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k–MINIMAL TRIANGULATIONS OF SURFACES

A. MALNIČ[1] AND R. NEDELA[2]

ABSTRACT. A triangulation of a closed surface is k-minimal ($k \geq 3$) if each edge belongs to some essential k-cycle and all essential cycles have length at least k. It is proved that the class of k-minimal triangulations is finite (up to homeomorphism). As a consequence it follows, without referring to the Robertson-Seymour's theory, that there are only finitely many minor-minimal graph embeddings of given representativity. In the topological part, certain separation properties of homotopic simple closed curves are presented.

1. Introduction

Let G be a **graph** (possibly with loops and multiple edges) **embedded** in a **closed surface** $\Sigma \not\approx S^2$ (cf. [**10, 24**]). The **representativity** $rp_\Sigma G$ [**21, 24**] is defined as $\min |\{z \in S^1 : G \cap \gamma(z) \neq \emptyset\}|$, where the minimum is taken over all homotopically nontrivial (essential) closed paths $\gamma\colon S^1 \to \Sigma$. The minimum can be taken just over simple paths which intersect G in vertices only, and which moreover traverse each face of the embedding at most once. If $\mathcal{T}$ is a **triangulation** of Σ (a **triangular embedding** of a **simplicial graph**), then $rp_\Sigma \mathcal{T}$ is the length of the shortest essential cycle of $\mathcal{T}$.

Let $k \geq 3$ be a natural number. Triangulations with $rp_\Sigma \mathcal{T} \geq k$ can be described as an **inductive class** where the **generating rule** is the standard **vertex-splitting** operation as illustrated in Figure 1. The **base** of this inductive class is the class of k**-minimal triangulations of** Σ. One easily proves the following proposition.

Proposition 1.1. *A triangulation $\mathcal{T}$ is k-minimal if and only if $rp_\Sigma \mathcal{T} = k$ and each edge of $\mathcal{T}$ belongs to some essential k-cycle of $\mathcal{T}$.*

These triangulations do not have nice symmetry if k is sufficiently large; for instance, their automorphism groups are not arc-transitive. Barnette [**3**] found the 3-minimal base for the projective plane, Lavrenchenko [**13**] computed the 3-minimal base for the torus. Barnette and Edelson [**4, 5**] proved that for each

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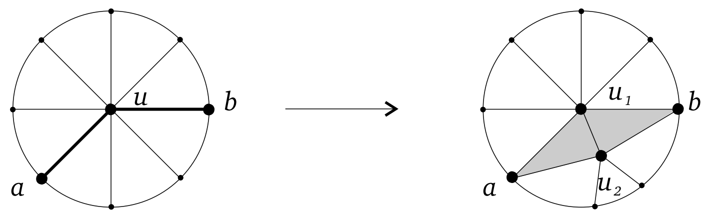

Figure 1. The vertex-splitting.

closed surface the 3-minimal base is finite. A triangulation is **locally-cyclic** ([**8, 11, 14, 18, 20**]) if, for each vertex, the induced subgraph on the set of neighbours is isomorphic to some cycle. By vertex-splitting, where producing vertices of degree 3 is forbidden, the locally-cyclic triangulations are generated from the **irreducible** ones. It is easy to see that a k-minimal triangulation, where $k \geq 4$, is locally-cyclic. Moreover, it is 4-minimal if and only if it is irreducible locally-cyclic. In the present terminology, the result proved in [**14**] states that the class of 4-minimal triangulations of orientable closed surfaces is finite. Fisk, Mohar and Nedela [**8**] computed the 4-minimal base for the projective plane. Our main result is the following.

Theorem 1.2 (Main Theorem). *The class of k-minimal triangulations $(k \geq 3)$ is finite (up to homeomorphism of embeddings) for each closed surface $\Sigma \not\approx S^2$.*

The proof is similar to that of [**14**] using homotopy techniques. However, some key steps are different and moreover, a more accurate study of certain separation properties of simple closed curves on surfaces is needed (see Section 2). These topological results seem not to appear in literature. At the end we present an application of Main Theorem. We give an elementary proof that every closed surface $\Sigma \not\approx S^2$ admits only finitely many minor-minimal embeddings of given representativity (which otherwise follows from the Robertson-Seymour's proof of the Wagner's conjecture).

2. Simple Curves on Surfaces

It is assumed that the reader is familiar with standard textbooks on analysis, algebraic topology and general topology (cf. [**1, 7, 16, 19**]), with the theory of covering spaces and with the homotopy theory in particular. Throughout the paper, $\mathcal{M}$ denotes an arbitrary (connected) 2-**manifold** (and **compact**, with the exception of an open disc/plane). By $\chi_{\mathcal{M}}$ we denote its **Euler characteristic**, by $g_{\mathcal{M}}$ its **genus** (orientable or nonorientable), by $\partial\mathcal{M}$ its boundary and by int $\mathcal{M}$ the interior of $\mathcal{M}$. If $\Omega \subset \mathcal{M}$ is a subset then $\bar{\Omega}$ denotes its **closure**, Ω° its **interior** and fr Ω its **frontier**. It is a consequence of the Schoenflies' theorem that a graph $G \subset \operatorname{int} \mathcal{M}$ has a **regular neighbourhood** N_G in $\mathcal{M}$. This is a compact surface with boundary obtained by "small" disjoint discs around vertices plus disjoint "strips" along the edges. The **separating regions** (or **faces**) are the components of $\mathcal{M} \backslash G$ while the corresponding components of $\mathcal{M} \backslash \operatorname{int} N_G$ are the **dissecting surfaces** obtained by **cutting** $\mathcal{M}$ **along** G. If $\mathcal{M} \not\approx S^2$ is not the 2-sphere and if a separating region R is an open disc, we write $R = R_W = R_{W^{-1}}$, where W is the closed walk in G "representing its boundary". By $\gamma_1 \cong \gamma_2$ we denote **free homotopy** of **closed paths** $\gamma_1, \gamma_2 \colon S^1 \to \mathcal{M}$, where $S^1 \subset \boldsymbol{C}$ is the unit circle, while by $\gamma_1 \cong \gamma_2 \,|_{x_0}$ and $\gamma_1 \cong \gamma_2 \,|_{x_0,x_1}$ we denote the **relative homotopy** of **paths** $\gamma_1, \gamma_2 \colon (S^1, 1) \to (\mathcal{M}, x_0)$ and $\gamma_1, \gamma_2 \colon ([0,1], 0, 1) \to (\mathcal{M}, x_0, x_1)$, respectively. A **simple (open,closed) curve** is the **image** of a **simple (open,closed) path**. **Different simple curves** are therefore understood as distinct in the set-theoretical sense. By an abuse of language, **homotopic simple closed curves** are to be understood as having homotopic simple closed paths as representatives (i.e., the first is homotopic to the second, or is homotopic to the inverse of the second representative path). Homotopically nontrivial (noncontractible) curves (paths) are called **essential**. The next two propositions are from [**14**].

Proposition 2.1. *Let Γ be a family of pairwise disjoint, pairwise freeley nonhomotopic essential simple closed curves on a closed surface $\Sigma \not\approx S^2$. Then the cardinality $|\Gamma| = 1$ if Σ is the torus or the projective plane, and $|\Gamma| = 3(g_\Sigma - 1)$ otherwise.*

Proposition 2.2. *Let Γ be a "bouquet" of pairwise "internally disjoint", pairwise relatively nonhomotopic essential simple closed curves at a common point $x_0 \in \Sigma$. Then the cardinality $|\Gamma| = 1$ if Σ is the projective plane, and $|\Gamma| = 3(1 - \chi_\Sigma)$ otherwise.*

Let γ_1, γ_2 be two different simple closed curves in $\mathcal{M}$. The connected components of their intersection $\gamma_1 \cap \gamma_2$ are arcs (possibly degenerated to points) and their cardinal number is the **intersection number** $\operatorname{int}(\gamma_1, \gamma_2)$. Assuming that $\operatorname{int}(\gamma_1, \gamma_2)$ is finite (or at least that the intersecting arcs (points) are "separated" by disjoint neighbourhoods when $\mathcal{M}$ is the plane), it is a consequence of the Schoenflies' theorem, that each intersection can be classified as either a **touching** or a **crossing**. Moreover, there are three types of intersections relative to the inherited orientation from S^1: a **crossing (cr)**, a **coherent touching (ct)** and a **noncoherent touching (nt)**. The definitions are obvious. Let $x_0 \in \operatorname{int} \mathcal{M}$ be an isolated point of intersection of $\gamma_1, \gamma_2 \colon (S^1, 1) \to (\mathcal{M}, x_0)$. The **angle between the paths** $\operatorname{ang}(\gamma_1, \gamma_2) = \operatorname{ang}(\gamma_2, \gamma_1)$ in case x_0 is a crossing or a coherent touching is informally shown in Figure 2(cr) and Figure 2(ct), respectively; if x_0 is a noncoherent touching then we distinguish between $\operatorname{ang}(\gamma_1, \gamma_2)$ and $\operatorname{ang}(\gamma_2, \gamma_1)$ as in Figure 2(nt) (note that in this case, the distinction which angle is which

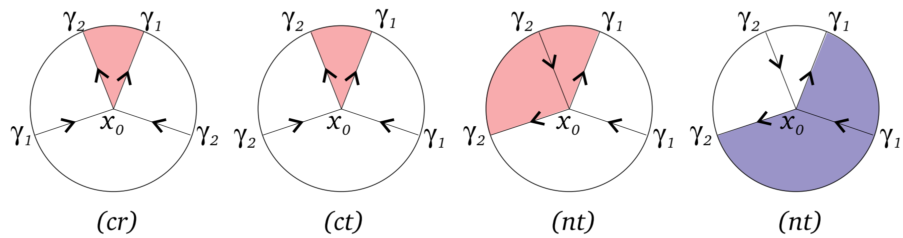

Figure 2. The angle between the paths.

depends on the preselected local orientation of the regular neighbourhood). The formal definition is left to the reader.

There are some well-known results regarding separation properties of simple closed curves with finite intersection number, which date back to Baer, Dehn, Schoenflies and Poincaré. Apart from the fact that a contractible simple closed curve bounds a disc we mention the following (cf. [**7**]): "between" freely homotopic essential simple closed curves with finite positive crossing intersection number there is a disc bounded by two arcs, one of each curve; when the curves are disjoint (and necessarily 2-sided), then the curves bound a cylinder. From this it follows that two 1-sided homotopic simple closed curves cross an odd number of times, while 2-sided an even number of times. But we shall need a more detailed result.

Theorem 2.3. *Let $a_1, a_2 \colon (S^1, 1) \to (\mathcal{M}, u)$ be essential homotopic simple closed paths with u as the single intersection. According to the type of intersection at u, one of the following cases occurs (on the projective plane we only have the case* **(cr)***, and both regions are open discs):*

- **(ct)** *The curves are 2-sided. There are either 2 or 3 separating regions; $R_{a_1 a_2^{-1}}$ is an open disc.*
- **(cr)** *The curves are 1-sided. There are 2 separating regions; $R_{a_1 a_2^{-1}}$ is an open disc.*

Proof. We first show that u cannot be a noncoherent touching. Suppose $u = (nt)$. Then $\mathcal{M}$ is not the projective plane. Consider the universal covering $p \colon \boldsymbol{R}^2 \to \mathcal{M}$. The connected components $C_i \subset p^{-1}(a_i)$ $(i = 1, 2)$ at $\tilde{u}_0 \in p^{-1}(u)$ are 2-way infinite paths because the fundamental group $\pi(\mathcal{M}, u)$ does not have elements of finite order (cf. [**7**] for the proof). Now the lifts $\tilde{a}_i \subset C_i$ of a_i $(i = 1, 2)$ originating at $\tilde{u}_0$ have the same terminal point $\tilde{u}_1 \in p^{-1}(u)$ (cf. [**16**]), and $\tilde{a}_1 \tilde{a}_2^{-1}$ bounds a disc D. Because p is a local homeomorphism, C_1 and C_2 have a noncoherent touching at $\tilde{u}_0$ and so one of C_1, C_2 has a continuation to the interior of D. This continuation in D cannot meet fr D since $C_1 \cap C_2 \subset p^{-1}(u)$. Hence $p^{-1}(u) \cap D$ is a discrete, infinite and bounded set. But this is a contradiction since an infinite and bounded set of an euclidean space has a limit point.

Suppose u is either a coherent touching or a crossing. Then the curves are 2-sided or 1-sided, respectively, and the dissecting surfaces have in all either three boundary components $a_1a_2^{-1}$, a_1 and a_2, or two boundary components $a_1a_2^{-1}$ and a_1a_2, respectively. There is a simple closed path $\delta \cong a_1a_2^{-1} \cong_u 1$ which bounds a disc in $\mathcal{M}$ and is entirely contained in the dissecting surface having $a_1a_2^{-1}$ as the boundary component. Hence the corresponding region must be a disc since a_1 and a_2 are essential. Consequently, there are at least two regions (and clearly not more than three). □

Corollary 2.4. *With notation of Theorem* 2.3*,* ang(a_1, a_2) *always belongs to an open disc having empty intersection with the curves. The same holds for* ang(a_1^{-1}, a_2^{-1}).

Theorem 2.5. *Let* $\gamma_1, \gamma_2 \colon (S^1, 1) \to (\mathcal{M}, u)$ *be essential homotopic simple closed paths with two intersecting points* $\gamma_1 \cap \gamma_2 = \{u, v\}$*. Set* $\gamma_i = a_ib_i$ *where* a_i, b_i *are the* $u - v$ *and* $v - u$ *subpaths of* γ_i $(i = 1, 2)$*. Then one of the following cases occurs, according to the type of intersection at* u *and* v *(on the projective plane we only have cases* **(cr-ct)***,* **(cr-nt)***,* **(ct-cr)***,* **(nt-cr)***, and all regions are open discs):*

- **(ct-ct)** *The curves are* 2*-sided. There are* 3 *or* 4 *separating regions;* $R_{a_1a_2^{-1}}$ *and* $R_{b_1b_2^{-1}}$ *are open discs.*
- **(ct-nt)** *The curves are* 2*-sided. There are* 2 *separating regions;* $R_{a_1b_1b_2^{-1}a_2^{-1}}$ *is an open disc.*
- **(cr-ct)** *The curves are* 1*-sided. There are* 3 *separating regions;* $R_{a_1a_2^{-1}}$ *and* $R_{b_1b_2^{-1}}$ *are open discs.*
- **(cr-nt)** *The curves are* 1*-sided. There are* 2 *or* 3 *separating regions, where* $R_{a_1b_1b_2^{-1}a_2^{-1}}$ *is an open disc.*
- **(ct-cr)** *The curves are* 1*-sided. There are* 3 *separating regions;* $R_{a_1a_2^{-1}}$ *and* $R_{b_1^{-1}b_2}$ *are open discs.*
- **(cr-cr)** *The curves are* 2*-sided. There are* 3 *or* 4 *separating regions.* $R_{a_1a_2^{-1}}$ *and* $R_{b_1b_2^{-1}}$ *are open discs.*
- **(nt-cr)** *The curves are* 1*-sided. There are* 3 *separating regions;* $R_{a_1a_2^{-1}b_2^{-1}b_1}$ *is an open disc. If* $\mathcal{M}$ *is not the projective plane, then exactly one of* $R_{a_1b_2}$*,* $R_{a_2b_1}$ *is an open disc.*

Proof. The proof is done by case to case analysis according to the type of intersection at points u and v. As we shall prove, only seven out of the possible nine cases may actually occur. Also, we shall assume that $\mathcal{M}$ is not the projective plane since this case is easy to check.

Case u = (ct) or (cr). We distinguish two subcases.

Subcase v = (ct) or (nt). First make the two paths disjoint in a small regular neighbourhood of v and then apply Theorem 2.3. If there is a coherent

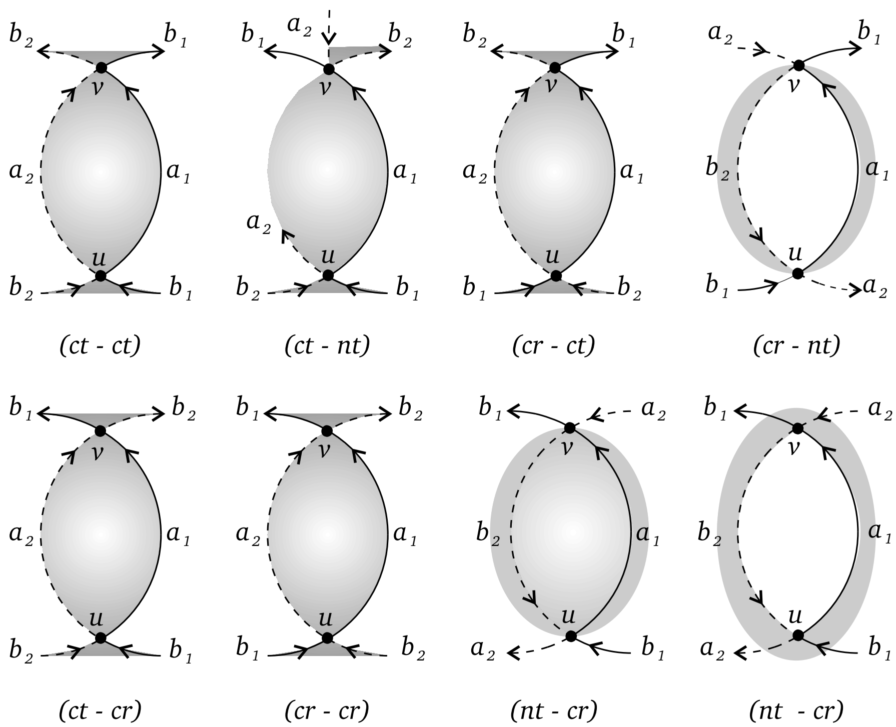

Figure 3. The case int $= 2$.

touching at u we have cases **(ct-ct), (ct-nt)** and if u is a crossing we have cases **(cr-ct), (cr-nt)**. See Figure 3.

Subcase v = (cr). Consider the universal covering $p\colon \boldsymbol{R}^2 \to \mathcal{M}$ and let $\tilde{a}_i\tilde{b}_i$ be the lifted paths of a_ib_i $(i = 1, 2)$ originating at $\tilde{u}_0 \in p^{-1}(u)$. Denote by $\tilde{u}_1 \in p^{-1}(u)$ their common end, and by $\tilde{v}_i \in p^{-1}(v)$ the ends of $\tilde{a}_i$, $(i = 1, 2)$. Clearly, $\tilde{u}_0 \neq \tilde{u}_1$. Suppose $\tilde{v}_1 \neq \tilde{v}_2$.

Then the "quadrilateral" $\tilde{a}_1\tilde{b}_1\tilde{b}_2^{-1}\tilde{a}_2^{-1}$ is a simple closed curve in $\boldsymbol{R}^2$ which bounds a disc D. Consider the connected components $C_i \subset p^{-1}(\gamma_i)$ $(i = 1, 2)$ at $\tilde{u}_0$, $C_2' \subset p^{-1}(\gamma_2)$ at $\tilde{v}_1$ and $C_1' \subset p^{-1}(\gamma_1)$ at $\tilde{v}_2$. These are all 2-way infinite paths. Moreover, by avoiding the limit point contradiction and by the unique path lifting, we have $C_2' = C_2$ and $C_1' = C_1$. Also, C_1 and C_2 continue to the interior of D at exactly one of the points $\tilde{u}_0, \tilde{u}_1$, say, after meeting $\tilde{u}_1$. In the interior, C_1 and C_2 either do not meet or they intersect at $\tilde{u}_2 \in p^{-1}(u)$ (the common end of the lifts of a_ib_i originating at $\tilde{u}_1$). In the second case $\tilde{u}_1$ and $\tilde{u}_2$ are opposite corners of a quadrilateral $D_1 \subset D$. Moreover, since C_1 and C_2 come to $\tilde{u}_1$ from the exterior of D_1, C_1 and C_2 continue to the interior of D_1 at $\tilde{u}_2$. Hence either C_1 and C_2 form an infinite sequence of nested quadrilaterals $D = D_0 \supset D_1 \supset D_2 \ldots$, or we can find a quadrilateral $D_k \subset D$ $(k \geq 0)$ such that C_1 and C_2 do not meet in the interior of D_k. In the first case we have a contradiction via the limit point argument. In the second case, u is necessarily a crossing, and the curves C_1 and C_2 join $\tilde{u}_{k+1} \in p^{-1}(u)$ across the interior of D_k to the two lifts of v on fr D_k. Hence there exist two "digons" whose boundaries project $1-1$ onto a_1b_2 and a_2b_1. Consequently, these projections are contractible simple closed curves in $\mathcal{M}$. This implies $(a_1a_2^{-1})^2 \cong_u 1$, a final contradiction.

It follows that $\tilde{v}_1 = \tilde{v}_2$. But this means that $\tilde{a}_1\tilde{a}_2^{-1}$ and $\tilde{b}_1^{-1}\tilde{b}_2$ project $1-1$ onto $a_1a_2^{-1}$ and $b_1^{-1}b_2$, respectively, and that these projections are contractible simple closed curves. The two possibilities are covered by **(ct-cr), (cr-cr)**. See Figure 3. The number of regions is clear by first performing a homotopic switch of arcs across the disc to obtain cases **(cr-ct), (ct-ct)**.

Case u = (nt). If the second intersection is not a crossing, we first make the curves disjoint in a small neighbourhood of v and then use Theorem 2.3 to obtain a contradiction. Hence there must be a crossing at v and therefore the curves 1-sided. Consider the universal covering $p\colon \boldsymbol{R}^2 \to \mathcal{M}$. We retain the notation as in the previous case. By the limit point argument we have $\tilde{v}_1 \neq \tilde{v}_2$. Therefore, the "quadrilateral" $\tilde{a}_1\tilde{b}_1\tilde{b}_2^{-1}\tilde{a}_2^{-1}$ is a simple closed curve which bounds a disc D. At points $\tilde{u}_0, \tilde{u}_1$, exactly one of C_1, C_2 continues to the interior of D (but clearly not the same one at both points).

Suppose C_1 has a continuation to the interior at $\tilde{u}_0$ and C_2 at $\tilde{u}_1$. If $\tilde{u}_2$ is the common end of the lifts of a_ib_i $(i = 1, 2)$ originating at $\tilde{u}_1$, then $\tilde{u}_2$ is in the exterior of D (otherwise C_1 connects $\tilde{u}_1$ with $\tilde{u}_2$ by crossing fr D at $\tilde{v}_2$, and then continues to $\tilde{u}_0$; thus C_1 is a closed curve, a contradiction). Hence C_2 joins $\tilde{u}_1$ and $\tilde{v}_1$ across the interior of D not meeting C_1, and C_1 joins $\tilde{v}_2$ and $\tilde{u}_0$ not meeting C_2. It follows that D° contains no points of $p^{-1} \cup p^{-1}(v)$. The disc D is composed of a "quadrilateral" $\tilde{R}$ and two "digons" $\tilde{R}', \tilde{R}''$. The covering projection on fr $\tilde{R}'$, fr $\tilde{R}''$ is $1-1$. Hence their projection a_2b_1 is a contractible simple closed curve in $\mathcal{M}$. The projection on fr $\tilde{R}$ fails to be $1-1$ at points $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{v}_0$ and $\tilde{v}_1$. By a small homotopic perturbation at points $\tilde{u}_1$ and $\tilde{v}_2$ it follows that the projection $a_1a_2^{-1}b_2^{-1}b_1$ of fr $\tilde{R}$ bounds a disc with two points of identification at u and v.

In the dual case (when C_1 has a continuation to the interior of D at $\tilde{u}_1$) we have two discs in $\mathcal{M}$ bounded by a_1b_2 and $a_1a_2^{-1}b_2^{-1}b_1$. See Figure 3. This completes the proof of Theorem 2.5. □

Corollary 2.6. *Let $\gamma_1 \cong_u \gamma_2$ be 2-sided essential simple closed paths with at most two intersections. Then they cannot have a noncoherent touching at u.*

Corollary 2.7. *With assumptions and notation as in Theorem 2.5, let u be either a coherent touching or a crossing. Then the angle* $\operatorname{ang}(\gamma_1, \gamma_2)$ *belongs to an open disc whose interior has empty intersection with the curves. The same holds for* $\operatorname{ang}(\gamma_1^{-1}, \gamma_2^{-1})$.

Corollary 2.8. *With assumptions and notation as in Theorem* 2.5 *we have* $a_1b_2 \cong_u a_2b_1$ *(still in the same homotopy class as the original paths) in the two cases* **(ct-cr)**, **(cr-cr)**, *while either* $b_2^{-1}b_1 \cong_u a_2a_1^{-1}$ *or* $a_1a_2^{-1} \cong_u b_1^{-1}b_2$ *in the case* **(nt-cr)**.

Corollary 2.9. *With assumptions and notation as in Theorem* 2.5, *let* u *be a noncoherent touching. Then either* $\operatorname{ang}(\gamma_1, \gamma_2)$ *or* $\operatorname{ang}(\gamma_2, \gamma_1)$ *belongs to an open disc. The same holds for* $\operatorname{ang}(\gamma_1^{-1}, \gamma_2^{-1})$ *or* $\operatorname{ang}(\gamma_2^{-1}, \gamma_1^{-1})$, *respectively.*

3. Essential Edges of a Relative Homotopy Class

We first introduce some nonstandard notation regarding an arbitrary triangulation $\mathcal{T}$ of a fixed closed surface $\Sigma \not\approx S^2$. If $x \neq y$ are points in Σ, let $\#(x, y)$ denote the minimal number of intersections $\mathcal{T} \cap \gamma(0,1)$, where $\gamma\colon [0,1] \to \Sigma$ ranges over all paths joining x and y (note that the endpoints never contribute to this number). Let dist : $\Sigma \times \Sigma \to \boldsymbol{R}$ be a function defined as

$$\operatorname{dist}(x, y) := \begin{cases} 0, & x = y \\ 1 + \#(x, y), & x \neq y. \end{cases}$$

This is a **metric** on Σ and for any pair of vertices $u, v \in V(\mathcal{T})$, $\operatorname{dist}(u, v)$ agrees with the **standard metric** in $\mathcal{T}$. If $u \in V(\mathcal{T})$, then $D(u) = D_1(u) = \{x \in \Sigma \mid \operatorname{dist}(u, x) \leq 1\}$ is a disc, and its frontier is the link cycle $N(u) = N_1(u)$. By $\mathcal{C}_u$ and $\mathcal{C}_u(n)$ we denote the sets of cycles (respectively, n-cycles) in $\mathcal{T}$ at $u \in V(\mathcal{T})$. The respective subsets of the essential ones are denoted by Ess_u and $\operatorname{Ess}_u(n)$, and ess_u is the length of the shortest cycle in Ess_u. The respective subsets with cycles in some nontrivial relative homotopy class Γ at u is denoted by $\mathcal{C}_u(\Gamma)$ and $\mathcal{C}_u(\Gamma, n)$. The edges of $\mathcal{T}$ at u used by $\mathcal{C}_u(\Gamma, n)$ are called (Γ, n)-**essential** and are denoted by $E_u(\Gamma, n)$.

Theorem 3.1. *Let* $\mathcal{T}$ *be a triangulation of* $\Sigma \not\approx S^2$ *and let* Γ *be a nontrivial relative homotopy class at the vertex* u, *where* $\operatorname{ess}_u = rp_\Sigma\mathcal{T} = k \geq 3$. *If* $|E_u(\Gamma, k)| \geq 3$, *there exist cycles* $C_1, C_2 \in \mathcal{C}_u(\Gamma, k)$ *and open discs* R_1, R_2, *each bounded by segments of* C_1 *and* C_2, *such that*

$$E_u(\Gamma, k) \subset R_1 \cup R_2 \cup \{C_1, C_2\}.$$

(Possibly, one of R_1, R_2 *is empty or* $R_1 = R_2$*; also,* R_1, R_2 *have empty intersection with* C_1, C_2*.) The set* $E_u(\Gamma, k)$ *may be written as* $E_u(\Gamma, k) = E_u^1(\Gamma, k) \cup E_u^2(\Gamma, k)$, *where* $|E_u^1(\Gamma, k) \cap E_u^2(\Gamma, k)| \leq 1$, *such that each cycle in* $\mathcal{C}_u(\Gamma, k)$ *uses exactly one edge of* $E_u^1(\Gamma, k)$ *and one of* $E_u^2(\Gamma, k)$.

The proof is performed by induction on k. The induction step consists in contracting the graph within $D(u)$ homotopically to u to obtain the triangulation

$\mathcal{T}' = \mathcal{T}/_{D(u)=u}$. However, some details of this contraction must be carefully analyzed, and we do this in the next lemma. Let $v \in V(\mathcal{T})$ be a vertex such that $\mathrm{ess}_u = n \geq k = rp_\Sigma\mathcal{T} \geq 3$, and let

$$r_u = \begin{cases} 1, & n = 3, 4 \\ \max(2, \lfloor (k-1)/2 \rfloor), & n \geq 5. \end{cases}$$

By $N_r(u)$ we denote the cycle in $\mathcal{T}$ which satisfies three conditions: firstly, all points (as points of Σ) are at distance dist $= r$ from u, secondly, the cycle is planarly embedded in Σ, and thirdly, its bounding disc $D_r(u)$ contains the vertex u. If such a cycle exists, it is unique. (It is not unique if we drop the third requirement.) Observe that points in $D_r(u)$ can have arbitrarily large distance from u.

Lemma 3.2. *Let $\mathcal{T}$ be a triangulation of a closed surface $\Sigma \not\approx S^2$ and let u be a vertex with $\mathrm{ess}_u = n \geq k = rp_\Sigma\mathcal{T} \geq 3$. Then for each $1 \leq r \leq r_u$ the cycle $N_r(u)$ exists. Moreover, if $n \geq 5$ then the triangulation $\mathcal{T}' = \mathcal{T}/_{D(u)=u}$ exists and if Γ is a nontrivial relative homotopy class at u, the following holds:*

- (a) *If $C \in \mathcal{C}_u(\Gamma, n)$, then $C' = C/_{D(u)=u}$ is a cycle in $\mathcal{T}'$ and we have $C' \in \mathcal{C}'_u(\Gamma, n-2)$.*
- (b) *$rp_\Sigma\mathcal{T}' = \min(n-2,\ k)$.*
- (c) *If $n \leq k+2$ then $\mathrm{ess}'_u = n-2$.*
- (d) *If $n \leq k+1$ then the converse to* (a) *is true: to each $C' \in \mathcal{C}'_u(\Gamma, n-2)$ there exists a unique cycle $C \in \mathcal{C}_u(\Gamma, n)$ such that $C' = C/_{D(u)=u}$.*

Proof. The cycle $N_1(u) = N(u)$ exists for all values of $n \geq k \geq 3$. Moreover, if $N_2(u)$ exists then $n \geq 5$. Hence the first part of the lemma holds for $3 \leq k \leq n \leq 4$. We now first prove the existence of $N_2(u)$ for $n \geq 5$.

Let $Z = \{z \in V(C) \mid \mathrm{dist}(z, u) = 2,\ C \in \mathcal{C}_u(4)\}$. This set is nonempty. For each $z \in Z$ there exist vertices $x_z, y_z \in N(u)$ such that the 4-cycle $Q_z = z - x_z - u - y_z - z$ encompasses all other 2-paths $z - u$. Denote by $\alpha_z = x_z - y_z$ the arc on $N(u)$ encompassed by Q_z. Now the following is true. Firstly, the set of arcs $\mathcal{A} = \{\alpha_z \mid z \in Z\}$ covers $N(u)$. Secondly, if two arcs $\alpha_z, \alpha_{z'} \in \mathcal{A}$ have an interior point in common, then one is contained in the other, say $\alpha_z \subseteq \alpha_{z'}$, and $Q_{z'}$ encompasses Q_z (it may happen that $z = z'$). And thirdly, if two arcs in $\alpha_z, \alpha_{z'} \in \mathcal{A}$ have disjoint interiors then none of Q_z, $Q_{z'}$ encompasses the other one. Also, $z \neq z'$. It follows that there exists a (unique) minimal subset $\mathcal{A}_0 \subseteq \mathcal{A}$ (of cardinality greater than 1) still covering $N(u)$, and no arc in $\mathcal{A}_0$ being contained in some "larger"one from $\mathcal{A}$. Now relabel the arcs of $\mathcal{A}_0$ (and their endvertices) as $\alpha_1 = x_1 - y_1$, $\alpha_2 = x_2 - y_2, \ldots$, consistently with some preselected orientation of $N(u)$. For each α_i choose $z_i \in Z$ such that $\alpha_{z_i} = \alpha_i$. If more than one such vertex exists, let z_i be the one for which Q_{z_i} encompasses all other such vertices in Z. The graph formed by the edges $x_i z_i$, $y_i z_i$, $i = 1, 2, \ldots$, is a planar cycle.

For each x_i take the path $z_{i-1} - z_i$ in $N(x_i)$ outside the above planar cycle. The graph formed by the paths $z_{i-1} - z_i$, $i = 1, 2, \dots$, is the required cycle $N_2(u)$.

By contracting the edges of $D(u)$ to u and after replacing all subgraphs bounded by the double adjacencies $u'z_i$, each by a single edge, a simplicial triangulation $\mathcal{T}' = \mathcal{T}/_{D(u)=u}$ is obtained. We now prove statements (a), (b), (c) and (d).

To see (a), let $C \in \mathcal{C}_u(\Gamma, n)$. Since C intersects $N(u)$ in exactly two vertices, it contracts exactly by the two edges incident at u and the homotopy class is clearly preserved. To prove (b) we first establish the inequality $rp_\Sigma \mathcal{T}' \geq \min(n-2, k)$ by showing that any cycle $C' \subset \mathcal{T}'$ of shorter length is planar. Namely, if there is a cycle $C \subset \mathcal{T}$ such that $C' = C$ after contraction, then $|C| = |C'| < k$. If not, then necessarily $u \in C'$. Moreover, the edges of C' in $\mathcal{T}$ must form a "path of attachment" at two different vertices on $N(u)$ in $\mathcal{T}$. Therefore, there is a cycle $C \in \mathcal{C}_u$ which contracts to C' and is of length $|C| = |C'| + 2 < n$. In both cases C is planar and so is C'. To show the reverse inequality note that $\mathrm{Ess}'_u(n-2)$ is not empty by (a). So $rp_\Sigma \mathcal{T}' \leq n-2$. Also, $rp_\Sigma \mathcal{T}' \leq k$ since contraction cannot increase the representativity. To see (c), observe that (b) implies $\mathrm{ess}'_u \geq rp_\Sigma \mathcal{T}' = n-2$. The reverse inequality follows from (a). Next, we prove (d). Since $n \leq k+1$, the edges of $C' \in \mathcal{C}'_u(\Gamma, n-2)$ form a "path of attachment" at two different vertices on $N(u)$ in $\mathcal{T}$. Hence there is a (unique) $C \in \mathcal{C}_u(\Gamma, n)$ which contracts to C'.

Finally, we show the existence of $N_r(u)$ for all $(1 \leq r \leq r_u)$. The proof is done by induction on k. In view of what has been proved above, the statement holds for $3 \leq k \leq 6$ and arbitrary $n \geq k$. In the inductive step we consecutively perform some homotopic contractions at u to obtain a triangulation $\mathcal{T}'$ with $k-2 \leq \mathrm{ess}'_u = rp_\Sigma \mathcal{T}' \leq k-1$. This is guaranteed by (b) and (c) above. The proof is completed after expanding back to $\mathcal{T}$. □

Proof of Theorem 3.1. Assume for the moment that the theorem has already been proved for $k = 3, 4$ and let $k \geq 5$. The statement of the theorem is clear if there is a pair of vertices $a, b \in N_2(u)$ in $\mathcal{T}$ such that $\mathcal{C}_u(\Gamma, k) \cap N_2(u) = \{a, b\}$. If not, we perform the contraction $\mathcal{T}' = \mathcal{T}/_{D(u)=u}$. By Lemma 3.2 we have $\mathrm{ess}'_u = rp_\Sigma \mathcal{T}' = k-2 \geq 3$, and each cycle in $\mathcal{C}'_u(\Gamma, k-2)$ is a contraction of a cycle in $\mathcal{C}_u(\Gamma, k)$. By the induction hypothesis there are cycles $C'_1, C'_2 \in \mathcal{C}'_u(\Gamma, k-2)$ and at most two open discs R'_1, R'_2 satisfying the requirements with respect to $\mathcal{T}'$. The required cycles C_1, C_2 in $\mathcal{T}$ are the ones that contract to C'_1, C'_2. (Note that one of R_1, R_2 may be empty even if none of R'_1, R'_2 is.) The precise description of R_1 and R_2 is left to the reader. It remains to show that the statement of the theorem is true for the starting cases $k = 3, 4$. As already mentioned, we shall here make use of the general topological results of Section 2.

Case $k = 3$**.** We shall adopt the following notation: if $T_i \in \mathcal{C}_u(\Gamma, 3)$, let $T_i = e_i - f_i - g_i$ where e_i is the 1-arc (i.e., an oriented edge) originating at u and

g_i is the one terminating at u. Since $|E_u(\Gamma, 3)| \geq 3$, at least two different such 3-cycles exist. Any two of them have intersection number $= 1$ (possibly along one edge at u).

Let Γ be 2-sided. Each pair of distinct 3-cycles (T_1, T_2) in Γ intersects in a coherent touching by Theorem 2.3. Thus the local rotation ρ_u of 1-arcs originating at u may be expressed, up to cyclic permutation or taking the inverse, as $\rho_u = (e_1, A, e_2, B, g_2^{-1}, C, g_1^{-1}, D)$, where A, B, C and D are chains of 1-arcs originating at u. Possibly, $e_1 = e_2$ or $g_1 = g_2$, but not simultaneously. Assuming that the pair (T_1, T_2) has been chosen so that the sum of the cardinalities $|A| + |C|$ is maximal, we claim that (T_1, T_2) is the required pair. Suppose there is some $T_3 \in \mathcal{C}_u(\Gamma, 3)$ which uses an arc in $B \cup B^{-1} \cup D \cup D^{-1}$ (note that B and D must be nonempty). Reenumerating the cycles and by symmetry (i.e., using the inverse local rotation), or considering Γ^{-1} instead of Γ, we may assume $e_3 \in B$. Since T_3 has a coherent touching both with T_1 and T_2 we essentially have only one possibility according to where in the local rotation the arc g_3^{-1} appears; $\rho_u = (e_1, A, e_2, B_1, e_3, B_2, g_3^{-1}, B_3, g_2^{-1}, C, g_1^{-1}, D)$ (possibly, $g_3 = g_2$). But then (T_1, T_3) contradicts the maximality of (T_1, T_2).

Let Γ be 1-sided. Then each pair of distinct 3-cycles (T_1, T_2) in Γ intersects in a crossing by Theorem 2.3. Hence $\rho_u = (e_1, A, e_2, B, g_1^{-1}, C, g_2^{-1}, D)$. Possibly, $e_1 = e_2$ or $g_1 = g_2$, but not simultaneously. Also, it may happen that $e_1 = g_2^{-1}$ or $e_2 = g_1^{-1}$, but not simultaneously. Let (T_1, T_2) be the maximal pair as before, and suppose that some $T_3 \in \mathcal{C}_u(\Gamma, 3)$ uses an arc of $B \cup B^{-1} \cup D \cup D^{-1}$ (it may happen that $B = D = \emptyset$ in which case there is nothing to prove). Again we may assume $e_3 \in B$. Since T_3 must cross both T_1 and T_2 we again have just one possibility for the local rotation $\rho_u = (e_1, A, e_2, B_1, e_3, B_2, g_1^{-1}, C, g_2^{-1}, D_1, g_3^{-1}, D_2)$ (possibly, $g_3 = g_2$ or $g_3 = e_1^{-1}$; if $e_1 = g_2^{-1}$ then $g_3 = g_2$). Now (T_1, T_3) contradicts the maximality of (T_1, T_2).

Case $k = 4$. We shall use the following notation: if $Q_i \in \mathcal{C}_u(\Gamma, 4)$, let $Q_i = e_i - f_i - g_i - h_i$ where e_i is the 1-arc originating at u and h_i is the one terminating at u. Since $|E_u(\Gamma, 4)| \geq 3$, at least two different such 4-cycles exist, which moreover use different couples of edges at u. Any pair of different 4-cycles has intersection number ≤ 2. If the intersection number $= 1$ then they meet in a path of length ≤ 2 (containing u), and if the intersection number $= 2$ then they meet at two "opposite" vertices.

Let Γ be 2-sided. First of all, we claim that the set of pairs of distinct 4-cycles in Γ coherently touching at u (possibly along one edge containing u) is not empty. Namely, take an arbitrary pair $Q_1, Q_2 \in \mathcal{C}_u(\Gamma, 4)$ using different couples of edges at u. If $\text{int}\,(Q_1, Q_2) = 1$ then by Theorem 2.3 there is nothing to prove. So assume $\text{int}\,(Q_1, Q_2) = 2$ and let u be a crossing. As in Corollary 2.8 we perform a homotopic switch of the paths $g_1 - h_1$ and $g_2 - h_2$ keeping the intersections fixed to obtain 4-cycles $Q'_1 = e_1 - f_1 - g_2 - h_2$ and $Q'_2 = e_2 - f_2 - g_1 - h_1$ in Γ. Of course

Q'_1, Q'_2 coherently touch at u. Since by Corollary 2.6 the noncoherent touching at u cannot occur, the claim is proved.

Let us now consider the set of distinct 4-cycles in Γ, using distinct couples of edges at u. Up to cyclic permutation or taking the inverse, the local rotation of 1-arcs originating at u can be expressed as $\rho_u = (e_1, A, e_2, B, h_2^{-1}, C, h_1^{-1}, D)$. If $\operatorname{int}(Q_1, Q_2) = 1$ we possibly have $e_1 = e_2$ or $h_1 = h_2$, but not simultaneously. Also, the sets B and D must be nonempty. Assume that (Q_1, Q_2) is a maximal pair in the sense that $|A| + |C|$ is maximal. Then (Q_1, Q_2) is the required pair. Indeed, let some $Q_3 \in \Gamma$ use an arc in $B \cup B^{-1} \cup D \cup D^{-1}$. We may assume $e_3 \in B$. Since Q_3 cannot have a noncoherent touching with Q_1 or Q_2 at u, we essentially distinguish three possibilities according to where in the local rotation the arc h_3^{-1} appears. In each case we shall find a pair of 4-cycles contradicting the maximality of (Q_1, Q_2).

Let $\rho_u = (e_1, A, e_2, B_1, e_3, B_2, h_3^{-1}, B_3, h_2^{-1}, C, h_1^{-1}, D)$ (including $h_3 = h_2$). Regardless of the intersection number of (Q_1, Q_2) or that of (Q_1, Q_3) (if it is 2, then the second intersection may either be a coherent or a noncoherent touching), the contradictory pair is (Q_1, Q_3).

Let $\rho_u = (e_1, A, e_2, B_1, e_3, B_2, h_2^{-1}, C_1, h_3^{-1}, C_2, h_1^{-1}, D)$ (including $h_3 = h_1$). Note that $\operatorname{int}(Q_3, Q_2) = 2$ (which means that $h_3 \neq h_2$; thus this case cannot occur if $h_1 = h_2$). By Corollary 2.8, $Q'_3 = e_3 - f_3 - g_2 - h_2$ is in Γ. Regardless of the intersection number of (Q_1, Q_2) we always have (Q_1, Q'_3) as the contradictory pair.

Finally, let $\rho_u = (e_1, A, e_2, B_1, e_3, B_2, h_2^{-1}, C, h_1^{-1}, D_1, h_3^{-1}, D_2)$. Here Q_3 has intersection number 2 with both Q_1 and Q_2. The reader may verify that if $\operatorname{int}(Q_1, Q_2) = 1$ then Q_1 and Q_2 meet in a path of length 2, and if $\operatorname{int}(Q_1, Q_2) = 2$ then the second intersection is a coherent touching as well. By Corollary 2.8, $Q'_3 = e_3 - f_3 - g_2 - h_2$ is in Γ. In all cases (Q_1, Q'_3) is the contradictory pair.

Let Γ be 1-sided. First of all, we claim that there are pairs of distinct 4-cycles in Γ, using different couples of edges at u, which cross at u (possibly along a subpath containing u). Take an arbitrary pair (Q_1, Q_2) in Γ with different couples of edges at u. If $\operatorname{int}(Q_1, Q_2) = 1$, then Q_1 and Q_2 cross and there is nothing to prove. So let $\operatorname{int}(Q_1, Q_2) = 2$ and suppose that u is a coherent touching. By Corollary 2.8, the 4-cycles $Q'_1 = e_1 - f_1 - g_2 - h_2$ and $Q'_2 = e_2 - f_2 - g_1 - h_1$ are in Γ, and they cross at u. It remains to consider the case with a noncoherent touching of Q_1 and Q_2 at u. By Corollary 2.8, either $Q'_1 = h_2^{-1} - g_2^{-1} - g_1 - h_1 \cong_u e_2 - f_2 - f_1^{-1} - e_1^{-1} = Q'_2$ or $Q'_1 = e_1 - f_1 - f_2^{-1} - e_2^{-1} \cong_u h_1^{-1} - g_1^{-1} - g_2 - h_2 = Q'_2$ are in Γ. In both cases (Q'_1, Q'_2) cross at u, and the proof of the claim is complete.

Consider now the set of pairs of distinct 4-cycles (Q_1, Q_2) in Γ which use different couples of edges at u and have a crossing at u. The local rotation of 1-arcs originating at u can be expressed, without loss of generality, as $\rho_u = (e_1, A, e_2, B, h_1^{-1}, C, h_2^{-1}, D)$. Assume that (Q_1, Q_2) is a maximal pair as above, and let $Q_3 \in \Gamma$ use an arc in $B \cup B^{-1} \cup D \cup D^{-1}$ (if $B = D = \emptyset$ there is nothing to prove). Again we may assume $e_3 \in B$. According to where in the local rotation ρ_u the arc h_3^{-1} appears we essentially distinguish 5 cases. In all cases we derive a contradiction by finding a pair of 4-cycles in Γ which contradicts the maximality of (Q_1, Q_2).

Let $\rho_u = (e_1, A, e_2, B_1, e_3, B_2, h_1^{-1}, C, h_2^{-1}, D_1, h_3^{-1}, D_2)$ (including $h_3 = h_2$ or $h_3 = e_1^{-1}$; possibly, $e_1 = h_2^{-1}$). Then (Q_1, Q_3) is the contradictory pair regardless of the intersection number of Q_1 and Q_2, or that of Q_3 with Q_1 or Q_2.

Let $\rho_u = (e_1, A, e_2, B_1, e_3, B_2, h_3^{-1}, B_3, h_1^{-1}, C, h_2^{-1}, D)$. Since the curves are 1-sided Q_3 has intersection number 2 with both Q_1 and Q_2 (note that possibly $\operatorname{int}(Q_1, Q_2) = 1$, where Q_1 and Q_2 meet in a path of length 2; we then have either $e_1 = e_2$, $f_1 = f_2$ or $h_1 = h_2$, $g_1 = g_2$, but not $e_1 = h_2^{-1}$, $f_1 = g_2^{-1}$; also, if $\operatorname{int}(Q_1, Q_2) = 2$ then Q_1 and Q_2 cannot have a noncoherent touching at the second intersection). By Corollary 2.8, $Q_3' = e_3 - f_3 - g_2 - h_2$ in Γ. In all cases the contradictory pair is (Q_1, Q_3').

Let $\rho_u = (e_1, A, e_2, B_1, h_3^{-1}, B_2, e_3, B_3, h_1^{-1}, C, h_2^{-1}, D)$. Again, Q_3 must have intersection number 2 with both Q_1 and Q_2 (i.e., Q_3 crosses both Q_1 and Q_2 at the common second intersection). The reader may verify that if $\operatorname{int}(Q_1, Q_2) = 1$, then either $e_1 = e_2$, $f_1 = f_2$ or $h_1 = h_2$, $g_1 = g_2$, and if $\operatorname{int}(Q_1, Q_2) = 2$, then the second intersection of Q_1 and Q_2 is a coherent touching. By Corollary 2.8, either $Q_1' = e_1 - f_1 - f_3^{-1} - e_3^{-1}$ or $Q_2' = h_3^{-1} - g_3^{-1} - g_2 - h_2$ is in Γ. Therefore either (Q_1', Q_2) or (Q_1, Q_2') is a contradictory pair.

Let $\rho_u = (e_1, A, e_2, B_1, e_3, B_2, h_1^{-1}, C_1, h_3^{-1}, C_2, h_2^{-1}, D)$ (including $h_3 = h_1$). The touching of Q_3 with Q_2 must be a coherent one. Consequently, $\operatorname{int}(Q_2, Q_3) = 2$ and Q_3 must cross Q_2 at the second intersection. Note that $h_3 \neq h_2$, and so this case cannot occur if $h_1 = h_2$. By Corollary 2.8, $Q_3' = e_3 - f_3 - g_2 - h_2$ is in Γ. Regardless of what is the intersection number $\operatorname{int}(Q_1, Q_2)$, the contradictory pair is always (Q_1, Q_3').

Let $\rho_u = (e_1, A_1, h_3^{-1}, A_2, e_2, B_1, e_3, B_2, h_1^{-1}, C, h_2^{-1}, D)$ (including $h_3 = e_2^{-1}$). The touching of Q_3 with Q_1 must be a noncoherent one. Note that Q_3 has intersection number 2 with Q_1 (so this case cannot occur if $e_1 = e_2$). Now $Q_3' = e_3 - f_3 - f_1^{-1} - e_1^{-1}$ is in Γ, and (Q_1, Q_3') is the contradictory pair regardless of $\operatorname{int}(Q_1, Q_2)$.

This completes the proof of case $k = 4$ and hence, of the theorem. □

4. Minimal Paths Across a Disc

We shall prove an auxiliary lemma for later reference. Let $\mathcal{T}$ be an arbitrary triangulation of a closed disc D. A path in $\mathcal{T}$ between two boundary vertices $u_1, u_2 \in \partial D$ is **minimal** (**with respect to** u_1, u_2) if there is no shorter $u_1 - u_2$ path in $\mathcal{T}$. Clearly, all minimal paths $\wp(u_1, u_2)$ between two fixed vertices are simple paths, and if w is an intersection of $P_1, P_2 \in \wp(u_1, u_2)$, then $\mathrm{dist}_{P_1}(u_1, w) = \mathrm{dist}_{P_2}(u_1, w)$.

Lemma 4.1. *Let $\mathcal{T}$ be a triangulation of a closed disc D and let $u \in \partial D$ be a vertex such that the "link path" $N(u)$ has exactly 2 vertices on ∂D. If $u_1, u_2 \in \partial D$ ($u_1, u_2 \neq u$) are vertices, then the set of minimal paths $\wp(u_1, u_2)$ covers at most two edges on $N(u)$.*

Proof. Assume that the edges of $N(u)$ are strictly in the interior of D, and let the paths in $\wp(u_1, u_2)$ have length $m \geq 3$ (otherwise there is nothing to prove). Choose the orientation of $N(u)$ coherently with $u_1 - u_2$ paths in D. If $e = xy \in N(u)$ (where the 1-arc xy is coherent with respect to the orientation of $N(u)$) then any minimal path $P \in \wp(u_1, u_2)$ via e meets x before y. Let $e_1 = x_1y_1$ and $e_2 = x_2y_2$ (in this order) be two disjoint edges on $N(u)$ such that there exist paths $P_i \in \wp(u_1, u_2)$, $e_i \in P_i$ ($i = 1, 2$). We claim that $P_1 \neq P_2$. For if $P_1 = P_2 = P$ then P meets e_1 before e_2. Hence $\mathrm{dist}_P(x_1, y_2) \geq 3$, and by the obvious rerouting of P through u we have a contradiction. The claim is proved. Consider now the subpaths $u_1 - x_2 \subset P_2$ and $y_1 - u_2 \subset P_1$. These subpaths must have an intersection, say w. Therefore, $\mathrm{dist}_{P_1}(x_1, w) + \mathrm{dist}_{P_2}(w, y_2) \geq 3$ and hence $\mathrm{dist}_{P_1}(u_1, x_1) + \mathrm{dist}_{P_2}(y_2, u_2) \leq m - 3$. Again, the obvious rerouting through u leads to a contradiction. □

5. k-Minimal Triangulations: Bounding the Vertex Degree

Theorem 5.1. *Let $\mathcal{T}$ be a k-minimal triangulation ($k \geq 3$) of a closed surface $\Sigma \not\approx S^2$. There exists a function* $\mathrm{const}\,(k, \chi_\Sigma)$ *which bounds from above the maximal vertex degree Δ of $\mathcal{T}$:*

$$\Delta \leq \mathrm{const}\,(k, \chi_\Sigma).$$

Lemma 5.2. *Let $\mathcal{T}$ be a k-minimal triangulation ($k \geq 3$) of a closed surface $\Sigma \not\approx S^2$, and let $\Gamma \neq 1$ be a relative homotopy class at $u \in V(\mathcal{T})$. There exists a function* $\mathrm{const}\,(k) = k(k-1)$ *such that the number of (Γ, k)-essential edges at u is bounded by $2 \cdot (1 + \mathrm{const}\,(2k))$. More precisely,*

$$|E_u^i(\Gamma, k)| \leq 1 + \mathrm{const}\,(2k), \quad (i = 1, 2).$$

Proof. Let C_1, C_2 be the extremal pair of k-cycles in Γ and consider the bounding open disc(s) R_1, R_2 (or $R = R_1 = R_2$) as in Theorem 3.1. Assume none of these discs is empty. Cut R_1 and R_2 (or just R) out of Σ by dissecting along C_1, C_2 to obtain triangulated closed disc(s) $\hat{R}_1, \hat{R}_2$ (or $\hat{R}$). We retain the labeling of vertices and edges as in Σ. Those which are "duplicated" on the boundary are equipped with additional indices. This holds at least for u which gives rise to two distinct vertices $u_i \in \partial \hat{R}_i$ (or $u_i \in \partial \hat{R}$) $(i = 1, 2)$. The link cycle $N(u)$ in Σ gives rise to two disjoint simple paths $N(u_i)$ in $\hat{R}_i$ (or $\hat{R}$) $(i = 1, 2)$ having exactly two vertices on $\partial \hat{R}_i$ (or $\partial \hat{R}$).

Consider the connected components in $\hat{R}_i$ $(i = 1, 2)$ (or $\hat{R}$) which arise from an essential k-cycle in $\mathcal{T}$. Since $\mathcal{T}$ is k-minimal, it is easy to see each such component is a minimal path between a pair of boundary vertices of $\hat{R}_i$ (or $\hat{R}$). The boundaries of $\hat{R}_i$ $(i = 1, 2)$ (or $\hat{R}$) have length at most $2k$. Hence there are at most $\binom{2k}{2}$ classes of minimal paths in $\hat{R}_i$ $(i = 1, 2)$ (or $\hat{R}$). By Lemma 4.1, each class covers at most 2 edges on $N(u)$ in $\hat{R}_i$ $(i = 1, 2)$ (or on $N(u_i)$, $i = 1, 2$, in $\hat{R}$). But all such edges are covered by k-minimality of $\mathcal{T}$. Hence $N(u)$ of $\hat{R}_i$ (or $N(u_i)$ of $\hat{R}$) $(i = 1, 2)$ consists of at most $\operatorname{const}(2k) = 2\binom{2k}{2}$ edges. □

Proof of Theorem 5.1. Let u be a vertex of maximal degree Δ. We show that one can choose a suitable number of cycles $C_1, C_2, \ldots, C_N$ at u in $\mathcal{T}$ which give rise to N pairwise internally disjoint and pairwise nonhomotopic simple loops at u in Σ. We distinguish two cases according to whether k is odd or even.

Suppose k is odd. Then $r_u = \frac{1}{2}(k - 1)$. Each essential k-cycle at u has all its vertices in $D_{r_u}(u)$, with a unique edge joining the two vertices on $N_{r_u}(u)$ from the outside. The required cycles are constructed as follows: at the beginning let E_1 contain all the edges incident with u. Then at i^{th}-step:

- Choose an edge $uu_i \in E_i$, and $C_i \in \operatorname{Ess}_u(k)$ containing uu_i. Denote the relative homotopy class at u to which C_i belongs by Γ_i.
- $E_{i+1} = E_i \backslash E_u(\Gamma_i, k)$.

The procedure does not stop before $N \geq \Delta/(2\,(1{+}\operatorname{const}(2k)))$ steps by Lemma 5.2. The cycles $\mathcal{C} = \{C_1, C_2, \ldots, C_N\}$ are pairwise nonhomotopic since at each step all the k-essential edges for the current homotopy class are deleted. The required loops are obtained by contracting the edges of $\mathcal{C} \cap D_{r_u}(u)$ homotopically to a point u. Since N is bounded by a constant $O(\chi_\Sigma)$ by Proposition 2.2, we have the bound on Δ.

Suppose k is even. Then $r_u = \frac{1}{2}(k - 2)$. Each essential k-cycle at u has exactly one vertex outside $D_{r_u}(u)$, the "antipodal vertex". This time we have to be slightly more careful with our construction of cycles since antipodal vertices of different cycles may coincide.

First of all, the antipodal vertices of cycles C_i $(i = 1, 2, \ldots)$ to be constructed below will be denoted by w_i, their neighbours on $N_{r_u}(u)$ by $\{s_i, t_i\}$, and the intersections of C_i with the link cycle at u by $\{u_i, x_i\}$ (here $u_i \in u - s_i - w_i$ and $x_i \in u - t_i - w_i$). The edges $w_i s_i$ will be pairwise distinct and moreover, the contraction of D_{r_u} plus all the edges $w_i t_i$ will preserve the homotopy class of each C_i. At the beginning, let E_1 contain all the edges at u. Then at the i^{th}-step:

- Choose an edge $uu_i \in E_i$. If each cycle in $\operatorname{Ess}_u(k)$ containing uu_i has its antipodal vertex different form the antipodal vertices w_j of C_j for each $j < i$, let C_i be an arbitrary essential k-cycle containing uu_i. See Figure 4(a). Otherwise there exists $C \in \mathrm{Ess}_u(k)$ containing uu_i such that its antipodal vertex w_i coincides with some antipodal vertex w_j of C_j, $j < i$. Choose C_i to be the cycle formed by the paths $u - u_i - s_i - w_i \subset C$ and $w_j - t_j - x_j - u \subset C_j$ (therefore we set $t_i = t_j$ and $x_i = x_j$; below we shall prove that C_i is indeed an essential k-cycle and that $s_i \neq s_j$; note that possibly $w_i = w_j$ for different indices $j < i$ in which case $w_j - t_j - x_j - u$ is common for all such indices and the corresponding s_j are pairwise different). See Figure 4(b). Let Γ_i be the relative homotopy class of C_i.

- Delete the edges of E_i which are at most const $(2k)$ apart from uu_i and those which are at most const $(2k)$ apart from ux_i (here const $(2k)$ is as in Lemma 5.2 and the distance between edges incident with u is calculated on the link cycle $N(u)$).

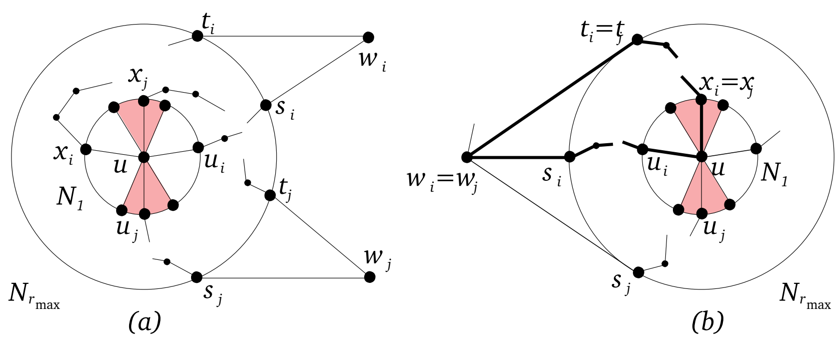

Figure 4. The construction of cycles $C_1, C_2, \ldots, C_N$ when k even.

Since at most $2\,(1+2\ \mathrm{const}\,(2k))$ edges are deleted at each step, the procedure does not stop before $N \geq \Delta/(2\,(1 + 2\ \mathrm{const}\,(2k)))$ steps. The constructed cycles are essential. This needs verification only in case $w_i = w_j$ for some $j < i$. First of all, it is immediate that C_i (as a closed walk – so far we have not yet proved that it should be a cycle, i.e., simple) is of length k. Moreover, it is homotopically nontrivial. For if not then the edges uu_i and ux_i belong to some planar cycle (determined by C_i) of length $\leq k$. Its bounding closed disc contains at most $1+\mathrm{const}\,(k) < 1+\mathrm{const}\,(2k)$ edges at u. But this is a contradiction since the edges uu_i and ux_i are more than const $(2k)$ apart by construction. From the fact that C_i is essential it also follows that C_i is indeed a cycle. Also, if $w_i = w_j$ then $s_i \neq s_j$. For if not then C_i is in the same homotopy class as C_j, which is impossible since at each step all the k-essential edges for the current homotopy class are deleted. By the very same reason the constructed cycles $\mathcal{C} = \{C_1, C_2, \ldots, C_N\}$ are pairwise nonhomotopic. As before, contract $\mathcal{C} \cap D_{r_u}(u)$ homotopically to u. Further, contract also the edges which were originally denoted by $t_i w_i$. Each cycle in $\mathcal{C}$ gives rise to exactly one loop at u and the N loops are internally disjoint (because each loop corresponds to an edge

$s_i w_i$ and these edges are pairwise distinct). Also, the contraction preserves the homotopy class of each cycle which contracts to the corresponding loop (because after the contraction of $D_{r_u}(u)$, each of the resulting curves is further contracted only by an arc on that curve). Again, the bound on Δ follows by Proposition 2.2.□

6. k-Minimal Triangulations: Bounding the Number of Edges

Theorem 6.1. *Let $\mathcal{T}$ be a k-minimal triangulation of $\Sigma \not\approx S^2$, let Δ denote the maximum vertex degree of $\mathcal{T}$. There exists a function* const(Δ, k, χ_Σ)*, polynomial in Δ, such that*

$$|E(\mathcal{T})| \leq \text{const}\,(\Delta, k, \chi_\Sigma).$$

Lemma 6.2. *Let $\mathcal{T}$ be a k-minimal triangulation of $\Sigma \not\approx S^2$ and let Γ be a nontrivial (free) homotopy class on Σ. Then no $k+1$ k-cycles in Γ (if k is even) and no $k+2$ k-cycles in Γ (if k is odd) can be pairwise disjoint.*

Proof. Denote by $r = k+1$ if k is even and $r = k+2$ if k is odd and suppose a family $C_1, C_2, \ldots, C_r$ of r pairwise disjoint k-cycles in Γ exists. Every pair (C_i, C_j) bounds a cylinder in Σ and Γ is necessarily 2-sided (cf. [**7**]). Consequently, we may assume that all these cycles belong to the bounding cylinder A_{1r} of (C_1, C_r) and that $C_{(r+1)/2}$ is the "middle one". Then each k-cycle at u, where u is a vertex of $(C_{(r+1)/2}$, is contained in A_{1r}.

Among essential k-cycles at u there exists at least one, say C, such that the distance (measured on $N(u)$) between the crossings $\{a_1, a_2\} = C \cap N(u)$ is minimal. The minimal arc $a_1 - a_2$ must contain a vertex, say v. Consider an essential k-cycle C_{uv} containing uv. Then $C_{uv} \cap N(u) = \{v, w\}$, where $v \neq a_1, a_2$ and $w \neq a_1, a_2$. By the minimality of $a_1 - a_2$, C and C_{uv} must cross at u, and since Γ is 2-sided, we have $\text{int}\,(C, C_{uv}) > 1$. Choose the intersection $x \in C \cap C_{uv}$ such that the path $u - v - x$ on C_{uv} does not contain other points of C. Cut the cylinder $A = A_{1r}$ out of Σ and attach a disc D to one of the boundary components of A to obtain a disc D_A. Then $C \subset D_A$ bounds a disc D_C which must contain D in its interior. We may as well assume that the minimal arc $a_1 - a_2$ on $N(u)$ also belongs to the interior of D_C (otherwise, D_A is defined by "filling up the other hole" of A). The simple path $u - v - x$ divides D_C into two discs D_1, D_2 and exactly one of them contains D, say $D \subset D_2$, where fr $D_1 = u - v - x - a_1 - u$ and fr $D_2 = u - v - x - a_2 - u$. The cycle fr D_2 on $A_{1r} \subset \Sigma$ is essential and has length k, a contradiction with the minimality of C. □

Proof of Theorem 6.1. For each edge $e \in E(\mathcal{T})$ choose an essential k-cycle $e \in C_e$. Observe the set of pairs $\mathcal{E} = \{(e, C_e) \mid \in E(\mathcal{T})\}$, and let $\mathcal{E}_u \subseteq \mathcal{E}$ be the subset of pairs (e, C_e) where C_e contains a fixed vertex $u \in V(\mathcal{T})$. If (e, C_e) is in this subset then e has at least one of its endvertices at distance $\leq r_u = \lfloor \frac{1}{2}(k-1) \rfloor$

from u. Therefore $|\mathcal{E}_u| \leq 2\,\Delta^{\lfloor (k+1)/2 \rfloor} =: h(k,\Delta)$. Fix a pair $(e, C_e) \in \mathcal{E}$ and let $C_e = u_1 - u_2 - \cdots - u_k - u_1$. Then

$$|\{(f, C_f) \in \mathcal{E} \mid C_f \cap C_e \neq \emptyset\}| \leq 1 + \sum_{i=1}^{k}(|\mathcal{E}_{u_i}| - 1) < k\,h(k,\Delta).$$

It follows that in $\mathcal{T}$ there are at least $|\mathcal{E}|/(k\,h(k,\Delta)) = |E(\mathcal{T})|/(k\,h(k,\Delta))$ pairwise disjoint essential k-cycles. At least $|E(\mathcal{T})|/(k(k+1)\,h(k,\Delta))$ are also pairwise nonhomotopic by Lemma 6.2. But this number is bounded above by some constant $O(\chi_\Sigma)$ by Proposition 2.1. This gives a bound on $|E(\mathcal{T})|$. □

7. Proof of the Main Theorem

Let $\mathcal{T}$ be a k-minimal triangulation of Σ and let Δ be its maximal vertex degree. By Theorem 6.1 there exists a function such that $|E(\mathcal{T})| \leq \mathrm{const}\,(\Delta, k, \chi_\Sigma)$. As this function is strictly increasing in $\Delta > 1$ and since $\Delta \leq \mathrm{const}\,(k, \chi_\Sigma)$ by Theorem 5.1, we have the upper bound on the number of edges of $\mathcal{T}$ in terms of the representativity and the Euler characteristic of the surface. Hence there exists a bound on the number of vertices as well and therefore, of triangulations (up to homeomorphism). The bound is $O((c\chi_\Sigma)^k)$. □

8. Minor-minimal Embeddings

A **surface minor** of an embedded graph is obtained by successive deletions of edges, edge contractions (without contracting loops), or removal of isolated vertices (cf. [**24**] for details). By $\mathcal{G}_\Sigma(\geq k)$ we denote all graph embeddings in Σ (up to homeomorphism) with representativity $\geq k \geq 0$. By $\mathcal{G}^m_\Sigma(= k)$ we denote the subclass of **minor-minimal** embeddings in $\mathcal{G}_\Sigma(\geq k)$ $(k \geq 1)$, that is, every edge deletion or edge contraction gives rise to an embedding of representativity $< k$. Since a single edge deletion or edge contraction lowers the representativity by at most 1, embeddings in $\mathcal{G}^m_\Sigma(= k)$ indeed have representativity k.

Proposition 8.1. *Let G be an embedded graph into $\Sigma \not\approx S^2$ with $rp_\Sigma G = k \geq 2$. If G is 2-connected then its barycentric subdivision B_G in Σ is a triangulation with $rp_\Sigma B_G = 2k$. If $G \in \mathcal{G}^m_\Sigma(= k)$ then G is 2-connected and B_G is a $2k$-minimal triangulation. Conversely, if G is 2-connected and B_G a $2k$-minimal triangulation, then $G \in \mathcal{G}^m_\Sigma(= k)$.*

Proof. Since G is 2-connected and $rp_\Sigma G \geq 2$, the embedding is a closed-cell embedding [**24**]. Hence B_G is a triangulation (i.e., simplicial). Let $C \subset B_G$ be some essential cycle of length $rp_\Sigma B_G$, and suppose C contains a vertex $e \in V(B_G)$ which represents the edge $e = uv$ of G. Let x and y be the vertices in B_G representing faces of G such that e lies in the common boundary of their closures.

Clearly $x \neq y$. Now C contains either the vertices $u, e, v \in V(B_G)$ or the vertices $x, e, y \in V(B_G)$. In both cases there is a cycle in B_G, homotopic to C and of the same length, which avoids the vertex e. Consequently, there is an essential cycle $C' \subset B_G$ with $|C'| = |C|$, using no vertices which represent edges of G, and those representing vertices and faces of G alternate on C. Hence $rp_\Sigma B_G = |C| = |C'| = 2l$ and $k \leq l$. In fact, we have equality. Indeed, take an essential simple closed curve γ on Σ which intersects G in k vertices and traverses each face of G at most once. Then γ is free isotopic to some $2k$-cycle in B_G.

We now prove the second part of the proposition. Let the embedding be minor-minimal. First of all, it is easily verified that a minor-minimal embedding of representativity ≥ 2 must be 2-connected. So by the first part of this proposition its barycentric subdivision is indeed a triangulation of representativity $2k$. We show that each edge of B_G is contained in an essential $2k$-cycle. Typical edges to be considered are ue, ux and ex. Since G is minor-minimal and since contraction of an edge drops the representativity by at most 1, the embedding G/e obtained by contracting the edge e has representativity $k-1$. Take an essential simple closed curve γ which intersects G/e in exactly $k-1$ vertices, traversing each face of G/e at most once. Clearly, γ contains the vertex of G/e to which e has collapsed. Therefore, there is a simple $u-v$ path $\delta\colon [0,1] \to \Sigma$ which intersects G in exactly k vertices, using each face of G at most once and such that $\delta \cup e$ represents an essential simple closed curve. This curve can be moved isotopically to two $2k$-cycles of B_G, one containing the edge ue, and the other one containing the edge ux. Finally, consider $G - e$. There is a $(k-1)$-representative simple closed curve γ intersecting $G - e$ in $k-1$ vertices and traversing each face of $G - e$ at most once. Clearly, γ intersects e of G (in its interior!). It is again trivial to show that γ can be moved isotopically to a $2k$-cycle of B_G containing the edge ex.

The converse statement is proved in the same way. □

It follows from the Robertson-Seymour's proof of the Wagner's conjecture that the class $\mathcal{G}^m_\Sigma(=k)$ $(k \geq 1)$ is finite. This fact follows trivially also from our Main Theorem.

Corollary 8.2. *Let $\Sigma \not\approx S^2$ be a closed surface. Then the class of minor-minimal embeddings with representativity $k \geq 1$ is finite (up to homeomorphism).*

Proof. Clearly, $\mathcal{G}^m_\Sigma(=1)$ consists of a bouquet of circles with a fixed number of loops. If $G \in \mathcal{G}^m_\Sigma(=k)$ $(k \geq 2)$ then B_G is a $2k$-minimal triangulation by Proposition 8.1. Since a triangulation is the barycentric subdivision of at most 2 different embeddings, the claim follows from our Main theorem. □

Note added in proof. Recently, a shorter proof of our Main Theorem was obtained by Gao, Richter and Seymour [**9**]. As they point out, this theorem is indeed equivalent to Corollary 8.2. They also list some unpublished references not included here. Another very short proof is found in [**12**].

References

1. Ahlfors L. V., Sario L., *Riemann Surfaces*, Princeton Univ. Press, Princeton NJ, 1960.
2. Archdeacon D., *Densely embedded graphs*, J. Combin. Theory Ser. B **54** (1992), 13–36.
3. Barnette D. W., *Generating the triangulations of the projective plane*, J. Combin. Theory Ser. B **33** (1982), 222–230.
4. Barnette D. W. and Edelson A., *All orientable 2-manifolds have finitely many minimal triangulations*, Isr. J. Math. **62** (1988), 90–98.
5. ———, *All 2-manifolds have finitely many minimal triangulations*, Isr. J. Math. **67**(2) (1989), 123–128.
6. Batagelj V., *Inductive classes of graphs*, Proc. 6^{th} Yugoslav Seminar on Graph Theory (Dubrovnik 1985), Univ. Novi Sad, Novi Sad, 1986, pp. 43–56.
7. Epstein D. B. A., *Curves on 2-manifolds and isotopies*, Acta Math. **115** (1966), 83–107.
8. Fisk S., Mohar B. and Nedela R., *Minimal locally cyclic triangulations of the projective plane*, J. Graph Theory **18**(1) (1994), 25–35.
9. Gao Z., Richter R. B. and Seymour P. D., *Irreducible triangulations of surfaces*, preprint 1994.
10. Gross J. L. and Tucker T. W., *Topological graph theory*, Wiley – Interscience, New York, 1987.
11. Hartsfield N. and Ringel G., *Clean triangulations*, Combinatorica **11**(2) (1991), 145–155.
12. Juvan M., Malnič A. and Mohar B., *Systems of curves on surfaces*, preprint, 1995.
13. Lavrenchenko S. A., *The irreducible triangulations for the torus*, Geom. Sb. **30** (1987), 52–62.
14. Malnič A. and Mohar B., *Generating locally-cyclic triangulations of surfaces*, J. Combin. Theory Ser. B **56** (1992), 147–164.
15. Malnič A. and Nedela R., *k-Minimal triangulations of surfaces*, Preprint ser. Univ. Ljubljana **405** (1993).
16. Massey W. S., *Algebraic topology: An introduction*, Harcourt, Brace and World, 1967.
17. Mohar B., *Combinatorial local planarity and the width of graph embeddings*, Canad. J. Math. **44**(6) (1992), 1272–1288.
18. Nedela R., *Locally-cyclic graphs covering complete tripartite graphs*, Math. Slovaca **42**(2) (1992), 143–146.
19. Neuman M. H. A., *Elements of the topology of plane sets of points* (1961, ed.), Cambridge Univ. Press, 1939.
20. Parsons T. D. and Pisanski T., *Graphs which are locally paths*, Combinatorics and Graph Theory, Warsaw, Banach Center Publications 25, 1989, pp. 127–135.
21. Robertson N. and Seymour P. D., *Graph minors VII: Disjoint paths on a surface*, J. Combin Theory Ser. B **45** (1988), 212–254.
22. ———, *Graph minors VIII: A Kuratowski theorem for general surfaces*, J. Combin. Theory Ser. B **48** (1990), 225–288.
23. ———, *Graph minors XX: Wagner's conjecture*, in preparation.
24. Robertson N. and Vitray R. P., *Representativity of surface embeddings*, Paths, Flows and VLSI-Layout (B. Korte, L. Lovász, H. J. Prömel, and A. Schrijver, eds.), Springer-Verlag, Berlin-New York, 1990.
25. Thomassen C., *Embeddings of graphs with no short noncontractible cycles*, J. Combin. Theory, Ser. B **48** (1990), 155–177.

A. Malnič, Pedagoška fakulteta, Univerza v Ljubljani, Kardeljeva pl. 16, 61000 Ljubljana, Slovenija

R. Nedela, Department of Mathematics, M. Bel University, 97549 Banská Bystrica, Slovakia