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# UNIVERSAL $q$–DIFFERENTIAL CALCULUS AND $q$–ANALOG OF HOMOLOGICAL ALGEBRA

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Abstract. We recall the definition of $q$-differential algebras and discuss some representative examples. In particular we construct the $q$-analog of the Hochschild coboundary. We then construct the universal $q$-differential envelope of a unital associative algebra and study its properties. The paper also contains general results on $d^N = 0$.

## 1. Introduction and Algebraic Preliminaries

At the origin of this paper there is the long-standing physically-motivated interest of one of the authors (R.K.) on $\mathbb{Z}_3$-graded structures and differential calculi [**RK**] although here the point of view is somehow different. There is also the observation that the simplicial (co)-homology admits $\mathbb{Z}_N$ versions leading to cyclotomic homology [**Sark**] and that, more generally, this suggests that one can introduce "$q$-analog of homological algebra" for each primitive root $q$ of the unity [**Kapr**]. Moreover the occurrence of various notions of "$q$-analog" in connection with quantum groups suggests to include in the formulation the general case where $q$ is not necessarily a root of unity but is an arbitrary invertible complex number [**Kapr**]. It is our aim here to go further in this direction.

Throughout this paper, we shall be interested in complex associative graded algebras equipped with endomorphisms $d$ of degree 1 satisfying a twisted Leibniz rule, **the $q$-Leibniz rule**, of the form $d(\alpha\beta) = d(\alpha)\beta + q^{\partial\alpha}\alpha d(\beta)$, where $q$ is a given complex number distinct of 0 and where $\partial\alpha$ denotes the degree of $\alpha$. Furthermore, whenever $q^N = 1$, for an integer $N \geq 1$, we shall add the rule $d^N = 0$. We shall refer to $d$ as **the $q$-differential** of the graded algebra. Thus an ordinary differential on a graded algebra is just a $(-1)$-differential in this terminology. Our aim is to produce universal objects in this class. Before entering the subject we want to discuss two problems connected with the case $q^N = 1$, i.e. the case where $q$ is a primitive root of the unity.

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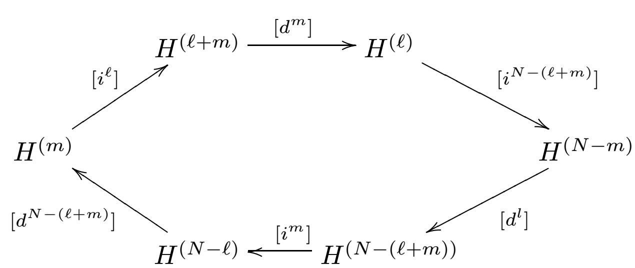
We shall be concerned here with the case of $\mathbb{N}$-graded algebras. However when $q^N = 1$, it is very natural to consider graduation over $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ instead of over $\mathbb{N}$. So let us recall how one can identify a $\mathbb{Z}_N$-graded algebra with a $\mathbb{N}$-graded one. Let $\mathfrak{A} = \oplus_{p\in\mathbb{Z}_N} \mathfrak{A}^p$ be a $\mathbb{Z}_N$-graded algebra and let $n \mapsto p(n)$ be the canonical projection of $\mathbb{N}$ onto $\mathbb{Z}_N$. We associate to $\mathfrak{A}$ a $\mathbb{N}$-graded algebra $p^*\mathfrak{A} = \underset{n\in\mathbb{N}}{\oplus} p^*\mathfrak{A}^n$ in the following manner. A homogeneous element of $p^*\mathfrak{A}$ is a pair $(n,\alpha) \in \mathbb{N}\times\mathfrak{A}$ of an integer $n \in \mathbb{N}$ and an homogeneous element $\alpha$ of $\mathfrak{A}$ such that $\partial\alpha = p(n)$ and we identify $p^*\mathfrak{A}^n = (n, \mathfrak{A}^{p(n)})$ with the vector space $\mathfrak{A}^{p(n)}$. The product in $p^*\mathfrak{A} = \oplus p^*\mathfrak{A}^n$ is then defined by $(m,\alpha)(n,\beta) = (m+n, \alpha\beta)$. The canonical projection $\pi\colon p^*\mathfrak{A} \to \mathfrak{A}$ defined by $\pi(n,\alpha) = \alpha$ is an algebra homomorphism which is graded in the sense that one has $\pi(p^*\mathfrak{A}^n) \subset \mathfrak{A}^{p(n)}$. The $\mathbb{N}$-graded algebra $p^*\mathfrak{A}$ is characterized, up to an isomorphism, by the following universal property: Any graded homomorphism $\varphi$ of a $\mathbb{N}$-graded algebra $\Omega$ into $\mathfrak{A}$ factorizes through a unique homomorphism of $\mathbb{N}$-graded algebra $\bar\varphi\colon \Omega \to p^*\mathfrak{A}$ as $\varphi = \pi\circ\bar\varphi$. Let $D$ be a homogeneous linear mapping of $\mathfrak{A}$ into itself and let $k$ be the unique positive integer strictly smaller than $N$ such that $p(k)$ is the degree of $D$. Then there is a unique linear mapping $p^*(D)$ of $p^*\mathfrak{A}$ into itself which is homogeneous of degree $k$ and satisfies $\pi\circ p^*(D) = D\circ\pi$. The construction of $p^*(D)$ is obvious.

As already stressed, we impose $d^N = 0$ whenever $q^N = 1$. More generally let $E$ be a vector space equipped with an endomorphism $d$ satisfying $d^N = 0$, $N$ being an integer greater than or equal to 2. For each integer $k$ with $0 \le k \le N$, one has $\mathrm{Im}(d^{N-k}) \subset \ker(d^k)$ so the vector space $H^{(k)} = \ker(d^k)/\mathrm{Im}(d^{N-k})$ is well defined. One has $H^{(0)} = H^{(N)} = 0$ and the $H^{(k)}$ for $1 \le k \le N-1$ are the generalized homologies of $E$. Let $\ell$ and $m$ be two positive integers such that $\ell + m \le N$. The inclusion $i^\ell\colon \ker(d^m) \subset \ker(d^{\ell+m})$ induces a linear mapping $[i^\ell]\colon H^{(m)} \to H^{(\ell+m)}$ since $\mathrm{Im}(d^{N-m}) \subset \mathrm{Im}(d^{N-(\ell+m)})$. On the other hand, one has $d^m(\ker(d^{\ell+m})) \subset \ker(d^\ell)$ and $d^m(\mathrm{Im}(d^{N-(\ell+m)})) \subset \mathrm{Im}(d^{N-\ell})$ and therefore $d^m$ induces a linear mapping $[d^m]\colon H^{(\ell+m)} \to H^{(\ell)}$. One has the following result.

**Lemma 1.** *The hexagon* $(\mathcal{H}^{\ell,m})$ *of homomorphisms*




*is exact.*

*Proof.* It is clearly sufficient to show that the sequences $H^{(m)} \xrightarrow{[i^\ell]} H^{(\ell+m)} \xrightarrow{[d^m]} H^{(\ell)}$ and $H^{(\ell+m)} \xrightarrow{[d^m]} H^{(\ell)} \xrightarrow{[i^{(N-(\ell+m))}]} H^{(N-m)}$ are exact. It is straightforward that $[d^m]\circ[i^\ell]$ is the zero mapping of $H^{(m)}$ into $H^{(\ell)}$ and that $[i^{N-(\ell+m)}]\circ[d^m]$ is the zero mapping of $H^{(\ell+m)}$ into $H^{(N-m)}$. Let $c\in\ker(d^{(\ell+m)})$ be such that $d^m c = d^{N-\ell}c'$ for some $c'\in E$; Then $d^m(c-d^{N-(\ell+m)}c')=0$ i.e. $[c]\in H^{(\ell+m)}$ is in $[i^\ell](H^{(m)})$ which achieves the proof of the exactness of the first sequence. Let $c\in\ker(d^\ell)$ be such that $c=d^m c'$ for some $c'\in E$; then one has $d^{\ell+m}c'=0$ which means that $[c]\in H^{(\ell)}$ is in $[d^m](H^{(\ell+m)})$ which achieves the proof of the exactness of the second sequence. □

Notice that the content of this lemma is nontrivial only if $\ell\geq 1$, $m\geq 1$ and $N-(\ell+m)\geq 1$ which implies $N\geq 3$. In the case $N=3$ the only nontrivial choice is $(\ell,m)=(1,1)$ so, in this case there is only one (nontrivial) hexagon, namely $(\mathcal{H}^{1,1})$. In the case $N=4$, there are 3 possible choices for $(\ell,m)$ namely $(\ell,m)=(1,1),(\ell,m)=(1,2)$ and $(\ell,m)=(2,1)$. However it is readily seen that, (for $N=4$), $(\mathcal{H}^{1,1}),(\mathcal{H}^{1,2})$ and $(\mathcal{H}^{2,1})$ are identical; one passes from one to the others by applying "rotations of $2\pi/3$". Thus, for a given integer $N\geq 3$, it is not completely obvious to count the number of independent nontrivial hexagons. In any case, this lemma is very useful for the computations. Practically we shall apply it in the graded case where $E=\oplus_{n\in\mathbb{Z}}E^n$ is a $\mathbb{Z}$-graded vector space and where $d$ is homogeneous of degree 1, (i.e. $d(E^n)\subset E^{n+1}$). In this case, the hexagon $(\mathcal{H}^{\ell,m})$ of the lemma splits into $N$ long exact sequences $(\mathcal{S}^{\ell,m}_p)$, $p\in\{0,1,\dots,N-1\}$.

$$(\mathcal{S}^{\ell,m}_p)\qquad \cdots\longrightarrow H^{(m),Nr+p}\xrightarrow{[i^\ell]}H^{(\ell+m),Nr+p}\xrightarrow{[d^m]}H^{(\ell),Nr+p+m}$$
$$\xrightarrow{[i^{N-(\ell+m)}]}H^{(N-m),Nr+p+m}\xrightarrow{[d^\ell]}H^{(N-(\ell+m)),Nr+p+\ell+m}$$
$$\xrightarrow{[i^m]}H^{(N-\ell),Nr+p+\ell+m}\xrightarrow{[d^{N-(\ell+m)}]}H^{(m),N(r+1)+p}\xrightarrow{[i^\ell]}\cdots$$

where $H^{(k),n}=\{x\in E^n|d^k(x)=0\}/d^{N-k}(E^{n+k-N})$. Notice that, if instead of being graded over $\mathbb{Z}$, $E$ is graded over $\mathbb{Z}_N$ then the $N$ exact sequences $(\mathcal{S}^{\ell,m}_p)$ are again $N$ exact hexagons because in $\mathbb{Z}_N$ one has $Nr+p=N(r+1)+p=p$.

In degree 0, the $q$-Leibniz rule reduces to the ordinary Leibniz rule. Thus a $q$-differential induces a derivation of the subalgebra of elements of degree 0 into the space of elements of degree 1 which is a bimodule over the algebra of elements of degree 0. In this context, let us recall the construction of the universal derivation [**CE**]. Let $\mathcal{A}$ be a unital associative algebra and let $\Omega^1(\mathcal{A})$ be the kernel of the product $\mathfrak{m}\colon\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ of $\mathcal{A}$, $\mathfrak{m}(x\otimes y)=xy$. The mapping $\mathfrak{m}$ is a bimodule homomorphism so $\Omega^1(\mathcal{A})$ is a bimodule over $\mathcal{A}$. One defines a derivation $d$ of $\mathcal{A}$ into $\Omega^1(\mathcal{A})$ by $d(x)=\mathbf{1}\otimes x-x\otimes\mathbf{1}$ for $x\in\mathcal{A}$. The derivation $d$ is universal in the sense that **for any derivation $X$ of $\mathcal{A}$ into a bimodule $\mathcal{M}$ over $\mathcal{A}$, there is a unique bimodule homomorphism $i_X$ of $\Omega^1(\mathcal{A})$ into $\mathcal{M}$ such that** $X=i_X\circ d$. This universal property characterizes the pair $(\Omega^1(\mathcal{A}),d)$ uniquely, up to an isomorphism. We proceed now to recall the construction of the universal differential calculus over $\mathcal{A}$ [**Kar**]. Set $\Omega^0(\mathcal{A})=\mathcal{A}$ and $\Omega^n(\mathcal{A})=\otimes^n_{\mathcal{A}}\Omega^1(\mathcal{A})$.

The direct sum $\Omega(\mathcal{A}) = \oplus_n \Omega^n(\mathcal{A})$ is an associative graded algebra for the tensor product over $\mathcal{A}$; it is in fact the tensor algebra over $\mathcal{A}$ of the bimodule $\Omega^1(\mathcal{A})$. The derivation $d\colon \mathcal{A} \to \Omega^1(\mathcal{A})$ extends uniquely into a differential (i.e. a $(-1)$-differential) of $\Omega(\mathcal{A})$ which will be again denoted by $d$. Thus, $\Omega(\mathcal{A})$ is a graded differential algebra, (i.e. a graded $(-1)$-differential algebra in the sense of the definition of next section). This graded differential algebra is characterized, up to an isomorphism, by the following universal property: **Any homomorphism of associative unital algebra $\varphi$ of $\mathcal{A}$ into the algebra $\mathfrak{A}^0$ of the elements of degree 0 of a graded differential algebra $\mathfrak{A} = \oplus_{n\in\mathbb{N}} \mathfrak{A}^n$ extends uniquely into a homomorphism of graded differential algebra $\bar{\varphi}\colon \Omega(\mathcal{A}) \to \mathfrak{A}$.** This is why $\Omega(\mathcal{A})$ is called **the universal differential envelope** of $\mathcal{A}$ or **the universal differential calculus over $\mathcal{A}$**. It is one of the aims of this paper to generalize this construction (corresponding to $q = -1$) for the $q$-differential calculus.

## 2. $q$-differential Calculus

In the rest of the paper $q$ is a complex number with $q \neq 0$ and we shall use the following definition.

**Definition 1.** A graded $q$-differential algebra is a $\mathbb{N}$-graded unital $\mathbb{C}$-algebra $\mathfrak{A} = \underset{n\in\mathbb{N}}{\oplus} \mathfrak{A}^n$ equipped with an endomorphism $d$ of degree one satisfying $d(\alpha\beta) = d(\alpha)\beta + q^a \alpha d(\beta)$, $\forall \alpha \in \mathfrak{A}^a$ and $\forall \beta \in \mathfrak{A}$, and such that $d^N = 0$ whenever $q^N = 1$ for $N \in \mathbb{N}$ with $N \neq 0$. Let $\mathcal{A}$ be a unital $\mathbb{C}$-algebra. **A $q$-differential calculus over $\mathcal{A}$** is a graded $q$-differential algebra $\mathfrak{A} = \underset{n}{\oplus} \mathfrak{A}^n$ such that $\mathcal{A}$ is a subalgebra of $\mathfrak{A}^0$.

Notice that a graded 1-differential algebra is just a $\mathbb{N}$-graded algebra $(d = 0)$, that a graded $(-1)$-differential algebra is just a $\mathbb{N}$-graded differential algebra in the usual sense and that, if $\mathfrak{A} = \oplus \mathfrak{A}^n$ is a $q$-differential calculus over $\mathcal{A}$ with $q \neq 1$, then the restriction of $d$ to $\mathcal{A}$ is just a derivation of $\mathcal{A}$ into the bimodule $\mathfrak{A}^1$ over $\mathcal{A}$.

Let us introduce, as usual, the $q$-analogs of basic numbers, of factorials and of binomial coefficients

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1},$$
$$[n!]_q = [1]_q [2]_q \ldots [n]_q \text{ and}$$
$$\begin{bmatrix} n \\ p \end{bmatrix}_q = \frac{[n!]_q}{[p!]_q [(n-p)!]_q}$$

where $n, p \in \mathbb{N}$ and $n \geq p$. By induction on $n$, it follows from the $q$-Leibniz rule $d(\alpha\beta) = d(\alpha)\beta + q^a \alpha d(\beta)$ that one has:

$$d^n(\alpha\beta) = \sum_{p=0}^{n} q^{ap} \begin{bmatrix} n \\ p \end{bmatrix}_q d^{n-p}(\alpha) d^p(\beta) \tag{1}$$

for $\alpha \in \mathfrak{A}^a$ and $\beta \in \mathfrak{A}$, $(n \in \mathbb{N})$. It is worth noticing here that the consistency of $d^N = 0$ whenever $q^N = 1$ with the $q$-Leibniz rule follows from the fact that (1) implies for $q^N = 1$ that one has $d^N(\alpha\beta) = d^N(\alpha)\beta + \alpha d^N(\beta)$.

There is an obvious notion of homomorphism of $q$-differential algebra. Given a unital algebra $\mathcal{A}$, a morphism of a $q$-differential calculus over $\mathcal{A}$ into another one is a homomorphism of the corresponding $q$-differential algebra which induces the identity mapping of $\mathcal{A}$ onto itself. It is the aim of Section 4 to produce an initial object in the category of $q$-differential calculi over $\mathcal{A}$, (i.e. a universal graded $q$-differential envelope for $\mathcal{A}$). In the remaining part of this section, we present some examples.

**Example 1: Matrix Algebra** $M_N(\mathbb{C})$

Let $N \in \mathbb{N}$ with $N \geq 2$ and let $q$ be a primitive $N$-root of the unity, (e.g. $q = \exp\left(\frac{2\pi i}{N}\right)$). Let us introduce the usual standard basis $E^k_\ell$, $(k, \ell \in \{1, \dots, N\})$, of the matrix algebra $M_N(\mathbb{C})$ defined by $(E^k_\ell)^i_j = \delta^k_j \delta^i_\ell$. One has the relations

$$E^k_\ell E^r_s = \delta^k_s E^r_\ell \quad \text{and} \quad \sum_{n=1}^{N} E^n_n = \mathbf{1} \tag{2}$$

It follows from (2) that $M_N(\mathbb{C})$ is a $\mathbb{Z}_N$-graded algebra if one equips it with the $\mathbb{Z}_N$-graduation defined by giving the degree $k - \ell \pmod{N}$ to $E^k_\ell$; $M_N(\mathbb{C}) = \bigoplus_{p \in \mathbb{Z}_N} (M_N(\mathbb{C}))^p$. Let $e = \lambda_1 E^2_1 + \cdots + \lambda_{N-1} E^N_{N-1} + \lambda_N E^1_N$, $(\lambda_i \in \mathbb{C})$, be an arbitrary element of degree 1, $(e \in (M_N(\mathbb{C}))^1)$, i.e.

$$e = \begin{pmatrix} 0 & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & & & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 0 \\ 0 & & & & \lambda_{N-1} \\ \lambda_N & 0 & 0 & \dots & 0 \end{pmatrix}$$

One has in view of (2)

$$e^N = \lambda_1 \lambda_2 \dots \lambda_N \mathbf{1}. \tag{3}$$

One defines a linear mapping of degree 1 of $M_N(\mathbb{C})$ into itself by setting $d(A) = eA - q^a Ae$ for $A \in (M_N(\mathbb{C}))^a$. The linear mapping $d$ satisfies the $q$-Leibniz rule

$$d(AB) = d(A)B + q^a A d(B), \quad \forall A \in (M_N(\mathbb{C}))^a, \forall B \in M_N(\mathbb{C}).$$

Moreover (3) implies that $d^N = 0$, (since $d^N = ad(e^N)$ as easily verified). Thus $M_N(\mathbb{C}) = \bigoplus_p (M_N(\mathbb{C}))^p$ equipped with $d$ satisfies the axioms of graded $q$-differential algebra except that it is $\mathbb{Z}_N$-graded instead of being $\mathbb{N}$-graded. However the $\mathbb{N}$-graded covering $p^*M_N(\mathbb{C})$ equipped with $p^*(d)$, (see in Section 1), is a graded $q$-differential algebra. The algebra $(p^*M_N(\mathbb{C}))^0 = (M_N(\mathbb{C}))^0$ of diagonal matrices identifies with the algebra $\mathbb{C}^N$ of complex functions on a set with $N$ elements and therefore the above graded $q$-differential algebra is a $q$-differential calculus over the commutative algebra $\mathbb{C}^N$. Notice that for $N = 2$, $p^*M_2(\mathbb{C})$ is an ordinary graded differential algebra which is isomorphic to the universal differential envelope $\Omega(\mathbb{C}^2)$ of the commutative algebra $\mathbb{C}^2$ whenever $\lambda_1\lambda_2 \neq 0$.

**Example 2: Hochschild Cochains**

Let $\mathcal{A}$ be a unital associative $\mathbb{C}$-algebra and let $\mathcal{M}$ be a bimodule over $\mathcal{A}$. Recall that a $\mathcal{M}$-valued Hochschild cochain of degree $n \in \mathbb{N}$ is a $n$-linear mapping of $\underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{n}$ into $\mathcal{M}$, (i.e. a linear mapping of $\otimes^n \mathcal{A}$ into $\mathcal{M}$). The vector space of $\mathcal{M}$-valued Hochschild cochains of degree $n$ is denoted by $C^n(\mathcal{A}, \mathcal{M})$. The vector space $C(\mathcal{A}, \mathcal{M}) = \oplus_{n\in\mathbb{N}} C^n(\mathcal{A}, \mathcal{M})$ of all $\mathcal{M}$-valued Hochshild cochains is a $\mathbb{N}$-graded vector space. If $\mathcal{M}'$ is another bimodule over $\mathcal{A}$, one defines a graded bilinear mapping of $C(\mathcal{A}, \mathcal{M}) \times C(\mathcal{A}, \mathcal{M}')$ into $C(\mathcal{A}, \mathcal{M} \otimes_{\mathcal{A}} \mathcal{M}')$, $(\alpha, \alpha') \mapsto \alpha \cup \alpha'$, **the cup product**, by setting for $\alpha \in C^a(\mathcal{A}, \mathcal{M})$ and $\alpha' \in C^{a'}(\mathcal{A}, \mathcal{M}')$

$$(\alpha \cup \alpha')(x_1, \dots, x_{a+a'}) = \alpha(x_1, \dots, x_a) \otimes_{\mathcal{A}} \alpha'(x_{a+1}, \dots, x_{a+a'}), \quad \forall x_i \in \mathcal{A}.$$

The cup product if associative in the sense that if $\mathcal{M}''$ is a third bi-module over $\mathcal{A}$, one has for $\alpha \in C(\mathcal{A}, \mathcal{M}), \alpha' \in C(\mathcal{A}, \mathcal{M}')$ and $\alpha'' \in C(\mathcal{A}, \mathcal{M}'')$: $(\alpha \cup \alpha') \cup \alpha'' = \alpha \cup (\alpha' \cup \alpha'')$. By taking $\mathcal{M} = \mathcal{M}' = \mathcal{A} (= \mathcal{M}'')$ and by making the identification $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} = \mathcal{A}$, one sees that, equipped with the cup product, $C(\mathcal{A}, \mathcal{A})$ is a unital $\mathbb{N}$-graded algebra with $C^0(\mathcal{A}, \mathcal{A}) = \mathcal{A}$. Let $q$ be a complex number with $q \neq 0$. One defines a linear endormorphism $\delta_q$ of degree one of $C(\mathcal{A}, \mathcal{M})$ by setting for $\omega \in C^n(\mathcal{A}, \mathcal{M})$, $\delta_1 \omega = 0$ and, for $q \neq 1$:

$$\begin{aligned} (4) \quad \delta_q(\omega)(x_0, \dots, x_n) = x_0\omega(x_1, \dots, x_n) + \sum_{k=1}^{n} q^k \omega(x_0, \dots, x_{k-1}x_k, \dots, x_n) \\ - q^n \omega(x_0, \dots, x_{n-1})x_n \end{aligned}$$

$\forall x_i \in \mathcal{A}$. One verifies that $\delta_q^N = 0$ whenever $q^N = 1$ ($N \neq 0$) and that, if $\beta \in C(\mathcal{A}, \mathcal{M}')$, one has: $\delta_q(\omega \cup \beta) = \delta_q(\omega) \cup \beta + q^n \omega \cup \delta_q(\beta)$. This implies in particular that $C(\mathcal{A}, \mathcal{A})$ equipped with $\delta_q$ is a graded $q$-differential algebra and that it is a $q$-differential calculus over $\mathcal{A}$. Notice that $\delta_{(-1)}$ is the usual Hochschild coboundary $\delta$ so, when $q^N = 1$ ($N \geq 2$), the $H^{(k)}(C(\mathcal{A}, \mathcal{M}), \delta_q)$ defined as in Section 1 are $q$-analog of Hochschild cohomology.

**Example 3: $q$-differential Dual of a Product**

Let $\mathcal{A}$ be an associative $\mathbb{C}$-algebra and let $C(\mathcal{A}) = \underset{n\in\mathbb{N}}{\oplus} C^n(\mathcal{A})$ be the graded vector space of multilinear forms on $\mathcal{A}$; i.e. $C^n(\mathcal{A}) = (\otimes^n\mathcal{A})^*$ is the ($\mathbb{C}$-) dual of $\otimes^n\mathcal{A}$ and $C^0(\mathcal{A}) = \mathbb{C}$. By making the natural identifications $C^n(\mathcal{A})\otimes C^m(\mathcal{A}) \subset C^{n+m}(\mathcal{A})$ one sees that $C(\mathcal{A})$ is canonically a $\mathbb{N}$-graded unital $\mathbb{C}$-algebra, (the product being the tensor product over $\mathbb{C}$). By duality, the product $\mathfrak{m} : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ of $\mathcal{A}$ gives a linear mapping $\mathfrak{m}^*$ of $\mathcal{A}^*$ into $(\mathcal{A}\otimes\mathcal{A})^*$ i.e. $\mathfrak{m}^* \colon C^1(\mathcal{A}) \to C^2(\mathcal{A})$. For $q \in \mathbb{C}\backslash\{0, 1\}$, $\mathfrak{m}^*$ extends into a linear mapping $\mathfrak{m}^*_q \colon C(\mathcal{A}) \to C(\mathcal{A})$ which satisfies the graded $q$-Leibniz rule with

$$\mathfrak{m}^*_q(\omega)(x_0, \dots, x_n) = \sum_{k=1}^{n} q^{k-1}\omega(x_0, \dots, x_{k-1}x_k, \dots, x_n) \tag{5}$$

for $\omega \in C^n(\mathcal{A})$ and $x_i \in \mathcal{A}$. It follows then from the associativity of the product of $\mathcal{A}$ that one has $(\mathfrak{m}^*_q)^N = 0$ whenever $q^N = 1$, $N \in \mathbb{N}\backslash\{0\}$. Thus $C(\mathcal{A})$ equipped with $\mathfrak{m}^*_q$ is a graded $q$-differential algebra. It is worth noticing here that the $\delta_q$ defined by (4) on $C(\mathcal{A}, \mathcal{M})$ in Example 2 is up to a factor $q$ the $\mathfrak{m}^*_q$ defined by (5), (i.e. "the dual" of the product of $\mathcal{A}$) combined with a "$q$-twisted bimodule action" on $\mathcal{M}$. It should also be stressed that the results in Example 2 are true if $\mathcal{A}$ is not unital except that then $C(\mathcal{A}, \mathcal{A})$ is also not unital.

Let us now drop the assumption that $\mathcal{A}$ is associative, i.e. let $\mathcal{A}$ be a complex vector space equipped with a bilinear product $(x, y) \mapsto xy$. Then the formula (5) still defines a homogeneous linear mapping $\mathfrak{m}^*_q$ of degree 1 of $C(\mathcal{A})$ into itself satisfying the $q$-Leibniz rule which extends the dual of the product, but now $q^N = 1$ does not imply $(\mathfrak{m}^*_q)^N = 0$. Let $q$ be a $N$-th primitive root of the unity with $N \geq 2$. For $N = 2$, $(\mathfrak{m}^*_{(-1)})^2 = 0$ is equivalent to the associativity of the product of $\mathcal{A}$. For $N \geq 3$, $(\mathfrak{m}^*_q)^N = 0$ is equivalent to a generalization of degree $N + 1$ of the associativity of the product of $\mathcal{A}$ which is of the form $R_q(x_0 \otimes x_1 \otimes \cdots \otimes x_N) = 0$, $\forall x_i \in \mathcal{A}$, where $R_q$ is a linear mapping of $\otimes^{N+1}\mathcal{A}$ into $\mathcal{A}$. However, it was remarked by Peter W. Michor [**PWM**] that, if $\mathcal{A}$ has a unit, then the relation $R_q = 0$ implies the associativity of the product of $\mathcal{A}$, i.e. $R_{(-1)} = 0$. Let us prove this fact. So let us assume that there is a $\mathbf{1} \in \mathcal{A}$ such that $\mathbf{1}x = x\mathbf{1} = x$, $\forall x \in \mathcal{A}$, and let $q$ be a $N$-th primitive root of the unity with $N \geq 3$. Then one has for $x, y, z \in \mathcal{A}$ and $\omega \in C^1(\mathcal{A})(= \mathcal{A}^*)$, $(\mathfrak{m}^*_q)^N\omega(x, y, \underbrace{\mathbf{1}, \dots, \mathbf{1}}_{N-2}, z) = [N - 2]_q q^{N-2}\omega((x, y)z - x(yz))$. Since $\omega$ is arbitrary this shows that $(\mathfrak{m}^*_q)^N = 0$ implies the associativity of the product of $\mathcal{A}$, (we already know that the associativity of the product of $\mathcal{A}$ implies $(\mathfrak{m}^*_q)^N = 0$ whenever $q^N = 1$, $q \neq 1$ and $N \in \mathbb{N}\backslash\{0\}$).

## 3. The Tensor Algebra over $\mathcal{A}$ of $\mathcal{A} \otimes \mathcal{A}$

In this section $\mathcal{A}$ is a unital associative $\mathbb{C}$-algebra. The tensor product (over $\mathbb{C}$)

$\mathcal{A}\otimes\mathcal{A}$ is in a natural way a bimodule over $\mathcal{A}$. The tensor algebra over $\mathcal{A}$ of the bimodule $\mathcal{A}\otimes\mathcal{A}$ will be denoted by $\mathfrak{T}(\mathcal{A}) = \underset{n\in\mathbb{N}}{\oplus}\mathfrak{T}^n(\mathcal{A})$. This is a unital graded algebra with $\mathfrak{T}^n(\mathcal{A}) = \otimes^{n+1}\mathcal{A}$ and product defined by

$$(x_1\otimes\cdots\otimes x_m)(y_1\otimes\cdots\otimes y_n) = x_1\otimes\cdots\otimes x_{m-1}\otimes x_m y_1\otimes y_2\otimes\cdots\otimes y_n \text{ for } x_i, y_j\in\mathcal{A}.$$

In particular $\mathcal{A}$ coincides with the subalgebra $\mathfrak{T}^0(\mathcal{A})$. As a tensor algebra over $\mathcal{A}$, $\mathfrak{T}(\mathcal{A})$ satisfies a universal property. Here, since $\mathcal{A}$ is unital, $\mathcal{A}\otimes\mathcal{A}$ is the free bimodule generated by $\tau = \mathbf{1}\otimes\mathbf{1}$. Hence $\mathfrak{T}(\mathcal{A})$ is the $\mathbb{N}$-graded algebra generated by $\mathcal{A}$ in degree 0 and by a free generator $\tau$ of degree 1. In fact one has $x_0\otimes\cdots\otimes x_n = x_0\tau x_1\dots\tau x_n$, $\forall x_i\in\mathcal{A}$. Thus the graded algebra $\mathfrak{T}(\mathcal{A})$ is also characterized by the following property.

**Lemma 2.** *Let $\mathfrak{A} = \oplus\mathfrak{A}^n$ be a unital $\mathbb{N}$-graded $\mathbb{C}$-algebra, then for any homomorphism $\varphi\colon \mathcal{A}\to\mathfrak{A}^0$ of unital algebras and for any $\alpha\in\mathfrak{A}^1$, there is a unique homorphism $\mathfrak{T}_{\varphi,\alpha}\colon \mathfrak{T}(\mathcal{A})\to\mathfrak{A}$ of graded algebras which extends $\varphi$, (i.e. $\mathfrak{T}_{\varphi,\alpha}\restriction\mathcal{A} = \varphi$), and is such that $\mathfrak{T}_{\varphi,\alpha}(\tau) = \alpha$.*

As an example of application of this lemma, let us take $\mathfrak{A} = C(\mathcal{A},\mathcal{A})$, i.e. the algebra of $\mathcal{A}$-valued cochains of $\mathcal{A}$ (see Example 2 of Section 2), take for $\varphi$ the identity mapping of $\mathcal{A}$ onto itself considered as a homomorphism of $\mathcal{A}$ into $C^0(\mathcal{A},\mathcal{A})$ and take (again) for $\alpha$ the identity mapping of $\mathcal{A}$ onto itself considered as an element of $C^1(\mathcal{A},\mathcal{A})$. Let $\Psi = \mathfrak{T}_{\varphi,\alpha}\colon \mathfrak{T}(\mathcal{A})\to C(\mathcal{A},\mathcal{A})$ be the corresponding graded-algebra homomorphism. This homomorphism which was considered in [**Mas**] is given by

$$\Psi(x_0\otimes\cdots\otimes x_n)(y_1,\dots,y_n) = x_0y_1x_1\dots y_nx_n \tag{6}$$

We now equip $\mathfrak{T}(\mathcal{A})$ with a structure of graded $q$-differential algebra. Let $q$ be a complex number different from 0 and 1. One has the following lemma

**Lemma 3.** *There is a unique linear mapping $d_q\colon \mathfrak{T}(\mathcal{A})\to\mathfrak{T}(\mathcal{A})$ homogeneous of degree 1 satisfying the q-Leibniz rule such that*

$$d_q(x) = \mathbf{1}\otimes x - x\otimes\mathbf{1} = \tau x - x\tau,\ \forall x\in\mathcal{A},$$

*and*

$$d_q(\tau) = \tau^2,\ (i.e.\ d_q(\mathbf{1}\otimes\mathbf{1}) = \mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}).$$

*Moreover $d_q$ satisfies $d_q^N = 0$ whenever $q^N = 1$ for $N\geq 2$, $N\in\mathbb{N}$.*

*Proof.* It follows from the very structure of $\mathfrak{T}(\mathcal{A})$ that for any derivation $D$ of $\mathcal{A}$ into the bimodule $\mathfrak{T}^1(\mathcal{A}) = \mathcal{A}\otimes\mathcal{A}$ and for any $\mu\in\mathfrak{T}^2(\mathcal{A}) = \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}$, there is a unique $D_q\colon \mathfrak{T}(\mathcal{A})\to\mathfrak{T}(\mathcal{A})$ satisfying the $q$-Leibniz rule and such that $D_q(x) = D(x)$ for $x\in\mathcal{A}$ and $D_q(\tau) = \mu$. The first part of the lemma follows since ad $(\tau)$ is a derivation of $\mathcal{A}$ into $\mathfrak{T}^1(\mathcal{A})$. On the other hand, by induction on $N \in \mathbb{N}$ with $N \geq 1$, one has for $x \in \mathcal{A}$

$$d_q^N(x) = [N!]_q \tau^{N-1} d_q(x) \quad \text{and} \quad d_q^N(\tau) = [N!]_q \tau^{N+1} \tag{7}$$

so the remaining part of the lemma follows from $[N]_q = 0$ whenever $q^N = 1$ for $N \in \mathbb{N}$ with $N \geq 2$. □

Thus $\mathfrak{T}(\mathcal{A})$ equipped with $d_q$ is a graded $q$-differential algebra and, in fact a $q$-differential calculus over $\mathcal{A}$. One verifies that, if $C(\mathcal{A}, \mathcal{A})$ is equipped with the $\delta_q$ given by (4), then the above homomorphism $\Psi\colon \mathfrak{T}(\mathcal{A}) \to C(\mathcal{A}, \mathcal{A})$ given by (6) is an homomorphism of graded $q$-differential algebra, i.e. one has $\Psi \circ d_q = \delta_q \circ \Psi$. This generalizes the result of [**Mas**] which is the case $q = -1$.

**Remark 1.** There is another natural $q$-differential $d'_q$ on $\mathfrak{T}(\mathcal{A})$ which is defined to be the unique linear mapping of $\mathfrak{T}(\mathcal{A})$ into itself satisfying the $q$-Leibniz rule such that $d'_q(x) = d_q(x) = \tau x - x\tau$ for $x \in \mathcal{A}$ and $d'_q(\tau) = -q\tau^2$. One verifies that $d'^N_q = 0$ whenever $q^N = 1$ for $N \geq 2$, $N \in \mathbb{N}$. Correspondingly, there is another $q$-differential $\delta'_q$ on $C(\mathcal{A}, \mathcal{A})$ which, instead of formula (4), is given by

$$\begin{aligned}\delta'_q(\omega)(x_0, \dots, x_n) = x_0\omega(x_1, \dots, x_n) - \sum_{k=1}^{n} q^{k-1}\omega(x_0, \dots, x_{k-1}x_k, \dots, x_n) \\ - q^n\omega(x_0, \dots, x_{n-1})x_n\end{aligned}$$

and is such that $\Psi \circ d'_q = \delta'_q \circ \Psi$. The same formula gives, more generally, an endomorphism of $C(\mathcal{A}, \mathcal{M})$ for any bimodule $\mathcal{M}$ which has the same properties as $\delta_q$. Notice that all these definitions coincide when $q = -1$.

**Remark 2.** It is worth noticing here that both $q$-differentials $d_q$ and $d'_q$ on $\mathfrak{T}(\mathcal{A})$ coincide on $\mathcal{A} = \mathfrak{T}^0(\mathcal{A})$ with the universal derivation $d\colon \mathcal{A} \to \Omega^1(\mathcal{A}) \subset \mathfrak{T}^1(\mathcal{A})$, (see in Section 1). This is natural in view of the fact that we shall represent the universal $q$-differential envelope of $\mathcal{A}$ as a $q$-differential subalgebra of $\mathfrak{T}(\mathcal{A})$. Now given an arbitrary $\mu \in \mathfrak{T}^2(\mathcal{A})$ there is a unique $\tilde{d}$ on $\mathfrak{T}(\mathcal{A})$ satisfying the $q$-Leibniz rule which extends the universal derivation $d$ and is such that $\tilde{d}(\tau) = \mu$. However in general one does not have $\tilde{d}^N = 0$ when $q$ is a $N$-th primitive root of the unity. The choices $\mu = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} = \tau^2$ and $\mu = -q\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} = -q\tau^2$ are the two choices for which this generically holds.

**Remark 3.** As a graded algebra, the universal differential envelope $\Omega(\mathcal{A})$ of $\mathcal{A}$ is a graded subalgebra of $\mathfrak{T}(\mathcal{A})$. The space $\Omega^n(\mathcal{A})$ is the subspace of $\mathfrak{T}^n(\mathcal{A})$ which is annihilated by applying the multiplication $\mathfrak{m}$ of $\mathcal{A}$ to two consecutive arguments. On the other hand, for $q = -1$, $d_{(-1)}$ is an ordinary differential on $\mathfrak{T}(\mathcal{A})$ for which $\Omega(\mathcal{A})$ is stable. In fact, **$\Omega(\mathcal{A})$ is the smallest differential subalgebra of $\mathfrak{T}(\mathcal{A})$ equipped with $d_{(-1)}$ which contains $\mathcal{A}$.** We shall generalize this result by showing that the universal $q$-differential envelope of $\mathcal{A}$ can be identified with the smallest $q$-differential subalgebra of $\mathfrak{T}(\mathcal{A})$ equipped with $d_q$ which contains $\mathcal{A}$ (i.e. the $q$-differential subalgebra of $\mathfrak{T}(\mathcal{A})$ generated by $\mathcal{A}$).

## 4. Universal $q$-differential Envelope

In this section $\mathcal{A}$ is a unital associative $\mathbb{C}$-algebra with unit denoted by $\mathbf{1}$ and $q \in \mathbb{C}\backslash\{0\}$ as before. If $q$ is a root of the unity, we define $N$ to be the smallest strictly positive integer such that $q^N = 1$, otherwise we set $N = \infty$. Let $d^k(\mathcal{A})$ for $k \in \{1, 2, \dots, N-1\}$ be $N-1$ copies of the vector space $\mathcal{A}/\mathbb{C}\mathbf{1}$, $d^k \colon \mathcal{A} \to d^k(\mathcal{A})$ being the corresponding canonical projections. We extend $d\colon \mathcal{A} \to d(\mathcal{A})$ as a linear mapping, again denoted by $d$, of $\mathcal{A} \oplus d(\mathcal{A}) \oplus d^2(\mathcal{A}) \oplus \cdots \oplus d^{N-1}(\mathcal{A})$ into itself by defining $d\colon d^k(\mathcal{A}) \to d^{k+1}(\mathcal{A})$ to be the canonical isomorphism for $k = 1, 2, \dots, N-2$ and by $d(d^{N-1}(\mathcal{A})) = 0$. The space $\mathcal{A} \oplus \oplus_{k=1}^{N-1} d^k(\mathcal{A})$ is equipped with a structure of graded vector space by giving the degree 0 to the elements of $\mathcal{A}$ and the degree $k$ to the elements of $d^k(\mathcal{A})$ for $k = 1, 2, \dots, N-1$. The endomorphism $d$ is homogeneous of degree 1 and the graded subspace $\mathcal{E} = \oplus_{k=1}^{N-1} d^k(\mathcal{A})$ is preserved by $d$. Notice that the canonical projection $d^k \colon \mathcal{A} \to d^k(\mathcal{A})$ coincides then with $\underbrace{d \circ \cdots \circ d}_{k} \colon \mathcal{A} \to d^k(\mathcal{A})$, etc. so the notations are coherent. Let $T(\mathcal{E})$ be the tensor algebra over the graded vector space $\mathcal{E} = \oplus_{k=1}^{N-1} d^k(\mathcal{A})$. On $T(\mathcal{E})$ there is a unique graduation compatible with the graduation of $\mathcal{E}$ such that it is a graded algebra and on this graded algebra there is a unique extension, again denoted by $d$, of the endormorphism $d$ of $\mathcal{E}$ which satisfies the $q$-Leibniz rule. Namely one has for $x_i \in \mathcal{A}$ and $k_i \in \{1, \dots, N-1\}$

$$
\begin{aligned}
\partial(d^{k_1}(x_1)\dots d^{k_n}(x_n)) &= k_1 + \cdots + k_n,\\
d(d^{k_1}(x_1)\dots d^{k_n}(x_n)) &= \sum_{i=1}^{n} q^{k_1+\cdots+k_{i-1}} d^{k_1}(x_1)\dots d^{k_{i-1}}(x_{i+1})\\
&\qquad \times d^{k_i+1}(x_i) d^{k_{i+1}}(x_{i+1}) \dots d^{k_n}(x_n)
\end{aligned}
$$

where $\partial$ denotes the degree and the product is the tensor product. Formula (1) is satisfied therefore, for $N < \infty$, $d^N = \underbrace{d \circ \cdots \circ d}_{N}$ is a derivation which vanishes on $T(\mathcal{E})$ since it vanishes on $\mathcal{E}$. Thus $T(\mathcal{E})$ is a graded $q$-differential algebra.

Let $\Omega_q(\mathcal{A})$ be defined by $\Omega_q(\mathcal{A}) = \mathcal{A} \otimes T(\mathcal{E})$. The space $\Omega_q(\mathcal{A})$ is a graded vector space with graduation given by $\partial(x \otimes t) = \partial(t)$ for $x \in \mathcal{A}$ and $t \in T(\mathcal{E})$. It is also canonically a left $\mathcal{A}$-module and a graded right $T(\mathcal{E})$-module for the above graduation. One extends all the previous definitions of $d$ to $\Omega_q(\mathcal{A})$ by setting $d(x \otimes t) = \mathbf{1} \otimes d(x)t + x \otimes d(t) = d(x)t + xd(t)$ for $x \in \mathcal{A}$, $t \in T(\mathcal{E})$ and where in the last equality $\mathbf{1} \otimes T(\mathcal{E})$ and $T(\mathcal{E})$ are identified. The endomorphism $d$ satisfies in $\Omega_q(\mathcal{A})$ $d((x\otimes t)t') = d(x\otimes t)t' + q^{\partial(x\otimes t)}(x\otimes t)d(t')$ for $x\otimes t\in\Omega_q(\mathcal{A})$ and $t'\in T(\mathcal{E})$ $(\subset \Omega_q(\mathcal{A}))$. Identifying $\mathcal{A}$ with $\mathcal{A}\otimes\mathbf{1}\subset\Omega_q(\mathcal{A})$, one has the following.

**Lemma 4.** *There is a unique associative product on $\Omega_q(\mathcal{A})$ which extends its structure of $(\mathcal{A}, T(\mathcal{E}))$-bimodule, for which $\Omega_q(\mathcal{A})$ is a graded algebra and for which $d$ satisfies the $q$-Leibniz rule. Then $\Omega_q(\mathcal{A})$ equipped with $d$ is a graded $q$-differential algebra, (i.e. $d^N = 0$ for $N<\infty$).*

*Proof.* What is needed is a product on the right by elements of $\mathcal{A}$. If the $q$-Leibniz is satisfied for $d$, one must have (formula (1))

$$d^n(x)b = d^n(xb) - \sum_{p=1}^{n} \begin{bmatrix} n \\ p \end{bmatrix}_q d^{n-p}(x)d^p(b) \in \Omega_q(\mathcal{A}),\quad \forall x,b\in\mathcal{A}$$

and $\forall n\geq 1$, from which the uniqueness of the product follows if it exists. Define $(yd^n(x))b$ for $y,x,b\in\mathcal{A}$ and $n\geq 1$ by the above formula i.e.

$$(yd^n(x))b = yd^n(xb) - \sum_{p=1}^{n} \begin{bmatrix} n \\ p \end{bmatrix}_q yd^{n-p}(x)d^p(b).$$

By definition, one has $(yd^n(x))b = y(d^n(x)b)$. On the other hand, it follows from the properties of the $q$-binomial coefficients that one has for $y,x,b,c\in\mathcal{A}$ and $n\geq 1$ $(yd^n(x))(bc) = ((yd^n(x))b)c$ and that therefore the product extends uniquely into an associative one by setting $(yt)(y't') = ((yt)y')t'$ for $y,y'\in\mathcal{A}$ and $t,t'\in T(\mathcal{E})$. The fact that $d$ satisfies the $q$-Leibniz rule follows from

$$d^n(x)b = d^k(d^{n-k}(x)b) - \sum_{p=1}^{k} q^{(n-k)p} \begin{bmatrix} k \\ p \end{bmatrix}_q d^{n-p}(x)d^p(b),$$

for $x,b\in\mathcal{A}$ and $n\geq k\geq 1$. Furthermore, for $N<\infty$, one has $d^N=0$ since $d^N$ vanishes on the generators. Thus $\Omega_q(\mathcal{A})$ is a graded $q$-differential algebra. □

**Theorem 1.** *Let $\mathfrak{A} = \bigoplus_{n\in\mathbb{N}} \mathfrak{A}^n$ be a graded $q$-differential algebra and let $\varphi\colon \mathcal{A}\to\mathfrak{A}^0$ be a homomorphism of unital algebras. Then there is a unique homomorphism $\bar\varphi\colon\Omega_q(\mathcal{A})\to\mathfrak{A}$ of graded $q$-differential algebras which induces $\varphi$.*

*Proof.* In any graded $q$-differential algebra one has $d(\mathbf{1}) = 0$, therefore one defines a linear mapping $\varphi_0\colon\mathcal{E}\to\mathfrak{A}$ by setting $\varphi_0(d^kx) = d^k\varphi(x)$ for $x\in\mathcal{A}$ and $k\geq 1$. By the universal property of the tensor algebra $\varphi_0$ extends uniquely into an algebra homomorphism $\varphi_1\colon T(\mathcal{E})\to\mathfrak{A}$. The homomorphism $\varphi_1$ is obviously a homomorphism of graded algebras satisfying $\varphi_1\circ d = d\circ\varphi_1$ so it is a homomorphism of graded $q$-differential algebras. Define the linear mapping $\bar\varphi\colon\Omega_q(\mathfrak{A})\to\mathfrak{A}$ by $\bar\varphi(xt) = \varphi(x)\varphi_1(t)$ for $x\in\mathcal{A}$ and $t\in T(\mathcal{E})$. One has $\bar\varphi(x) = \varphi(x)$ for $x\in\mathcal{A}$,

$\bar{\varphi}((xt)t') = \bar{\varphi}(xt)\bar{\varphi}(t')$ for $t, t' \in T(\mathcal{E})$, $\bar{\varphi} \circ d = d \circ \bar{\varphi}$ and $\bar{\varphi}$ is unique under these conditions. It follows that $\bar{\varphi}$ is in fact a homomorphism of graded $q$-differential algebras which is unique under the condition that $\bar{\varphi} \restriction \mathcal{A} = \varphi$. □

The graded $q$-differential algebra $\Omega_q(\mathcal{A})$ is characterized uniquely up to an isomorphism by the universal property stated in Theorem 1, this is why $\Omega_q(\mathcal{A})$ will be called **the universal $q$-differential envelope of $\mathcal{A}$** or **the universal $q$-differential calculus over $\mathcal{A}$**.

**Proposition 1.** *The canonical homomorphism $\overline{Id}$: $\Omega_q(\mathcal{A}) \to \mathfrak{T}(\mathcal{A})$ induced by the identity mapping of $\mathcal{A}$ onto itself (as in Theorem* 1*) is injective* ($q \neq 0$ *and* $q \neq 1$).

*Proof.* In $\mathfrak{T}(\mathcal{A})$ one has (7)

$$d_q^k(x) = [k!]_q(\mathbf{1}^{\otimes^k} \otimes x - \mathbf{1}^{\otimes^{k-1}} \otimes x \otimes \mathbf{1}) \text{ for } k \in \{1, 2, \dots, N-1\},\ x \in \mathcal{A}.$$

This implies that $\overline{\mathrm{Id}}$ induces an isomorphism of $T(\mathcal{E})$ onto the subalgebra of $\mathfrak{T}(\mathcal{A})$ generated by the $d_q^k(x)$ for $k \in \{1, \dots, N-1\}$ and $x \in \mathcal{A}$. The remaining follows from the fact that the left $\mathcal{A}$-submodule of $\mathfrak{T}(\mathcal{A})$ generated by the image of $T(\mathcal{E})$, is freely generated, i.e. is isomorphic to $\mathcal{A} \otimes T(\mathcal{E})$. □

Thus, one can identify $\Omega_q(\mathcal{A})$ with the $q$-differential subalgebra of $\mathfrak{T}(\mathcal{A})$ generated by $\mathcal{A}$. This generalizes the standard representation of the usual universal differential envelope of $\mathcal{A}$, [**Kar**], which is the case $q = -1$.

There is another approach of the construction of $\Omega_q(\mathcal{A})$ as $q$-differential subalgebra of $\mathfrak{T}(\mathcal{A})$ which we now sketch. This approach is based on the universal Hochschild cocycles [**CKMV**], [**CQ**]. Recall that a derivation $X$ of $\mathcal{A}$ into a bimodule $\mathcal{M}$ is a $\mathcal{M}$-valued Hochschild 1-cocycle. If $X_i$: $\mathcal{A} \to \mathcal{M}_i$ are $n$ derivations of $\mathcal{A}$ into bimodules $\mathcal{M}_i$, then their cup product $X_1 \cup \cdots \cup X_n : \otimes^n \mathcal{A} \to \mathcal{M}_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_n$ is a $\mathcal{M}_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_n$-valued Hochschild $n$-cocycle. This cocycle is normalized in the sense that it vanishes whenever one of its arguments is the unit $\mathbf{1}$ of $\mathcal{A}$. Consider in particular the universal derivation $d$: $\mathcal{A} \to \Omega^1(\mathcal{A})$, (see in Section 1). By taking the cup product $n$ times with itself of $d$, one obtains a normalized $n$-cocycle $d^{\cup^n}$: $\otimes^n \mathcal{A} \to \Omega^n(\mathcal{A})$ which is defined by $d^{\cup^n}(x_1, \cdots, x_n) = d(x_1) \cdots d(x_n)$. It turns out that this normalized $n$-cocycle is universal, [**CKMV**], [**CQ**], in the sense that **for any normalized Hochschild $n$-cocycle $c$: $\otimes^n \mathcal{A} \to \mathcal{M}$ of $\mathcal{A}$ into a bimodule $\mathcal{M}$, there is a unique bimodule homomorphism $i_c$ of $\Omega^n(\mathcal{A})$ into $\mathcal{M}$ such that** $c = i_c \circ d^{\cup^n}$. Furthermore one can characterize the triviality of $c$ in terms of the homomorphism $i_c$, [**CKMV**]. For that one notices that one has the inclusions of bimodules $\Omega^n(\mathcal{A}) \subset \mathcal{A} \otimes \Omega^{n-1}(\mathcal{A}) \subset \mathfrak{T}^n(\mathcal{A})$ for $n \geq 1$. More precisely one has an exact sequence ($n \geq 1$)

$$0 \to \Omega^n(\mathcal{A}) \xrightarrow{\subset} \mathcal{A} \otimes \Omega^{n-1}(\mathcal{A}) \xrightarrow{\mathrm{m}} \Omega^{n-1}(\mathcal{A}) \to 0.$$

where $\mathfrak{m}$ is the multiplication of $\Omega(\mathcal{A})$. The cocycle $d^{\cup^n}$ is non-trivial, however it is trivial if it is considered as a $\mathcal{A}\otimes\Omega^{n-1}(\mathcal{A})$-valued cocycle because one has there:

$$dx_1\dots dx_n = -\Big(x_1\otimes dx_2\cdots dx_n+\sum_{k=1}^{n-1}(-1)^k\mathbf{1}\otimes dx_1\dots d(x_kx_{k+1})\dots dx_n \\ +(-1)^n\mathbf{1}\otimes(dx_1\dots dx_{n+1})x_n\Big)$$

i.e. $d^{\cup^n}=\delta(-\mathbf{1}\otimes d^{\cup^{n-1}})$ in $\mathcal{A}\otimes\Omega^{n-1}(\mathcal{A})$, where $\delta$ is the Hochschild coboundary ($\delta=\delta_{(-1)}$). Therefore, if the $\mathcal{M}$-valued normalized $n$-cocycle $c$ is such that $i_c$ is the restriction to $\Omega^n(\mathcal{A})$ of a bimodule homomorphism $\varphi\colon\mathcal{A}\otimes\Omega^{n-1}(\mathcal{A})\to\mathcal{M}$, then it is trivial because one has $c=\delta(\varphi(-\mathbf{1}\otimes d^{\cup^{n-1}}))$. Conversely, if $c=\delta(c')$ then by setting $c'=\varphi(-\mathbf{1}\otimes d^{\cup^{n-1}})$ one defines an extension $\varphi$ of $i_c$ to $\mathcal{A}\otimes\Omega^{n-1}(\mathcal{A})$. Let us apply this to the construction of $\Omega_q(\mathcal{A})$. So let $q$ be a complex number different from 0 and 1 and let $\mathfrak{A}=\oplus\mathfrak{A}^n$ be an arbitrary graded $q$-differential algebra with $\mathfrak{A}^0=\mathcal{A}$. As already stressed, the $q$-differential $d_{\mathfrak{A}}$ of $\mathfrak{A}$ induces a derivation of $\mathcal{A}$ into $\mathfrak{A}^1$ so one must take $\Omega^1_q(\mathcal{A})=\Omega^1(\mathcal{A})$ and the $q$-differential of $\Omega_q(\mathcal{A})$ must induce the universal derivation $d\colon\mathcal{A}\to\Omega^1(\mathcal{A})$. Then the normalized 2-cocycle $d_{\mathfrak{A}}\cup d_{\mathfrak{A}}$ induces a unique bimodule homomorphism $i_2=i_{d_{\mathfrak{A}}\cup d_{\mathfrak{A}}}$ of $\Omega^2(\mathcal{A})$ into $\mathfrak{A}^2$ so $\Omega^2(\mathcal{A})\subset\Omega^2_q(\mathcal{A})$. However the $q$-Leibniz rules implies $d^2_{\mathfrak{A}}(xy)=xd^2_{\mathfrak{A}}(y)+d^2_{\mathfrak{A}}(x)y+(1+q)d_{\mathfrak{A}}(x)d_{\mathfrak{A}}(y)$. So if $q\neq-1$ then $d_{\mathfrak{A}}\cup d_{\mathfrak{A}}=\delta\left(-\frac{1}{1+q}d^2_{\mathfrak{A}}\right)=\delta\left(-\frac{1}{[2]_q}d^2_{\mathfrak{A}}\right)$. Therefore (if $q\neq-1$), by the above discussion, $i_2$ has a unique extension as a bimodule homomorphism $\varphi\colon\mathcal{A}\otimes\Omega^1(\mathcal{A})\to\mathfrak{A}^2$ such that $d^2_{\mathfrak{A}}(x)=\varphi([2]_q\mathbf{1}\otimes d(x))$, $\forall x\in\mathcal{A}$. It follows that, if $q\neq-1$, one must take $\Omega^2_q(\mathcal{A})=\mathcal{A}\otimes\Omega^1(\mathcal{A})$ $(\subset\mathfrak{T}^2(\mathcal{A}))$ and $d^2(x)=[2]_q\mathbf{1}\otimes d(x)=[2]_q\tau d(x)$ which is, in view of (7), the formula induced by the $q$-differential $d_q$ of $\mathfrak{T}(\mathcal{A})$. Although it becomes a little cumbersome, one can continue the construction of $\Omega_q(\mathcal{A})$ as $q$-differential subalgebra of $\mathfrak{T}(\mathcal{A})$ along this line, (by using the formula (1) and the universal cocycles, etc).

## 5. Conclusion

In this paper we have generalized several constructions of ordinary differential algebra to $q$-differential algebra. When $q$ is a primitive $N$-th root of the unity, (e.g. $q=\exp\left(\frac{2\pi i}{N}\right)$), with $N\geq 2$, it is natural to ask what is the generalized cohomology $H^{(p),n}$ ($p=1,\dots,N-1$, $n\in\mathbb{N}$) of the various graded $q$-differential algebras introduced here. The computation of these generalized cohomologies will be described in a separate paper [**D-V**], we just give here the results. For the graded $q$-differential algebras $(C(\mathcal{A}),\mathfrak{m}^*_q)$, $(\mathfrak{T}(\mathcal{A}),d_q)$ and $\Omega_q(\mathcal{A})$ of Example 3, of Section 3 and of Section 4, these generalized cohomologies are trivial as expected, i.e. one has $H^{(p),n}=0$ for $n\geq 1$ and $H^{(p),0}=\mathbb{C}$, $p\in\{1,2,\dots,N-1\}$. For the case of the generalized Hochschild cohomology i.e. of $(C(\mathcal{A},\mathcal{M}),\ \delta_q)$ of Example 2 the result is the following: If $\mathcal{A}$ **is unital**, then one has $H^{(p),Nk}=H^{2k}$ and

$H^{(p),N(k+1)-p} = H^{2(k+1)-1}$ for $p \in \{1,\dots,N-1\}$ and $k \in \mathbb{N}$, where $H^n$ denotes the usual Hochschild cohomology, and $H^{(p),r} = 0$ otherwise i.e. if $r \neq 0 \bmod(N)$ and $r+p \neq 0 \bmod(N)$. Thus, for unital algebras, the information contained in the generalized Hochschild cohomology is the same as the one of ordinary Hochschild cohomology.

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