in \{0, 1, \ldots, n-1\}$. Let

$$\sigma, \tau, \delta : V(K_n^3) \to V(K_n^3)$$

be permutations such that

$$\sigma(a_1, a_2, a_3) = (\alpha(a_1), a_2, a_3),$$
$$\tau(a_1, a_2, a_3) = (a_1, \alpha(a_2), a_3),$$

and

$$\delta(a_1, a_2, a_3) = (a_1, a_2, \alpha(a_3)).$$

Let Σ be the set of all permutations

$$f : V(K_n^3) \to V(K_n^3)$$

such that $f = \sigma^i \tau^j \delta^k$ for some $i, j, k \in \{0, 1, \ldots, n-1\}$.

Lemma 3.1. *For any $x, y \in V(K_n^3)$, there is $f \in \Sigma$ with $y = f(x)$.*

Proof. Assume that $x = (x_1, y_1, z_1)$, $y = (x_2, y_2, z_2)$ are two vertices of K_n^3. One can easily verify that $f(x_1, y_1, z_1) = (x_2, y_2, z_2)$ if

$$f = \sigma^{(x_2 - x_1) \bmod n} \tau^{(y_2 - y_1) \bmod n} \delta^{(z_2 - z_1) \bmod n}. \qquad \square$$

Let $f \in \Sigma$. Given a path $P = (u_1, u_2, \ldots, u_r)$ in K_n^3, let $f(P)$ be the path $(f(u_1), f(u_2), \ldots, f(u_r))$. Given a chain of paths

$$\mathcal{C} = (P_1, P_2, \ldots, P_s),$$

let $f(\mathcal{C})$ be the chain of paths

$$(f(P_1), f(P_2), \ldots, f(P_s)).$$

It is clear that each $f \in \Sigma$ is a bijection. In the following lemma, we show that every $f \in \Sigma$ is an isomorphism of K_n^3.

Lemma 3.2. *Let $f \in \Sigma$ and $u, v \in V(K_n^3)$. Then u and v are adjacent in K_n^3 if and only if $f(u)$ and $f(v)$ are adjacent in K_n^3.*

Proof. Let $f \in \Sigma$ and $u, v \in V(K_n^3)$. Assume that $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3)$ are adjacent in K_n^3, then u and v differ at exactly one position. Since α is a bijective function it follows that

$$f(u) = (f(a_1), f(a_2), f(a_3)),$$

and

$$f(v) = (f(b_1), f(b_2), f(b_3)),$$

are differing in exactly one position. Hence $f(u)$ and $f(v)$ are adjacent. Conversely, if $f(u)$ and $f(v)$ are adjacent in K_n^3, then similarly as above we show that u and v are adjacent. □

The remaining lemmas in this section follow from the fact that each $f \in \Sigma$ is an isomorphism and that the corresponding properties are preserved by isomorphism. We omit the proofs in the cases when the property involved is simple.

Lemma 3.3. *If P is an open snake in K_n^3 and $f \in \Sigma$, then $f(P)$ is also an open snake in K_n^3.*

Lemma 3.4. *If $\mathcal{C}$ is an openly separated chain of paths in K_n^3 and $f \in \Sigma$, then the chain $f(\mathcal{C})$ is also openly separated.*

Lemma 3.5. *If $f \in \Sigma$ and P is a path in K_n^3, then $f(-P) = -f(P)$.*

Lemma 3.6. *If $f \in \Sigma$ and $\mathcal{C}$ is a chain of paths in K_n^3, then $f(-\mathcal{C}) = -f(\mathcal{C})$.*

Lemma 3.7. *Let $\mathcal{L}$ be an sm-chain of paths in K_n^3, and let*

$$\mathcal{R} = (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m)$$

be the m-splitting of $\mathcal{L}$. If $\mathcal{R}$ is openly alternating and $f \in \Sigma$, then $f(\mathcal{R})$ is also openly alternating.

Proof. Let

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_1 \\ -\mathcal{L}_2 \\ \vdots \\ (-1)^{m-1}\mathcal{L}_m \end{pmatrix} = \begin{pmatrix} Q_1^1 & Q_1^2 & \cdots & Q_1^s \\ Q_2^1 & Q_2^2 & \cdots & Q_2^s \\ \vdots & \vdots & & \vdots \\ Q_m^1 & Q_m^2 & \cdots & Q_m^s \end{pmatrix}$$

be the alternate matrix of $\mathcal{R}$. Then

$$\mathcal{A}' = \begin{pmatrix} f(\mathcal{L}_1) \\ -f(\mathcal{L}_2) \\ \vdots \\ (-1)^{m-1}f(\mathcal{L}_m) \end{pmatrix} = \begin{pmatrix} f(Q_1^1) & f(Q_1^2) & \ldots & f(Q_1^s) \\ f(Q_2^1) & f(Q_2^2) & \ldots & f(Q_2^s) \\ \vdots & \vdots & & \vdots \\ f(Q_m^1) & f(Q_m^2) & \ldots & f(Q_m^s) \end{pmatrix}$$

is the alternate matrix of $f(\mathcal{R})$. If $\mathcal{R}$ is openly alternating, then for every odd j, $1 \le j \le m-1$, the paths Q_j^s and Q_{j+1}^s have exactly one vertex in common, for every even j, $2 \le j \le m-1$, the paths Q_j^1 and Q_{j+1}^1 have exactly one vertex in common, and otherwise the paths Q_j^i and Q_l^i are vertex disjoint, $1 \le i \le s$,

$1 \le j, l \le m$, $j \ne l$. Since f is a bijection, then for every odd j, $1 \le j \le m-1$, the paths $f(Q_j^s)$ and $f(Q_{j+1}^s)$ have exactly one vertex in common, for every even j, $2 \le j \le m-1$, the paths $f(Q_j^1)$ and $f(Q_{j+1}^1)$ have exactly one vertex in common, and otherwise the paths $f(Q_j^i)$ and $f(Q_l^i)$ are vertex disjoint, $1 \le i \le s$, $1 \le j, l \le m$, $j \ne l$. Hence $f(\mathcal{R})$ is also openly alternating.

Lemma 3.8. *Let $\mathcal{L}$ be a closed sm-chain of paths in K_n^3, and let*

$$\mathcal{R} = (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m)$$

be the m-splitting of $\mathcal{L}$. If $\mathcal{R}$ is closely alternating and $f \in \Sigma$, then $f(\mathcal{R})$ is also closely alternating.

Proof. Let

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_1 \\ -\mathcal{L}_2 \\ \vdots \\ (-1)^{m-1}\mathcal{L}_m \end{pmatrix} = \begin{pmatrix} Q_1^1 & Q_1^2 & \ldots & Q_1^s \\ Q_2^1 & Q_2^2 & \ldots & Q_2^s \\ \vdots & \vdots & & \vdots \\ Q_m^1 & Q_m^2 & \ldots & Q_m^s \end{pmatrix}$$

be the alternate matrix of $\mathcal{R}$. Then

$$\mathcal{A}' = \begin{pmatrix} f(\mathcal{L}_1) \\ -f(\mathcal{L}_2) \\ \vdots \\ (-1)^{m-1}f(\mathcal{L}_m) \end{pmatrix} = \begin{pmatrix} f(Q_1^1) & f(Q_1^2) & \ldots & f(Q_1^s) \\ f(Q_2^1) & f(Q_2^2) & \ldots & f(Q_2^s) \\ \vdots & \vdots & & \vdots \\ f(Q_m^1) & f(Q_m^2) & \ldots & f(Q_m^s) \end{pmatrix}$$

is the alternate matrix of $f(\mathcal{R})$. If $\mathcal{R}$ is closely alternating, then for every odd j, $1 \le j \le m-1$, the paths Q_j^s and Q_{j+1}^s have exactly one vertex in common, for every even j, $2 \le j \le m-1$, the paths Q_j^1 and Q_{j+1}^1 have exactly one vertex in common, the paths Q_1^1 and Q_m^1 have exactly one vertex in common, and otherwise the paths Q_j^i and Q_l^i are vertex disjoint, $1 \le i \le s$, $1 \le j, l \le m$, $j \ne l$. Since f is a bijection, then for every odd j, $1 \le j \le m-1$, the paths $f(Q_j^s)$ and $f(Q_{j+1}^s)$ have exactly one vertex in common, for every even j, $2 \le j \le m-1$, the paths $f(Q_j^1)$ and $f(Q_{j+1}^1)$ have exactly one vertex in common, the paths $f(Q_1^1)$ and $f(Q_m^1)$ have exactly one vertex in common, and otherwise the paths $f(Q_j^i)$ and $f(Q_l^i)$ are vertex disjoint, $1 \le i \le s$, $1 \le j, l \le m$, $j \ne l$. Hence $f(\mathcal{R})$ is also closely alternating. □

Lemma 3.9. *If $\mathcal{C}$ is an openly well distributed chain of paths in K_n^3 and $f \in \Sigma$, then $f(\mathcal{C})$ is also openly well distributed.*

Proof. We are going to use induction with respect to d. If $d = 1$, then $\mathcal{C}$ is an openly separated chain of open snakes so the lemma follows from Lemmas 3.3

and 3.4. Assume that the lemma is true for d, we show that it is true for $d+1$. Let

$$\mathcal{S} = (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n)$$

be the n-splitting of $\mathcal{C}$. Since $\mathcal{C}$ is openly well distributed, it follows that every chain $\mathcal{L}_i$, $i = 1, 2, \ldots, n$, is openly well distributed and $\mathcal{S}$ is openly alternating. Then

$$f(\mathcal{S}) = (f(\mathcal{L}_1), f(\mathcal{L}_2), \ldots, f(\mathcal{L}_n)),$$

is the n-splitting of $f(\mathcal{C})$. By Lemma 3.7, $f(\mathcal{S})$ is openly alternating and by the induction hypothesis $f(\mathcal{L}_i)$ is openly well distributed, for $i = 1, 2, \ldots, n$. Hence $f(\mathcal{C})$ is openly well distributed. $\square$

Lemma 3.10. *Let $\mathcal{C}$ be a chain of paths in K_n^3 joining c_1 to c_2 which is internally parallel to a chain of paths $\mathcal{L}$ joining l_1 to l_2. If $f \in \Sigma$, then $f(\mathcal{C})$ and $f(\mathcal{L})$ are also internally parallel chains of paths joining $f(c_1)$ to $f(c_2)$ and $f(l_1)$ to $f(l_2)$, respectively.*

Proof. Suppose that

$$\mathcal{C} = (C_1, C_2, \ldots, C_s)$$

and

$$\mathcal{L} = (L_1, L_2, \ldots, L_s)$$

are two internally parallel chains of paths joining c_1 to c_2 and l_1 to l_2, respectively, and let $f \in \Sigma$ be a given permutation. Since $\mathcal{C}$ and $\mathcal{L}$ are internally parallel chain of paths, then we have

(i) the paths $C_1 - c_1$ and L_1 are vertex disjoint,
(ii) the paths C_1 and $L_1 - l_1$ are vertex disjoint,
(iii) the paths $C_s - c_2$ and L_s are vertex disjoint,
(iv) the paths C_s and $L_s - l_2$ are vertex disjoint, and
(v) for every $i, 1 < i < s$, the paths C_i and L_i are vertex disjoint.

Since f is a bijection, it follows that

(i) the paths $f(C_1) - f(c_1)$ and $f(L_1)$ are vertex disjoint,
(ii) the paths $f(C_1)$ and $f(L_1) - f(l_1)$ are vertex disjoint,
(iii) the paths $f(C_s) - f(c_2)$ and $f(L_s)$ are vertex disjoint,
(iv) the paths $f(C_s)$ and $f(L_s) - f(l_2)$ are vertex disjoint, and
(v) for every $i, 1 < i < s$, the paths $f(C_i)$ and $f(L_i)$ are vertex disjoint.

Hence $f(\mathcal{C})$ and $f(\mathcal{L})$ are also internally parallel chains of paths joining $f(c_1)$ to $f(c_2)$ and $f(l_1)$ to $f(l_2)$, respectively. $\square$

4. Vertex Covering of K_n^d, for n Even

Assume that $n \geq 4$ is a fixed even integer, and for any integer $d \geq 1$, we define the closed $(n-1)n^d$-paths γ_n^{d+1} in K_n^{d+1}.

Let γ_n be the closed $(n-1)$-path $(0,1,\dots,n-2)$ in K_n. If $d \geq 1$, then let

$$\gamma_n^{d+1} = \gamma_n \otimes (\pi_n^d, -\pi_n^d, \pi_n^d, -\pi_n^d, \dots, \pi_n^d),$$

where π_n^d is the path in K_n^d defined at the end of Section 2. In this section we will prove Theorem 1.12 using a closed path in K_n^{d-3}. The role will be played by the path π_n^{d-3} (note that it is closed since n is even).

Lemma 4.1. *If $d \geq 1$ and $\mathcal{L}$ is a closely well distributed n^d-chain of paths in a graph H, then the path $\pi_n^d \otimes \mathcal{L}$ is a snake in the graph $K_n^d \times H$.*

Proof. Analogous to the proof of Lemma 2 in Wojciechowski [**14**]. □

We will be using Lemma 4.1 with K_n^3 as the graph H. Our aim now will be to construct a closely well distributed n^d-chain of paths in the graph K_n^3. The idea of this construction is similar to the idea of the construction, given by Wojciechowski [**14**], of a closely well distributed $(n-1)n^d$-chain of paths in K_n^3 in the case of n being odd; however, the details of these two constructions are different.

First we need to introduce some additional terminology.

Let H be a graph. An n-**net** in H is a pair $(U, \mathcal{M})$, where

$$U = \{u_1, u_2, \dots, u_{n+3}\}$$

is a set of $n+3$ vertices of H and

$$\mathcal{M} = \{\mathcal{N}_{s,t}^i : 0 \leq i \leq n-1; 1 \leq s,t \leq n+3; s \neq t\}$$

is a set of openly separated chains of open snakes in H such that $\mathcal{N}_{s,t}^i$ joins u_s to u_t,

$$\mathcal{N}_{s,t}^i = -\mathcal{N}_{t,s}^i,$$

and if $i \neq j$ then $\mathcal{N}_{s,t}^i$ and $\mathcal{N}_{v,w}^j$ are internally parallel, for $0 \leq i,j \leq n-1$ and $1 \leq s,t,v,w \leq n+3$ with $s \neq t$, $v \neq w$.

Let us assume that we are given an n-net $(U, \mathcal{M})$ in H. Let $\mathcal{L}$ be an n^d-chain of paths in H, $d \geq 1$. If every n-chain in the n^{d-1}-splitting of $\mathcal{L}$ belongs to $\mathcal{M}$, then we say that $\mathcal{L}$ is $\mathcal{M}$-**built**.

To get a closely well-distributed chain of paths in K_n^3 we show that, in general, the existence of an n-net $(U, \mathcal{M})$ in H implies that there is a closely well-distributed chain of paths in H which, moreover, is $\mathcal{M}$-built.

Let X_n be the following $n \times (n+3)$-matrix:

$$X_n = \begin{pmatrix}
5 & 6 & 3 & 4 & 7 & 8 & 9 & \cdots & n+3 & 1 & 2 \\
4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots & 1 & 2 & 3 \\
7 & 8 & 5 & 6 & 9 & 10 & 11 & \cdots & 2 & 3 & 4 \\
6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots & 3 & 4 & 5 \\
9 & 10 & 7 & 8 & 11 & 12 & 13 & \cdots & 4 & 5 & 6 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
n & n+1 & n+2 & n+3 & 1 & 2 & 3 & \cdots & n-3 & n-2 & n-1 \\
n+3 & 1 & n+1 & n+2 & 2 & 3 & 4 & \cdots & n-2 & n-1 & n \\
n+2 & n+3 & 1 & 2 & 3 & 4 & 5 & \cdots & n-1 & n & n+1
\end{pmatrix},$$

and Y_n be the following $n \times (n+3)$-matrix:

$$Y_n = \begin{pmatrix} 5 & 6 & 3 & 4 & 7 & 8 & 9 & \cdots & 1 & 2 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots & 2 & 3 \\ 7 & 8 & 5 & 6 & 9 & 10 & 11 & \cdots & 3 & 4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & n+1 & n+2 & n+3 & 1 & 2 & 3 & \cdots & n-2 & n-1 \\ n+3 & 1 & n+1 & n+2 & 2 & 3 & 4 & \cdots & n-1 & n \\ 1 & n+3 & 4 & 2 & 3 & 5 & 6 & \cdots & n+1 & n+2 \end{pmatrix}.$$

The matrix X_n can be obtained by taking as rows the n consecutive cyclic permutations of the sequence $(3, 4, \ldots, n+3, 1, 2)$, and then, for each odd row, exchanging the entries at the positions 1 and 3, and exchanging the entries at the positions 2 and 4. The matrix Y_n differs from X_n only at the last row.

For any i, $1 \leq i \leq n$, let

$$\varphi_i : \mathcal{M} \to \mathcal{M}$$

be defined by

$$\varphi_i(\mathcal{N}^j_{v,w}) = \mathcal{N}^{j+i-1 \mod n}_{x_{i,v},x_{i,w}},$$

where the bottom indices are taken from the matrix $X_n = (x_{p,r})$. Analogously, for each i, $1 \leq i \leq n$, let

$$\psi_i : \mathcal{M} \to \mathcal{M}$$

be defined by

$$\psi_i(\mathcal{N}^j_{v,w}) = \mathcal{N}^{j+i-1 \mod n}_{y_{i,v},y_{i,w}},$$

where the bottom indices are taken from the matrix $Y_n = (y_{p,r})$.

If $\mathcal{L}$ is any $\mathcal{M}$-built n^d-chain, then $\varphi_i(\mathcal{L})$ and $\psi_i(\mathcal{L})$ are the n^d-chains obtained by applying φ_i and ψ_i, respectively, to each of the n-chains in the n^{d-1}-splitting of $\mathcal{L}$.

Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are $\mathcal{M}$-built n^d-chains with n^{d-1}-splittings $(\mathcal{N}^{i_k}_{s_k,t_k})_{k=1}^{n^{d-1}}$ and $(\mathcal{N}^{j_k}_{v_k,w_k})_{k=1}^{n^{d-1}}$, respectively. We define that $\mathcal{H}_1$ and $\mathcal{H}_2$ are **internally $\mathcal{M}$-compatible** if $i_k = j_k$ for $1 \leq k \leq n^{d-1}$, $s_k = v_k$ for $1 < k \leq n^{d-1}$, and $t_k = w_k$ for $1 \leq k < n^{d-1}$. We define that $\mathcal{H}_1$ and $\mathcal{H}_2$ are **internally $\mathcal{M}$-parallel** if $i_k \neq j_k$ for $1 \leq k \leq n^{d-1}$, $s_k \neq v_k$ for $1 < k \leq n^{d-1}$, and $t_k \neq w_k$ for $1 \leq k < n^{d-1}$, also we define that $\mathcal{H}_1$ and $\mathcal{H}_2$ are **$\mathcal{M}$-parallel** if $i_k \neq j_k$ for $1 \leq k \leq n^{d-1}$, $s_k \neq v_k$ for $1 \leq k \leq n^{d-1}$, and $t_k \neq w_k$ for $1 \leq k \leq n^{d-1}$.

Let $\mathcal{L}$ be an $\mathcal{M}$-built n^d-chain, let

$$\mathcal{R} = (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n)$$

be the n-splitting of $\mathcal{L}$, and let

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_1 \\ -\mathcal{L}_2 \\ \mathcal{L}_3 \\ \vdots \\ \mathcal{L}_{n-1} \\ -\mathcal{L}_n \end{pmatrix}$$

be the alternate matrix of $\mathcal{R}$. If either $d = 1$, or $d > 1$ and the following conditions are satisfied:

(i) For each i, $1 \leq i \leq n$, the n^{d-1}-chain $\mathcal{L}_i$ is $\mathcal{M}$-well distributed.
(ii) Any two consecutive rows of $\mathcal{A}$ form a pair of internally $\mathcal{M}$-parallel chains.
(iii) Any two nonconsecutive rows of $\mathcal{A}$ form a pair of $\mathcal{M}$-parallel chains,

then we say that $\mathcal{L}$ is **$\mathcal{M}$-well distributed**.

Lemma 4.2. *If $\mathcal{L}$ is an $\mathcal{M}$-well distributed n^d-chain, then $\mathcal{L}$ is an openly well distributed chain.*

Proof. We are going to use induction on d. Assume first that $d = 1$ and $\mathcal{L}$ is an $\mathcal{M}$-well distributed n-chain. Since $\mathcal{L}$ is $\mathcal{M}$-built, the chain $\mathcal{L}$ belongs to $\mathcal{M}$ and so it is an openly separated chain of open snakes, hence it is openly well distributed.

Assume now that $d > 1$, that $\mathcal{L}$ is an $\mathcal{M}$-well distributed n^d-chain, and that any $\mathcal{M}$-well distributed chain shorter than $\mathcal{L}$ is openly well distributed. Let

$$\mathcal{R} = (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n),$$

be the n-splitting of $\mathcal{L}$ and let

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_1 \\ -\mathcal{L}_2 \\ \mathcal{L}_3 \\ \vdots \\ \mathcal{L}_{n-1} \\ -\mathcal{L}_n \end{pmatrix}$$

be the alternate matrix of the splitting $\mathcal{R}$. Since $\mathcal{L}$ is $\mathcal{M}$-well distributed, for each i with $1 \leq i \leq n$, the chain $\mathcal{L}_i$ is $\mathcal{M}$-well distributed, and hence $\mathcal{L}_i$ is openly well distributed by the inductive hypothesis.

Moreover, any two consecutive rows of $\mathcal{A}$ form a pair of internally $\mathcal{M}$-parallel chains and any two nonconsecutive rows of $\mathcal{A}$ form a pair of $\mathcal{M}$-parallel chains implying that $\mathcal{R}$ is openly alternating. Therefore $\mathcal{L}$ is openly well distributed chain and the proof is complete. $\square$

Lemma 4.3. *If $\mathcal{L}$ is an $\mathcal{M}$-well distributed n^d-chain, then for any i, $1 \le i \le n$, the chains $\varphi_i(\mathcal{L})$, $\psi_i(\mathcal{L})$ and $-\mathcal{L}$ are also $\mathcal{M}$-well distributed.*

Proof. We are going to use induction on d. If $d = 1$ and $\mathcal{L}$ is an $\mathcal{M}$-well distributed n-chain, then $\mathcal{L} = \mathcal{N}^j_{v,w}$ for some j, v, w with $0 \le j \le n-1$, $1 \le v, w \le n+3$, and $v \ne w$.

For any i with $1 \le i \le n$ we have

$$\varphi_i(\mathcal{L}) = \varphi_i(\mathcal{N}^j_{v,w}) = \mathcal{N}^{j+i-1 \mod n}_{x_{i,v},x_{i,w}} \in \mathcal{M},$$
$$\psi_i(\mathcal{L}) = \psi_i(\mathcal{N}^j_{v,w}) = \mathcal{N}^{j+i-1 \mod n}_{y_{i,v},y_{i,w}} \in \mathcal{M},$$

and

$$-\mathcal{L} = -\mathcal{N}^j_{v,w} = \mathcal{N}^j_{w,v} \in \mathcal{M}.$$

Therefore $\varphi_i(\mathcal{L})$, $\psi_i(\mathcal{L})$ and $-\mathcal{L}$ are $\mathcal{M}$-built n-chains, and so they are $\mathcal{M}$-well distributed as required.

Now assume that $d > 1$, that $\mathcal{L}$ is an $\mathcal{M}$-well distributed n^d-chain and that the lemma holds for any $\mathcal{M}$-well distributed chain that is shorter than $\mathcal{L}$.

Let

$$\mathcal{R} = (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n)$$

be the n-splitting of $\mathcal{L}$ and let

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_1 \\ -\mathcal{L}_2 \\ \mathcal{L}_3 \\ \vdots \\ \mathcal{L}_{n-1} \\ -\mathcal{L}_n \end{pmatrix}$$

be the alternate matrix of the splitting $\mathcal{R}$. Fix an i satisfying $1 \le i \le n$. Then

$$\mathcal{R}_1 = (\varphi_i(\mathcal{L}_1), \varphi_i(\mathcal{L}_2), \ldots, \varphi_i(\mathcal{L}_n)),$$

is the n-splitting of $\varphi_i(\mathcal{L})$ and

$$\mathcal{A}_1 = \begin{pmatrix} \varphi_i(\mathcal{L}_1) \\ -\varphi_i(\mathcal{L}_2) \\ \varphi_i(\mathcal{L}_3) \\ \vdots \\ \varphi_i(\mathcal{L}_{n-1}) \\ -\varphi_i(\mathcal{L}_n) \end{pmatrix}$$

is the alternate matrix of the splitting $\mathcal{R}_1$.

Since each $\mathcal{L}_j$ is $\mathcal{M}$-well distributed, it follows from the inductive hypothesis that $\varphi_i(\mathcal{L}_j)$ is $\mathcal{M}$-well distributed.

Since in the matrix X_n no integer occurs twice in any row, and since any two consecutive rows of $\mathcal{A}$ form a pair of internally $\mathcal{M}$-parallel chains, it follows that any two consecutive rows of $\mathcal{A}_1$ form a pair of internally $\mathcal{M}$-parallel chains. Similarly we conclude that any two nonconsecutive rows of $\mathcal{A}_1$ form a pair of $\mathcal{M}$-parallel chains. Therefore $\varphi_i(\mathcal{L})$ is $\mathcal{M}$-well distributed.

Similarly (using the matrix Y_n instead of X_n) we show that $\psi_i(\mathcal{L})$ is $\mathcal{M}$-well distributed.

Moreover,

$$\mathcal{R}_2 = (-\mathcal{L}_n, -\mathcal{L}_{n-1}, \ldots, -\mathcal{L}_1)$$

is the n-splitting of $-\mathcal{L}$ and

$$\mathcal{A}_2 = \begin{pmatrix} -\mathcal{L}_n \\ \mathcal{L}_{n-1} \\ -\mathcal{L}_{n-2} \\ \vdots \\ -\mathcal{L}_2 \\ \mathcal{L}_1 \end{pmatrix}$$

is the alternate matrix of the splitting $\mathcal{R}_2$. Using the inductive hypothesis and the fact that $\mathcal{A}_2$ is obtained from $\mathcal{A}$ by reversing the order of rows we can show that $-\mathcal{L}$ is $\mathcal{M}$-well distributed completing the proof. $\square$

Lemma 4.4. *There exist four internally $\mathcal{M}$-compatible n^d-chains $\mathcal{L}^d_{3,1}$, $\mathcal{L}^d_{3,2}$, $\mathcal{L}^d_{4,1}$ and $\mathcal{L}^d_{4,2}$, where $\mathcal{L}^d_{s,t}$ is $\mathcal{M}$-well distributed and joins u_s to u_t, for any $s \in \{3,4\}$, $t \in \{1,2\}$, and any $d \geq 1$.*

Proof. We are going to use induction with respect to d. If $d = 1$, then let

$$\mathcal{L}^1_{s,t} = \mathcal{N}^0_{s,t},$$

for any $s \in \{3,4\}$, $t \in \{1,2\}$. It is clear that the n-chains $\mathcal{L}^1_{3,1}$, $\mathcal{L}^1_{3,2}$, $\mathcal{L}^1_{4,1}$ and $\mathcal{L}^1_{4,2}$ have the required properties.

Assume that $d \geq 1$ and the n^d-chains $\mathcal{L}^d_{3,1}$, $\mathcal{L}^d_{3,2}$, $\mathcal{L}^d_{4,1}$ and $\mathcal{L}^d_{4,2}$ satisfy the specified conditions. Given $s \in \{3,4\}$ and $t \in \{1,2\}$, let $\mathcal{L}^{d+1}_{s,t}$ be the n^{d+1}-chain with the n-splitting

$$\mathcal{R} = (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n)$$

such that $\mathcal{L}_1 = \varphi_1(\mathcal{L}^d_{s,1})$,

$$\begin{aligned} \mathcal{L}_i &= \varphi_i(\mathcal{L}^d_{4,1}) && \text{for odd } i,\ 1 < i < n, \\ \mathcal{L}_i &= -\varphi_i(\mathcal{L}^d_{3,2}) && \text{for even } i,\ 1 < i < n, \end{aligned}$$

and $\mathcal{L}_n = -\varphi_n(\mathcal{L}^d_{t+2,2})$.

Let $m = n^{d-1}$. Since $\mathcal{L}^d_{3,1}$, $\mathcal{L}^d_{3,2}$, $\mathcal{L}^d_{4,1}$ and $\mathcal{L}^d_{4,2}$ are internally $\mathcal{M}$-compatible, there are $i_1, i_2, \ldots, i_m \in \{0, 1, \ldots, n\}$ and $z_1, z_2, \ldots, z_{m-1} \in \{1, 2, \ldots, n+3\}$, such that for any $v \in \{3, 4\}$ and $w \in \{1, 2\}$, the sequence

$$(\mathcal{N}^{i_1}_{v,z_1}, \mathcal{N}^{i_2}_{z_1,z_2}, \mathcal{N}^{i_3}_{z_2,z_3}, \ldots, \mathcal{N}^{i_{m-1}}_{z_{m-2},z_{m-1}}, \mathcal{N}^{i_m}_{z_{m-1},w}),$$

is the m-splitting of $\mathcal{L}^d_{v,w}$. Thus, if $\mathcal{A}$ is the alternate matrix of the splitting $\mathcal{R}$, then we have:

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_1 \\ -\mathcal{L}_2 \\ \mathcal{L}_3 \\ \vdots \\ \mathcal{L}_{n-1} \\ -\mathcal{L}_n \end{pmatrix} = \begin{pmatrix} \varphi_1(\mathcal{L}^d_{s,1}) \\ \varphi_2(\mathcal{L}^d_{3,2}) \\ \varphi_3(\mathcal{L}^d_{4,1}) \\ \vdots \\ \varphi_{n-1}(\mathcal{L}^d_{4,1}) \\ \varphi_n(\mathcal{L}^d_{t+2,2}) \end{pmatrix}$$

$$= \begin{pmatrix} \varphi_1(\mathcal{N}^{i_1}_{s,z_1}) & \varphi_1(\mathcal{N}^{i_2}_{z_1,z_2}) & \cdots & \varphi_1(\mathcal{N}^{i_m}_{z_{m-1},1}) \\ \varphi_2(\mathcal{N}^{i_1}_{3,z_1}) & \varphi_2(\mathcal{N}^{i_2}_{z_1,z_2}) & \cdots & \varphi_2(\mathcal{N}^{i_m}_{z_{m-1},2}) \\ \varphi_3(\mathcal{N}^{i_1}_{4,z_1}) & \varphi_3(\mathcal{N}^{i_2}_{z_1,z_2}) & \cdots & \varphi_3(\mathcal{N}^{i_m}_{z_{m-1},1}) \\ \vdots & \vdots & & \vdots \\ \varphi_{n-1}(\mathcal{N}^{i_1}_{4,z_1}) & \varphi_{n-1}(\mathcal{N}^{i_2}_{z_1,z_2}) & \cdots & \varphi_{n-1}(\mathcal{N}^{i_m}_{z_{m-1},1}) \\ \varphi_n(\mathcal{N}^{i_1}_{t+2,z_1}) & \varphi_n(\mathcal{N}^{i_2}_{z_1,z_2}) & \cdots & \varphi_n(\mathcal{N}^{i_m}_{z_{m-1},2}) \end{pmatrix}$$

We will prove now that $\mathcal{L}^{d+1}_{s,t}$ satisfies the required conditions. To verify that $\mathcal{L}^{d+1}_{s,t}$ is a chain of paths, note that the entries of the matrix $X_n = (x_{j,k})$ satisfy $x_{i,1} = x_{i+1,2}$ for i odd, and $x_{i,3} = x_{i+1,4}$ for i even, $1 \le i \le n-1$. Since the chains $\mathcal{L}^d_{v,w}$ are $\mathcal{M}$-well distributed, Lemma 4.3 implies that the chains $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n$ are also $\mathcal{M}$-well distributed. Since the integers 3 and 4 do not appear in the 3rd or 4th column of the matrix X_n except in the first row, the integers 1 and 2 do not appear in the 3rd or 4th column of X_n except in the last row, and no integer appears twice in any column of X_n, it follows from the definition of the function φ_i that any two consecutive rows of $\mathcal{A}$ are internally $\mathcal{M}$-parallel chains and any two nonconsecutive rows of $\mathcal{A}$ are $\mathcal{M}$-parallel chains. Hence the chain $\mathcal{L}^{d+1}_{s,t}$ is $\mathcal{M}$-well distributed. Since it is clear that the chains $\mathcal{L}^{d+1}_{3,1}$, $\mathcal{L}^{d+1}_{3,2}$, $\mathcal{L}^{d+1}_{4,1}$ and $\mathcal{L}^{d+1}_{4,2}$ are internally $\mathcal{M}$-compatible, the proof is complete. $\square$

Lemma 4.5. *For each $d \ge 2$, there exists a closely well distributed $\mathcal{M}$-built n^d-chain in H.*

Proof. We are going to assume that $d \ge 1$ and construct a closely well distributed $\mathcal{M}$-built n^{d+1}-chain in H. Let $\mathcal{D}$ be the closed n^{d+1}-chain with n-splitting

$$\mathcal{R} = (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n)$$

such that

$$\begin{aligned}\mathcal{L}_i &= \psi_i(\mathcal{L}^d_{4,1}) && \text{for odd } i,\ 1 \le i \le n,\\ \mathcal{L}_i &= -\psi_i(\mathcal{L}^d_{3,2}) && \text{for even } i,\ 1 \le i \le n,\end{aligned}$$

where $\mathcal{L}^d_{4,1}$ and $\mathcal{L}^d_{3,2}$ are as in Lemma 4.4. Thus if $\mathcal{A}$ is the alternate matrix of the splitting $\mathcal{R}$, and $m = n^{d-1}$, then we have

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_1 \\ -\mathcal{L}_2 \\ \mathcal{L}_3 \\ \vdots \\ \mathcal{L}_{n-1} \\ -\mathcal{L}_n \end{pmatrix} = \begin{pmatrix} \psi_1(\mathcal{L}^d_{4,1}) \\ \psi_2(\mathcal{L}^d_{3,2}) \\ \psi_3(\mathcal{L}^d_{4,1}) \\ \vdots \\ \psi_{n-1}(\mathcal{L}^d_{4,1}) \\ \psi_n(\mathcal{L}^d_{3,2}) \end{pmatrix}$$

$$= \begin{pmatrix} \psi_1(\mathcal{N}^{i_1}_{4,z_1}) & \psi_1(\mathcal{N}^{i_2}_{z_1,z_2}) & \dots & \psi_1(\mathcal{N}^{i_m}_{z_{m-1},1}) \\ \psi_2(\mathcal{N}^{i_1}_{3,z_1}) & \psi_2(\mathcal{N}^{i_2}_{z_1,z_2}) & \dots & \psi_2(\mathcal{N}^{i_m}_{z_{m-1},2}) \\ \psi_3(\mathcal{N}^{i_1}_{4,z_1}) & \psi_3(\mathcal{N}^{i_2}_{z_1,z_2}) & \dots & \psi_3(\mathcal{N}^{i_m}_{z_{m-1},1}) \\ \vdots & \vdots & & \vdots \\ \psi_{n-1}(\mathcal{N}^{i_1}_{4,z_1}) & \psi_{n-1}(\mathcal{N}^{i_2}_{z_1,z_2}) & \dots & \psi_{n-1}(\mathcal{N}^{i_m}_{z_{m-1},1}) \\ \psi_n(\mathcal{N}^{i_1}_{3,z_1}) & \psi_n(\mathcal{N}^{i_2}_{z_1,z_2}) & \dots & \psi_n(\mathcal{N}^{i_m}_{z_{m-1},2}) \end{pmatrix}$$

where, for some $i_1, i_2, \dots, i_m \in \{0, 1, \dots, n\}$ and $z_1, z_2, \dots, z_{m-1} \in \{1, 2, \dots, n+3\}$, the sequence

$$(\mathcal{N}^{i_1}_{4,z_1}, \mathcal{N}^{i_2}_{z_1,z_2}, \mathcal{N}^{i_3}_{z_2,z_3}, \dots, \mathcal{N}^{i_{m-1}}_{z_{m-2},z_{m-1}}, \mathcal{N}^{i_m}_{z_{m-1},1}),$$

is the m-splitting of the chain $\mathcal{L}^d_{4,1}$, and the sequence

$$(\mathcal{N}^{i_1}_{3,z_1}, \mathcal{N}^{i_2}_{z_1,z_2}, \mathcal{N}^{i_3}_{z_2,z_3}, \dots, \mathcal{N}^{i_{m-1}}_{z_{m-2},z_{m-1}}, \mathcal{N}^{i_m}_{z_{m-1},2}),$$

is the m-splitting of the chain $\mathcal{L}^d_{3,2}$.

Since the entries of the matrix $Y_n = (y_{j,k})$ satisfy $y_{i,1} = y_{i+1,2}$ for i odd, $y_{i,3} = y_{i+1,4}$ for i even, $1 \le i \le n-1$, and $y_{n,3} = y_{1,4}$, it follows that the sequence $\mathcal{D}$ of paths is a closed chain of paths. Since the chains $\mathcal{L}^d_{v,w}$ are $\mathcal{M}$-well distributed, Lemma 4.3 implies that the chains $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$ are also $\mathcal{M}$-well distributed. By Lemma 4.2, $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$ are openly well distributed. Since no integer appears twice in any column of Y_n, it follows that any two consecutive rows of $\mathcal{A}$ are internally $\mathcal{M}$-parallel chains and any two nonconsecutive rows of $\mathcal{A}$ are $\mathcal{M}$-parallel chains; hence, the splitting $\mathcal{R}$ is closely alternating. Thus, $\mathcal{D}$ is a closely well distributed $\mathcal{M}$-built chain. $\square$

To use Lemma 4.5 with H being the graph K_n^3, we need to construct an n-net in K_n^3.

The vertices of the graph K_n^3 are 3-tuples of digits from the set $\{0,1,\dots,n-1\}$. Given a vertex $v=(a_1,a_2,a_3)$ of K_n^3, we say that a_i appears at the i-th **position** of $v, 1\le i\le 3$. If $a\in\{0,1,\dots,n-1\}$ then let $\bar{a}=a+n/2 \mod n$, and $\bar{\bar{a}}=a+n/2+1 \mod n$. If $a_1,a_2\in\{0,1,\dots,n-1\}$, then let $[a_1,a_2]=\{a_1,a_1+1,a_1+2,\dots,a_2\}$, where addition is taken modulo n. Thus, for example, if $n=6$ then $[4,3]=\{4,5,0,1,2,3\}$.

Let's now define the n-net $(U_n,\mathcal{M}_n)$ in K_n^3. Let

$$\begin{aligned} U_n &= \{u_1,u_2,\dots,u_{n+3}\}\\ &= \Big\{(a_1,a_2,a_3) : a_1\in\Big\{0,1,\dots,\frac{n}{2}\Big\}, \{a_2,a_3\}=\{\bar{a_1},\bar{\bar{a_1}}\}\Big\}\\ &\quad\cup\Big\{(a_1,a_2,a_3) : a_1=\frac{n}{2}+1, a_2=\bar{a_1}, a_3=\bar{\bar{a_1}}\Big\}. \end{aligned}$$

For example, if $n=6$ then

$$U_n=\{034,043,145,154,250,205,301,310,412\},$$

where we write $a_1a_2a_3$ instead of (a_1,a_2,a_3).

Before constructing the chains in the set $\mathcal{M}_n$, we need to give some definitions. Let $v=(a_1,a_2,a_3)$ be a vertex from the set U_n, and let us assume that the digit $b\notin\{a_1,a_2,a_3\}$. The **adjunct** of the digit b in v is the digit $\bar{a_1}$ if $b\in[a_1,\bar{a_1}]$, and the digit $\bar{\bar{a_1}}$ otherwise. Let

$$\eta\colon\{1,2,3\}\to\{1,2,3\},$$

be the function defined by $\eta(i)=i+1$ if $1\le i<3$, and $\eta(3)=1$. For each i and s with $0\le i\le n-1$ and $1\le s\le n+3$, let Q_s^i be the path starting with u_s defined as follows. If the digit i appears at the j-th position in u_s, then let $Q_s^i=(u_s,u_s')$, where u_s' is obtained from u_s by replacing the digit at the $\eta(j)$-th position with the digit i. If the digit i does not appear in u_s, then let the adjunct of i appear at the j-th position in u_s. Let $Q_s^i=(u_s,u_s',u_s'')$, where u_s' is obtained from u_s by replacing the digit at the j-th position with the digit i, and let u_s'' be obtained from u_s' by replacing the digit at the $\eta(j)$-th position with the digit i. For example, if $n=6$ and the sequence $u_1,u_2,\dots,u_9$ of vertices in U_6 is

$$034,043,145,154,250,205,301,310,412,$$

then the path Q_s^i is the path in the s-th row and the $(i+1)$-st column of the following matrix.

(034,004)	(034,014,011)	(034,024,022)	(034,033)	(034,434)	(034,035,535)
(043,003)	(043,041,141)	(043,042,242)	(043,343)	(043,044)	(043,053,055)
(145,140,040)	(145,115)	(145,125,122)	(145,135,133)	(145,144)	(145,545)
(154,104,100)	(154,114)	(154,152,252)	(154,153,353)	(154,454)	(154,155)
(250,050)	(250,251,151)	(250,220)	(250,230,233)	(250,240,244)	(250,255)
(205,200)	(205,215,211)	(205,225)	(205,203,303)	(205,204,404)	(205,505)
(301,300)	(301,101)	(301,302,202)	(301,331)	(301,341,344)	(301,351,355)
(310,010)	(310,311)	(310,320,322)	(310,330)	(310,314,414)	(310,315,515)
(412,402,400)	(412,411)	(412,212)	(412,413,313)	(412,442)	(412,452,455)

Now we define the set

$$\mathcal{M}_n = \{\mathcal{N}^i_{s,t} : 0 \le i \le n-1; 1 \le s,t \le n+3; s \ne t\}$$

of n-chains of paths in K^3_n. Given $0 \le i \le n-1$, and $1 \le s,t \le n+3$ such that $s < t$, let

$$\mathcal{N}^i_{s,t} = (Q^i_s, P_1, P_2, \ldots, P_{n-2}, -Q^i_t),$$

and

$$\mathcal{N}^i_{t,s} = (Q^i_t, -P_{n-2}, -P_{n-3}, \ldots, -P_1, -Q^i_s),$$

where $P_1, P_2, \ldots, P_{n-2}$ are defined as follows. Let v_s, v_t be the last vertices of the paths Q^i_s and Q^i_t respectively, let $1 \le j_s, j_t \le 3$ be such that i does not appear neither at the j_s-th position of v_s, nor at the j_t-th position of v_t, and let a_s, a_t be the digits at the position j_s of v_s and the position j_t of v_t, respectively. Note that for $v \in \{v_s, v_t\}$, the digit i appears at exactly two positions of v; hence, j_s and j_t are well defined and $i \notin \{a_s, a_t\}$. Let us now consider two cases:

If $j_s = j_t$, then let $b_2, b_3, \ldots, b_{n-2}$ be a sequence of different digits from the set $\{0, 1, \ldots, n-1\} \setminus \{a_s, a_t, i\}$, let $b_1 = a_s$, $b_{n-1} = a_t$, and let P_j be the path (w_j, w_{j+1}) in K^3_n, $j = 1, 2, \ldots, n-2$, where w_k is obtained from v_s by replacing the digit at the i-th position with the digit $b_k, k = 1, 2, \ldots, n-1$. If $j_s \ne j_t$, then let $m = (n-1)/2$, let $b_1 = a_s$, let $b_2, b_3, \ldots, b_m$ be a sequence of different digits from the set $\{0, 1, \ldots, n-1\} \setminus \{a_s, i\}$, let $b_{m+1}, b_{m+2}, \ldots, b_{2m-1}$ be a sequence of different digits from the set $\{0, 1, \ldots, n-1\} \setminus \{a_t, i\}$, and let $b_{2m} = a_t$. For $j \in \{1, 2, \ldots, 2m-1\} \setminus \{m\}$ let P_j be the path (w_j, w_{j+1}) in K^3_n, and let $P_m = (w_m, (i,i,i), w_{m+1})$, where w_k is obtained from v_s by replacing the digit at the j_s-th position with the digit b_k, for $k = 1, 2, \ldots, m$, and w_k is obtained from v_t by replacing the digit at the j_t-th position with the digit b_k, for $k = m+1, m+2, \ldots, 2m$.

Assuming that in the construction of the paths $P_1, P_2, \ldots, P_{n-2}$, we choose the sequences of digits to be increasing and containing as small digits as possible, in the case $n = 6$, with $u_1, u_2, \ldots, u_9$ as above, we have for example:

$$\mathcal{N}^0_{1,2} = ((034, 004), (004, 001), (001, 002), (002, 005), (005, 003), (003, 043)),$$
$$\mathcal{N}^2_{2,5} = ((043, 042, 242), (242, 212), (212, 222, 223), (223, 225), (225, 220), (220, 250)),$$
$$\mathcal{N}^5_{1,4} = ((034, 035, 535), (535, 505), (505, 515), (515, 555, 455), (455, 155), (155, 154)).$$

It follows directly from the definition of the paths Q^i_s that the following lemma holds.

Lemma 4.6. *If $i, j \in \{0, 1, \ldots, n-1\}$, $i \ne j$, and $s, t \in \{1, 2, \ldots, n+3\}$, $s \ne t$, then the paths Q^i_s and Q^j_s have only the first vertex in common, and the paths Q^i_s and Q^j_t are vertex disjoint.*

Lemma 4.7. *The pair $(U_n, \mathcal{M}_n)$ is an n-net in K_n^3.*

Proof. Clearly, it follows from the construction and Lemma 4.6 that $\mathcal{N}_{s,t}^i$ is an openly separated chain of open snakes joining u_s to u_t, and

$$\mathcal{N}_{s,t}^i = -\mathcal{N}_{t,s}^i,$$

for any $i \in \{0, 1, \ldots, n-1\}$ and $s, t \in \{1, 2, \ldots, n+3\}$, $s \neq t$. Note that for every path P of $\mathcal{N}_{s,t}^i$ except the first and the last, the digit i appears at least at two positions in each vertex of P. From the above fact and Lemma 4.6 it follows that $\mathcal{N}_{s,t}^i$ and $\mathcal{N}_{v,w}^j$ are internally parallel, for any $i, j \in \{0, 1, \ldots, n-1\}$, $i \neq j$, and $s, t, v, w \in \{1, 2, \ldots, n+3\}$, such that $s \neq t$, $v \neq w$, hence the proof is complete. □

By Lemma 4.5, let $\mathcal{D}$ be a closely well distributed chain of paths in K_n^3. For each $f \in \Sigma$, let $\mathcal{D}_f = f(\mathcal{D})$ and let

$$\mathcal{P}_f = \pi_n^{d-3} \otimes \mathcal{D}_f.$$

Lemma 4.8. *The chain of paths $\mathcal{D}_f$ is closely well distributed for every $f \in \Sigma$.*

Proof. Since $\mathcal{D}$ is a closely well distributed chain of paths, every chain $\mathcal{D}_i$ in the n-splitting

$$S = (\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n)$$

of $\mathcal{D}$ is openly well distributed and S is closely alternating. By Lemma 3.9, every chain $f(\mathcal{D}_i)$ in the n-splitting

$$f(S) = (f(\mathcal{D}_1), f(\mathcal{D}_2), \ldots, f(\mathcal{D}_n)),$$

of $\mathcal{D}_f$ is openly well distributed. By Lemma 3.8, $f(S)$ is closely alternating. Hence $\mathcal{D}_f$ is closely well distributed. □

The following lemma follows immediately from Lemmas 4.1 and 4.8.

Lemma 4.9. *$\mathcal{P}_f$ is a snake in K_n^d for every $f \in \Sigma$.*

Lemma 4.10. *For every vertex v of K_n^d, there exist $f \in \Sigma$ such that v is a vertex of $\mathcal{P}_f$.*

Proof. Suppose that $v = (a_1, a_2, \ldots, a_d)$ is any vertex of K_n^d. By Lemma 2.1, $(a_1, a_2, \ldots, a_{d-3})$ is a vertex of π_n^{d-3}. Assume that $(a_1, a_2, \ldots, a_{d-3})$ is a vertex of π_n^{d-3} and $\pi_n^{d-3} = (v_1, v_2, \ldots, v_s)$, where $s = (n-1)n^{d-4}$. Then there is $i \in \{1, 2, \ldots, s\}$ with $v_i = (a_1, a_2, \ldots, a_{d-3})$. Assume that

$$\mathcal{D} = (P_1, P_2, \ldots, P_s)$$

and let (b_{d-2}, b_{d-1}, b_d) be a vertex of P_i. By Lemma 3.1, there is $f \in \Sigma$ with

$$(a_{d-2}, a_{d-1}, a_d) = f(b_{d-2}, b_{d-1}, b_d).$$

Then (a_{d-2}, a_{d-1}, a_d) is a vertex of $f(P_i)$. Since

$$\mathcal{D}_f = (f(P_1), f(P_2), \ldots, f(P_s)),$$

it follows that $(a_1, a_2, \ldots, a_d)$ is a vertex of

$$\mathcal{P}_f = \pi_n^{d-3} \otimes \mathcal{D}_f. \qquad \square$$

Now we are ready to proof Theorem 1.12.

Proof of Theorem 1.12. We have $K_n^d = K_n^3 \times K_n^{d-3}$. Let

$$\mathcal{S} = \{\mathcal{P}_f : f \in \Sigma\}.$$

By Lemma 4.9, the elements of $\mathcal{S}$ are snakes and by Lemma 4.10, they vertex cover K_n^d. Since $|\Sigma| = n^3$, it follows that $|\mathcal{S}| = n^3$ and the proof is complete. $\square$

5. Conclusion

In this paper we proved a weaker version of the Conjecture 1.10. It settle the part of Conjecture 1.10 that corresponds to the first question posed by Erds. It still remains open whether the snakes in Theorems 1.12 can be made vertex-disjoint. Also Conjecture 1.7 is an open problem.

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S. Y. Alsardary, Department of Mathematics, Physics, and Computer Science, Philadelphia College of Pharmacy and Science, 600 South 43rd Street, Philadelphia, PA 19104-4495, USA, *e-mail*: s.alsard@pcps.edu