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ERGODIC AVERAGES AND INTEGRALS OF COCYCLES

H. B. KEYNES, N. G. MARKLEY AND M. SEARS

Abstract. This paper concerns the structure of the space $\mathcal{C}$ of real valued cocycles for a flow $(X, \mathbb{Z}^m)$. We show that $\mathcal{C}$ is always larger than the set of cocycles cohomologous to the linear maps if the flow has a free dense orbit. By considering appropriate dual spaces for $\mathcal{C}$, we obtain the concept of an **invariant cocycle integral**. The extreme points of the set of invariant cocycle integrals parallel the role of ergodic measures and enable us to investigate different ergodic averages for cocycles and the uniform convergence of such averages. The cocycle integrals also enable us to characterize the subspace of the closure of the coboundaries in $\mathcal{C}$, and to show that $\mathcal{C}$ is the direct sum of this space with the linear maps exactly when the invariant cocycle integral is unique.

1. The Space of Cocycles

Let X be a compact metric space and let $\mathbb{Z}^m$ denote the integer lattice in $\mathbb{R}^m$, m-dimensional Euclidean space. We will assume that $\mathbb{Z}^m$ acts as a group of homeomorphisms on X, that is we have a flow $(X, \mathbb{Z}^m)$. A real valued (topological) **cocycle** for such a flow is a continuous function $h\colon X \times \mathbb{Z}^m \to \mathbb{R}$ such that for all $x \in X$ and all $a, b \in \mathbb{Z}^m$

$$h(x, a + b) = h(x, a) + h(ax, b)$$

where ax denotes the action of a on x. This equation is called the **cocycle equation**. The theme of this paper is to investigate ergodic averages of these cocycles and to describe an appropriate setting for invariant integrals of cocycles.

In earlier papers ([**2**], [**4**], and [**5**]), we have studied vector valued cocycles and their role in understanding the structure of $\mathbb{R}^m$ actions on compact metric spaces. Since the coordinates of a vector valued cocycle are real valued cocycles, the results in this paper provide additional tools for the analysis of $\mathbb{R}^m$ actions. Specifically they provide a natural definition of a nonsingular $\mathbb{R}^m$ valued cocycle. K. Madden and the second author [**7**] have shown that the suspension flow of a nonsingular cocycle is a time change of the constant one suspension and that every $\mathbb{R}^m$ action with a free dense orbit has an almost one-to-one extension which is the suspension

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of a nonsingular cocycle. However, the present work stands on its own as an extension of classical results about ergodic averages and coboundaries for a single homeomorphism.

Let $\mathcal{C}$ denote the set of real valued cocycles for $(X, \mathbb{Z}^m)$. Clearly $\mathcal{C}$ is a vector space over $\mathbb{R}$. Letting $|a| = \sum_{i=1}^m |a_i|$ for $a = (a_1, \cdots, a_m) \in \mathbb{Z}^m$,

$$\|h\| = \sup\left\{\frac{|h(x,a)|}{|a|} : x \in X \quad \text{and} \quad a \in \mathbb{Z}^m\right\}$$

defines a norm. Let $e_1, \cdots, e_m$ denote the usual generators of $\mathbb{Z}^m$. Using the cocycle equation it is not hard to show that

$$\|h\| = \sup\{|h(x, e_j)| : x \in X \quad \text{and} \quad 1 \le j \le m\},$$

and that with this norm $\mathcal{C}$ is a separable Banach space.

Remark 1.1. If $h \in \mathcal{C}$, then

$$|h(x,a) - h(x,b)| \le \|h\|\,|a-b|$$

for all $x \in X$ and $a, b \in \mathbb{Z}^m$.

Proof. Clearly $|h(y,c)| \le \|h\|\,|c|$ for any y and c. Since $h(x,b) + h(bx, a-b) = h(x,a)$, we have

$$|h(x,a) - h(x,b)| = |h(bx, a-b)| \le \|h\|\,|a-b|.$$

□

If $h \in \mathcal{C}$ and $h(x,a) = h(y,a)$ for all $x, y \in X$ and $a \in \mathbb{Z}^m$, then the map $a \to h(x,a)$ is linear. Conversely, if $T \in \mathcal{L}$, the linear maps from $\mathbb{R}^m$ into $\mathbb{R}$, then $h(x,a) = T(a)$ defines a cocycle in $\mathcal{C}$. Consequently, $\mathcal{L}$ is a closed subspace of $\mathcal{C}$ which is called the **constant cocycles**. In particular, the dual basis $e_1^*, \cdots, e_m^*$ of $e_1, \cdots, e_m$ are constant cocycles. We now seemingly have two norms on $\mathcal{L}$; namely the cocycle norm,

$$\|T\| = \sup\{|T(e_j)| : 1 \le j \le m\}$$

and the linear functional norm,

$$|T| = \sup\{|T(v)| : |v| \le 1\}$$

where as above $|v| = \sum_{i=1}^m |v_i|$. However, it is easily verified that in our context they are equal, and we will use the notation $\|T\|$.

Let $C(X)$ denote the Banach space of continuous real valued functions on X with the usual norm. Given $f \in C(X)$ we can construct $h \in \mathcal{C}$ by setting

$$h(x,a) = f(ax) - f(x).$$

Such a cocycle is called a **coboundary** and the coboundaries, $\mathcal{B}$, form another subspace of $\mathcal{C}$. It is important because two cocycles that differ by a coboundary are in many ways indistinguishable.

When $m = 1$ we have a single homeomorphism φ of X generating the $\mathbb{Z}$ action. In this case $\mathcal{C}$ is isometrically isomorphic to $C(X)$ because given $f \in C(X)$

$$h(x,n) = \begin{cases} \sum_{k=0}^{n-1} f(\varphi^k(x)) & n > 0 \\ 0 & n = 0 \\ -\sum_{k=1}^{-n} f(\varphi^{-k}(x)) & n < 0 \end{cases}$$

defines a cocycle with $h(x,1) = f(x)$. Notice that

$$\frac{h(x,n)}{n} = \frac{1}{n}\sum_{k=0}^{n-1} f(\varphi^k(x))$$

is an ergodic average for $f(x) = h(x,1)$. Moreover, these averages converge uniformly in x for all f if and only if (X,φ) is uniquely ergodic. (See §6.5 in Walters [**9**] for a general discussion of unique ergodicity.)

In general we can define a $\mathbb{Z}^m$ action on $\mathcal{C}$ by setting $ah(x,b) = h(ax,b) = h(x,a+b) - h(x,a)$ and then

$$\frac{1}{N^m}\sum_{0\le a_i<N} ah(x,b)$$

is a kind of an ergodic cocycle average and one expects its limits to be related to integration. But it is not clear how to interpret $h(x,a)/|a|$ as an ergodic average and fit it into an integration theory for cocycles. The main results in this paper establish the concept of a cocycle integral which firmly links these two types of averaging together for cocycles in the topological setting. Furthermore, these integrals provide a characterization of $\overline{\mathcal{B}}$, and the equivalence of uniform convergence and unique cocycle integral holds similarly to the equivalence of uniform convergence and unique invariant measure.

2. Existence of Non-Trivial Cocycles

A constant cocycle plus a coboundary is neither a constant cocycle nor a coboundary. Are there cocycles which are not of this form, that is, are there cocycles which are not cohomologous to a constant cocycle? In this section we will prove the unpublished folklore theorem that these cocycles are usually dense in $\mathcal{C}$ because without it the rest of our results are pointless.

The linear map $S\colon C(X) \to \mathcal{C}$ given by $(Sf)(x,a) = f(ax) - f(x)$ is bounded and has range $\mathcal{B}$. The kernel of S consists of the continuous functions which are constant on orbits and hence constant on orbit closures. Because $\mathcal{L}$ is a closed subspace of $\mathcal{C}$, we can form the quotient Banach space $\tilde{\mathcal{C}} = \mathcal{C}/\mathcal{L}$ and let π be the bounded linear projection of $\mathcal{C}$ onto $\tilde{\mathcal{C}}$. Note that $\mathcal{C} = \mathcal{B} + \mathcal{L}$ (every cocycle is cohomologous to a constant cocycle) if and only if $\pi \circ S$ is onto. Hence, the following theorem will answer the question:

Theorem 2.1. *If $(X, \mathbb{Z}^m)$ has a free dense orbit, then $\pi \circ S \colon C(X) \to \tilde{\mathcal{C}}$ is not onto.*

Proof. Since $(X, \mathbb{Z}^m)$ has a dense orbit, the kernel K of S is the constant functions and there is an induced bounded linear map $\tilde{S} \colon C(X)/K \to \tilde{\mathcal{C}}$. If $\pi \circ S$ is onto, then $\tilde{S}$ is an isomorphism by the open mapping theorem. In particular, $\tilde{S}^{-1}$ is continuous and if g_n is a sequence in $\tilde{\mathcal{C}}$ such that $\|g_n\|$ converges to 0, then $\|\tilde{S}^{-1} g_n\|$ also converges to 0. The following lemma establishes the existence of a sequence which violates this condition and completes the proof of the theorem by contradiction. (This proof is an adaption of a proof for $m = 1$ which R. Zimmer showed us.) □

Lemma 2.2. *If $(X, \mathbb{Z}^m)$ has a free orbit, then given $\epsilon > 0$ there exists $f \in C(X)$ such that $\|S(f)\| \leq \epsilon$ and $\inf\{\|f + c\| : c \in \mathbb{R}\} = 1/2$.*

Proof. Suppose $0 < \epsilon < 1$ and the orbit of x_0 is free, so $a \to a x_0$ is one-to-one. Choose M, a positive integer, such that $(1-\epsilon)^M < \epsilon$. There exists an open neighborhood U of x_0 such that $aU \cap bU = \phi$ when $-(M+1) \leq a_i, b_i \leq M+1$ and there exists $f_0 \in C(X)$ such that $f_0(x_0) = 1$, $f_0(x) = 0$ for $x \notin U$, and $0 \leq f_0(x) \leq 1$ for all x. Define f_a by $f_a(x) = (1-\epsilon)^{|a|} f_0((-a)x)$ and set

$$f(x) = \sum \{f_a(x) : |a_i| \leq M, i = 1, \cdots, m\}.$$

Note that $f(x) = 0$ unless $x \in aU$ for some a with $|a_i| \leq M$, $i = 1, \cdots, m$, in which case $f(x) = f_a(x)$.

Next we show that $\|Sf\| = \sup\{|f(e_i x) - f(x)| : x \in X \text{ and } i = 1, \cdots, m\} \leq \epsilon$. There are several cases for each e_i. First suppose $f(e_i x)$ and $f(x)$ are not zero. Hence $x \in aU$, $e_i x \in (a + e_i)U$, $|a_j| \leq M$ for $j \neq i$, and $-M \leq a_i < a_i + 1 \leq M + 1$. When $0 \leq a_i$ we have

$$\begin{aligned} f(e_i x) - f(x) &= (1-\epsilon)^{|a|+1} f_0((-a)x) - (1-\epsilon)^{|a|} f_0((-a)x) \\ &= -\epsilon (1-\epsilon)^{|a|} f_0((-a)x) \end{aligned}$$

and

$$|f(e_i x) - f(x)| \leq \epsilon\,.$$

A similar calculation holds for $a_i < 0$.

Now suppose $f(x) \neq 0$ and $f(e_i x) = 0$. In this case $x \in aU$ with $|a_j| \leq M$ for $j \neq i$ and $a_i = M$. Thus $|a| \geq M$ and

$$|f(e_i x) - f(x)| = | - (1-\epsilon)^{|a|} f_0((-a)x)| \leq (1-\epsilon)^M < \epsilon\,.$$

A similar calculation works when $f(e_i x) \neq 0$ and $f(x) = 0$.

Finally we calculate $\inf\{\|f+c\| : c \in \mathbb{R}\} = 1/2$. First note that $\|f+c\| = \sup\{|f+c| \colon x \in X\} \geq 1$ for $c \geq 0$ and $c \leq -1$. For $-1 < c < 0$ it is the maximum of $-c$ and $1+c$ which takes its minimum value when $c = -1/2$. □

Theorem 2.3. *Every open set in $\mathcal{C}$ contains cocycles which are not cohomologous to a constant cocycle.*

Proof. Let V be an open set in $\mathcal{C}$. Then $\pi(V)$ is open in $\tilde{\mathcal{C}}$. If $\pi(V) \subseteq \pi \circ S(C(X))$, then $\pi \circ S$ would be onto. □

The following result also follows readily from the above lemma.

Theorem 2.4. *If $(X, \mathbb{Z}^m)$ has a free dense orbit, then $\mathcal{B}$ is not closed.*

Proof. If $\mathcal{B}$ was closed, than the induced map of $C(X)/K$ onto $\mathcal{B}$ would be a Banach isomorphism. Lemma 2.2 shows that the inverse would not be continuous. Thus $\mathcal{B}$ is not closed. □

In contrast to Theorem 2.3, there is the question of what kind of functions are in $\mathcal{L} + \mathcal{B}$. For example, when the underlying system is an irrational rotation of the circle, then $\cos n\theta$ and $\sin n\theta$ are in $\mathcal{B}$ [**3**, p. 563] and consequently, all finite trigonometric polynomials are in $\mathcal{L} + \mathcal{B}$. In a recent paper Schmidt [**8**] has shown that for some subshifts of finite type cocycles with a summable variation are in $\mathcal{L} + \mathcal{B}$. He also gives some concrete examples of cocycles not in $\mathcal{L} + \mathcal{B}$.

3. Invariant Linear Functionals on $\mathcal{C}$

There is a natural map from $\mathcal{C}$ into $C(X)^m$ which we can use to study $\mathcal{C}^*$, the dual of $\mathcal{C}$. Define $\Theta \colon \mathcal{C} \to C(X)^m$ by $\Theta(h) = (h(\cdot, e_1), \cdots, h(\cdot, e_m))$ where $h(\cdot, e_j)$ denotes the function $x \to h(x, e_j)$. Using the norm $\|(f_1, \cdots, f_m)\| = \sup_{1 \leq j \leq m} \|f_j\|$, Θ is an isometric isomorphism of $\mathcal{C}$ onto the closed subspace

$$\{(f_1, \cdots, f_m) : f_i(e_j x) - f_i(x) = f_j(e_i x) - f_j(x) \quad \text{for} \quad i \neq j\}$$

of $C(X)^m$.

The dual of $C(X)^m$ can be identified with $C^*(X)^m$. Explicitly, given

$$\overline{f} = (f_1, \cdots, f_m) \in C(X)^m \quad \text{and} \quad \overline{\gamma} = (\gamma_1, \cdots, \gamma_m) \in C^*(X)^m,$$

set

$$\overline{\gamma}(\overline{f}) = \sum_{i=1}^{m} \gamma_i(f_i).$$

Moreover,

$$\|\overline{\gamma}\| = \sup\{|\overline{\gamma}(\overline{f})| : \|\overline{f}\| = \sup_{1\le j\le m} \|f_i\| \le 1\} = \sum_{i=1}^{m} \|\gamma_i\|.$$

Proposition 3.1. *If $\gamma \in \mathcal{C}^*$, then there exists $\overline{\gamma} = (\gamma_1, \cdots, \gamma_m) \in C^*(X)^m$ with the following properties*

a) $\gamma(h) = \overline{\gamma}(\Theta(h))$
b) $\|\gamma\| = \|\overline{\gamma}\|$
c) $\gamma(e_j^*) = \gamma_j(1)$
d) $|\gamma(e_j^*)| \le \|\gamma_j\|$.

Proof. Use the Hahn-Banach theorem for a) and b). For c) note that $\Theta(e_j^*) = (0, \cdots, 0, 1, 0, \cdots, 0)$, and then d) follows from c). □

Theorem 3.2. *If $\gamma \in \mathcal{C}^*$, then $\|\gamma \mid \mathcal{L}\| = \sum_{i=1}^{m} |\gamma(e_i^*)|$ and the following are equivalent*

a) $\|\gamma \mid \mathcal{L}\| = \|\gamma\|$
b) $\|\gamma\| = \sum_{j=1}^{m} |\gamma(e_j^*)|$
c) *there exist Borel probability measures $\mu_1, \cdots, \mu_m$ such that*

$$\gamma(h) = \sum_{j=1}^{m} \gamma(e_j^*) \int_X h(x, e_j)\, d\mu_j.$$

Proof. First we calculate $\|\gamma \mid \mathcal{L}\|$. Let $T \in \mathcal{L}$ with $\|T\| = \sup_{1\le j\le m} |T(e_j)| \le 1$ and write $T = \sum_{i=1}^{m} T(e_i)e_i^*$. Then

$$|\gamma(T)| = |\sum_{i=1}^{m} T(e_i)\gamma(e_i^*)| \le \sum_{i=1}^{m} |\gamma(e_i^*)|$$

and $\|\gamma \mid \mathcal{L}\| \le \sum_{i=1}^{m} |\gamma(e_i^*)|$. Now define T by setting $T(e_i) = 1, -1$, or 0 according as $\gamma(e_i^*)$ is positive, negative or zero. Then $\|T\| = 1$ and $\gamma(T) = \sum_{i=1}^{m} |\gamma(e_i^*)| \le \|\gamma \mid \mathcal{L}\|$. It follows that $\|\gamma \mid \mathcal{L}\| = \sum_{i=1}^{m} |\gamma(e_i^*)|$ and a) and b) are equivalent.

If c) holds, then it follows that $\|\gamma\| \le \sum_{j=1}^{m} |\gamma(e_j^*)| = \|\gamma \mid \mathcal{L}\| \le \|\gamma\|$ and a) holds. So it remains to show that b) implies c).

Assuming b), let $\overline{\gamma} = (\gamma_1, \cdots, \gamma_m) \in C^*(X)^m$ be given by Proposition 3.1. We now have

$$\|\gamma\| = \|\overline{\gamma}\| = \sum_{j=1}^{m} \|\gamma_j\| \geq \sum_{j=1}^{m} |\gamma(e_j^*)| = \|\gamma\|$$

and it follows that $|\gamma_j(1)| = |\gamma(e_j^*)| = \|\gamma_j\|$.

For $\gamma(e_i^*) \neq 0$ consider $(1/\gamma(e_i^*))\gamma_i = \mu_i \in C^*(X)$. Clearly $\|\mu_i\| = 1 = \mu_i(1)$. By the Riesz Representation Theorem μ_i is given by a Borel probability measure, that is, $\mu_i(f) = \int_X f(x)\, d\mu_i$. For $\gamma(e_j^*) = 0$ let μ_j be any Borel probability measure. Finally

$$\begin{aligned}
\gamma(h) = \overline{\gamma}(\Theta(h)) &= \sum_{j=1}^{m} \gamma_j(h(\cdot, e_j)) \\
&= \sum_{\gamma(e_j^*)\neq 0} \gamma(e_j^*) \frac{1}{\gamma(e_j^*)} \gamma_j(h(\cdot, e_j)) \\
&= \sum_{j=1}^{m} \gamma(e_j^*) \int_X h(x, e_j)\, d\mu_j
\end{aligned}$$

to complete the proof. □

For $\eta \in C^*(X)$ it is clear that $\eta(1) = \|\eta\|$ if and only if $\eta(f) = \int_X f(x)\, d\mu$ for some Borel measure μ. This can be rephrased as follows: $\eta(f) = \int_X f(x)\, d\mu$ for all $f \in C(X)$ or $\eta(f) = -\int_X f(x)\, d\mu$ for all $f \in C(X)$ if and only if $\|\eta \mid \text{constant functions}\| = \|\eta\|$. Hence the elements $\gamma \in \mathcal{C}^*$ such that $\|\gamma \mid \mathcal{L}\| = \|\gamma\|$ provide analogs of measures for $\mathcal{C}$ and can be viewed as integration in the $\sum_{i=1}^{m} \gamma(e_i^*) e_i$ direction. (An ordinary Borel measure providing integration in the positive direction of $\mathbb{R}$).

The natural action of $\mathbb{Z}^m$ on $\mathcal{C}$, given by $ah(x, b) = h(ax, b)$, is clearly norm preserving and hence continuous. We say $\gamma \in \mathcal{C}^*$ is **invariant** if $\gamma(ah) = \gamma(h)$ for all $a \in \mathbb{Z}^m$ and $h \in \mathcal{C}$.

Proposition 3.3. *If $\gamma \in \mathcal{C}^*$ is invariant, then there exists $\overline{\gamma} = (\gamma_1, \cdots, \gamma_m) \in C^*(X)^m$ such that*

a) $\gamma(h) = \overline{\gamma}(\Theta(h))$
b) $\|\gamma\| = \|\overline{\gamma}\|$
c) $\gamma(e_j^*) = \gamma_j(1)$
d) $|\gamma(e_j^*)| \leq \|\gamma_j\|$
e) γ_j *is invariant.*

Proof. Apply Proposition 3.1 to get $\overline{\gamma}_0$ satisfying a) through d), and observe that

$$\overline{\gamma}_N = \frac{1}{N^m} \sum_{0 \leq a_i < N} a\overline{\gamma}_0$$

also satisfies a) through d) because γ is invariant. There exists a subsequence of $\overline{\gamma}_N$ converging coordinate-wise in the weak* topology and its limit will satisfy a) through e). □

Given $w \in \mathbb{R}^m$ with $|w| = 1$, we define the invariant cocycle integrals in the w direction by

$$\mathcal{I}(w) = \{\gamma \in \mathcal{C}^* : \|\gamma\| = 1, \gamma(e_i^*) = w_i, \gamma \text{ is invariant}\}.$$

Theorem 3.4. *If $\gamma \in \mathcal{I}(w)$, then there exist invariant Borel probability measures $\mu_1, \cdots, \mu_m$ on X such that*

$$\gamma(h) = \sum_{j=1}^{m} w_j \int_X h(x, e_j)\, d\mu_j.$$

Furthermore, $\mathcal{I}(w)$ is non-empty, convex, and compact in the weak topology on $\mathcal{C}^*$.*

Proof. Let $\gamma \in \mathcal{I}(w)$. Clearly

$$\sum_{j=1}^{m} |\gamma(e_j^*)| = \sum_{j=1}^{m} |w_j| = |w| = 1 = \|\gamma\|$$

and Theorem 3.2 applies using the $\overline{\gamma}$ given by Proposition 3.3 to produce the required invariant measures. The rest is obvious. □

We now have a set of invariant cocycle integrals for each direction in $\mathbb{R}^m$ and it is natural to ask if there is a unifying concept of cocycle integral from which the $\mathcal{I}(w)'s$ can be extracted. This requires a different dual of $\mathcal{C}$.

4. Invariant Cocycle Integrals

In this section we define a new dual of $\mathcal{C}$ based on the idea that the integral of a cocycle should be a constant cocycle. Define $\mathcal{C}^\# = \{\sigma \colon \mathcal{C} \to \mathcal{L} : \sigma \text{ is bounded and linear}\}$ with norm

$$\|\sigma\| = \sup_{\|h\| \le 1} \|\sigma(h)\| = \sup_{\|h\| \le 1} \sup_{|w| \le 1} |\sigma(h)(w)|$$

where $h \in \mathcal{C}$ and $w \in \mathbb{R}^m$. Clearly $\mathcal{C}^\#$ is a Banach space.

We can also define a weak* topology on $\mathcal{C}^\#$ by $\sigma_n \to \sigma$ if and only if $\sigma_n(h) \to \sigma(h)$ for all $h \in \mathcal{C}$. As usual a neighborhood base at σ_0 is all sets of the form

$$\{\sigma : |\sigma(h_i) - \sigma_0(h_i)| < \epsilon,\ i = 1, \cdots, k\}.$$

The proof of Alaoglu's Theorem applies here and the unit ball in $\mathcal{C}^\#$ is weak* compact. For $x \in X$ we define $\sigma_x \in \mathcal{C}^\#$ by $\sigma_x(h) = \sum_{j=1}^{m} h(x, e_j) e_j^*$. It follows that $\|\sigma_x\| = 1$, $\sigma_x(T) = T$ for $T \in \mathcal{L}$, and $x \to \sigma_x$ is a homeomorphism of X into $\mathcal{C}^\#$ with the weak* topology.

Theorem 4.1. *Let $\sigma \in \mathcal{C}^{\#}$. If for all $T \in \mathcal{L}, \sigma(T) = \|\sigma\|T$, then there exist Borel probability measures $\mu_1, \cdots, \mu_m$ on X such that*

$$\sigma(h) = \|\sigma\| \sum_{j=1}^{m} \int_X h(x, e_j)\, d\mu_j e_j^*.$$

Proof. Set $\gamma_i(h) = \sigma(h)(e_i)$. Clearly $\gamma_i \in \mathcal{C}^*$ and $\sigma(h) = \sum_{j=1}^{m} \gamma_j(h) e_j^*$. We also have

$$\|\sigma\| e_j^* = \sigma(e_j^*) = \sum_{i=1}^{m} \gamma_i(e_j^*) e_i^*$$

and hence $\gamma_i(e_j^*) = \|\sigma\|\delta_{ij}$.

Now

$$\begin{aligned} \|\gamma_i\| &= \sup_{\|h\| \le 1} |\gamma_i(h)| = \sup_{\|h\| \le 1} |\sigma(h)(e_i)| \le \sup_{\|h\| \le 1} \|\sigma(h)\| = \|\sigma\| \\ &= \sum_{j=1}^{m} \|\sigma\|\delta_{ij} = \sum_{j=1}^{m} |\gamma_i(e_j^*)| \le \|\gamma_i\| \end{aligned}$$

and thus

$$\|\gamma_i\| = \sum_{j=1}^{m} |\gamma_i(e_j^*)|.$$

Therefore, by Theorem 3.2

$$\gamma_i(h) = \|\sigma\| \int_X h(x, e_i)\, d\mu_i$$

for some Borel probability measure μ_i on X. □

Theorem 4.2. *Let $\gamma \in \mathcal{C}^*$. Then $\|\gamma \mid \mathcal{L}\| = \|\gamma\|$ if and only if there exists $\sigma \in \mathcal{C}^{\#}$ and $w \in \mathbb{R}^m$ such that*

$$\sigma(T) = \|\sigma\|T$$

for all $T \in \mathcal{L}$ and

$$\gamma(h) = \sigma(h)(w)$$

for all $h \in \mathcal{C}$.

Proof. Suppose $\|\gamma \mid \mathcal{L}\| = \|\gamma\|$. Then by Theorem 3.2

$$\gamma(h) = \sum_{j=1}^{m} \gamma(e_j^*) \int_X h(x, e_j)\, d\mu_j$$

where $\mu_1, \cdots, \mu_m$ are Borel probability measures on X. Define $\sigma \in \mathcal{C}^{\#}$ by

$$\sigma(h) = \sum_{j=1}^{m} \int_X h(x, e_j)\, d\mu_j e_j^*,$$

set $w = (\gamma(e_1^*), \cdots, \gamma(e_m^*))$ and check the details.

For the converse observe that

$$\gamma(e_j^*) = \sigma(e_j^*)(w) = \|\sigma\| w_j$$

and

$$\|\gamma\| = \sup_{\|h\| \le 1} |\sigma(h)(w)| \le \|\sigma\|\, |w| = \sum_{j=1}^{m} |\gamma(e_j^*)| = \|\gamma \mid \mathcal{L}\| \le \|\gamma\|.$$

This completes the proof. □

Definition 4.3. An **invariant cocycle integral** is an element σ of $\mathcal{C}^{\#}$ satisfying:

1) $\|\sigma\| = 1$
2) $\sigma(ah) = \sigma(h)$ for all $h \in \mathcal{C}$ and $a \in \mathbb{Z}^m$
3) $\sigma(T) = T$ for all $T \in \mathcal{L}$.

The set of invariant cocycle integrals for $(X, \mathbb{Z}^m)$ will be denoted by $\mathcal{I}(X, \mathbb{Z}^m)$ or simply $\mathcal{I}$.

Theorem 4.4. *Let $\sigma \in \mathcal{C}^{\#}$. Then $\sigma \in \mathcal{I}$ if and only if there exist invariant Borel probability measures $\mu_1, \cdots, \mu_m$ such that*

$$\sigma(h) = \sum_{j=1}^{m} \int_X h(x, e_j)\, d\mu_j e_j^*.$$

Proof. Again define $\gamma_i(h) = \sigma(h)(e_i)$. Note that $\gamma_i(ah) = \sigma(ah)(e_i) = \sigma(h)(e_i) = \gamma_i(h)$ for all a and h. Consequently by Theorems 3.4 and 4.1 it has the required form. The converse is just a matter of checking that properties 1), 2) and 3) hold. □

We do not know whether or not the measures μ_j are uniquely determined by σ. In other words are there always enough cocycles so that the functions $h(\cdot, e_j)$ can distinguish invariant measures?

It is also now a routine calculation to prove the following:

Theorem 4.5. *If $w \in \mathbb{R}^m$ with $|w| = 1$, then*

$$\mathcal{I}(w) = \{\gamma : \exists\, \sigma \in \mathcal{I} \ni \gamma(h) = \sigma(h)(w)\}.$$

Theorem 4.6. *The set $\mathcal{I}$ of invariant cocycle integrals is a non-empty convex weak* compact subset of $\mathcal{C}^{\#}$. Moreover, there is an affine weak* homeomorphism Φ from $\mathcal{I}$ onto $\mathcal{I}(e_1) \times \cdots \times \mathcal{I}(e_m)$.*

The ideas and formulas in Sections 3 and 4 can also be expressed in the language of tensor products. In particular, $\mathcal{C}^{\#}$ is naturally identified with the continuous bilinear forms on $\mathcal{C} \times \mathbb{R}^m$ and hence is isomorphic to $(\mathcal{C} \otimes \mathbb{R}^m)^*$. Working from this point of view K. Madden [**6**] has shown that $\mathcal{C}^{\#}$ is naturally isometrically isomorphic to the Banach space of all bounded linear functions $\rho\colon \mathcal{C}_m \to \mathcal{L}_m$ such that $\rho(Tg) = T\rho(g)$ for $g \in \mathcal{C}_m$, the $\mathbb{R}^m$ valued cocycles, and $T \in \mathcal{L}_m$, the linear maps of $\mathbb{R}^m$ into itself.

5. Extreme Points and Ergodic Averages

Let $\mu \in \mathcal{M}$, the invariant Borel probability measures for $(X, \mathbb{Z}^m)$. So $\mathcal{M}$ is a compact convex set in the weak* topology and its extreme points are the ergodic measures $\mathcal{E}$. If $\mu \in \mathcal{E}$, then by the ergodic theorem there exists $x \in X$ such that for all $f \in C(X)$

$$\int_X f\, d\mu = \lim_{N\to\infty} \frac{1}{N^m} \sum_{\substack{0 \le a_i < N \\ 1 \le i \le m}} f(ax)$$

and the ergodic averages provide an access to the extreme points of $\mathcal{M}$ and hence to $\mathcal{M}$. In this section we establish two similar approaches to the extreme points of $\mathcal{I}(e_j)$.

For $N \in \mathbb{N}, x \in X$, and $h \in \mathcal{C}$ let $A_N\sigma_x$ denotes the element of $\mathcal{C}^{\#}$ defined by

$$A_N\sigma_x(h) = \frac{1}{N^m} \sum_{j=1}^{m} \sum_{\substack{0 \le a_i < N \\ 1 \le i \le m}} h(ax, e_j)e_j^*.$$

It is easily checked that $\|A_N\sigma_x\| = 1$, and $A_N\sigma_x(T) = T$ for $T \in \mathcal{L}$. If $A_{N_k}\sigma_{x_k}$ converges to σ in the weak* topology and N_k goes to infinity, then $\sigma \in \mathcal{I}$ because

$$|A_N\sigma_x(e_k h) - A_N\sigma_x(h)| \le \frac{2mN^{m-1}}{N^m}\|h\| = \frac{2m\|h\|}{N}.$$

Let σ be a weak* limit of a sequence $A_{N_k}\sigma_{x_k}$ with N_k going to infinity. $A_{N_k}\sigma_{x_k}$ can be regarded as acting on $C(X)$ so that $\sigma \in C^*(X)$. Thus there exists $\mu \in \mathcal{M}$ such that

$$\sigma(h) = \sum_{j=1}^{m} \int_X h(x, e_j)\, d\mu e_j^*.$$

The map $\Theta\colon \mathcal{C} \to C(X)^m$ has a dual $\Theta^*\colon C^*(X)^m \to \mathcal{C}^*$ defined by $\Theta^*(\overline{\gamma})(h) = \overline{\gamma}(\Theta(h))$ and we can look at this coordinate-wise, that is $\theta_j^*\colon C^*(X) \to \mathcal{C}^*$ by

$\theta_j^*(\eta) = \Theta^*(0, \cdots, \eta, \cdots, 0)$ or $\theta_j^*(\eta)(h) = \eta(h(\,\cdot\,, e_j))$. It follows from Theorem 3.4 that $\theta_j^*(\mathcal{M}) = \mathcal{I}(e_j)$, and the extreme points of $\mathcal{I}(e_j)$ all have the form $\gamma(h) = \int_X h(x, e_j)\, d\mu$ for some $\mu \in \mathcal{M}$. (In fact, we will see in a minute that we could even assume μ is ergodic.)

Because there is an affine homeomorphism from $\mathcal{I}$ onto $\mathcal{I}(e_1) \times \cdots \times \mathcal{I}(e_m)$, the extreme points of $\mathcal{I}$ have the same structure as the extreme points of a cartesian product of compact convex sets. As soon as two of the sets $\mathcal{I}(e_j)$ contain more than one point, there exists an extreme point σ of $\mathcal{I}$ that does not have the form of $\sigma(h)$ shown above with a single $\mu \in \mathcal{M}$. Consequently we turn our attention to the extreme points of $\mathcal{I}(e_j)$.

Theorem 5.1. *If γ is an extreme point of $\mathcal{I}(e_j)$, then there exists $x \in X$ such that $A_N\sigma_x$ converges in the weak* topology to $\sigma \in \mathcal{I}$ satisfying $\gamma(h) = \sigma(h)(e_j)$.*

Proof. Consider the compact convex set $\{\mu \in \mathcal{M} : \theta_j^*(\mu) = \gamma\}$. It is easily checked that its extreme points are also extreme point of $\mathcal{M}$. Hence there exists $\mu \in \mathcal{E}$ such that $\theta_j^*(\mu) = \gamma$ and the required σ is given by

$$\sigma(h) = \sum_{j=1}^{m} \int_X h(x, e_j)\, d\mu e_j^*.$$

□

For $m = 1$ and $a = ne_1, n > 0$ we have

$$\frac{h(x,a)}{|a|} = \frac{1}{n} \sum_{k=0}^{n-1} h((ke_1)x, e_1)$$

which is an ergodic average. How do we interpret $h(x, a)/|a|$ as an ergodic average in general? We will show that when a goes to infinity in a particular direction, $h(x, a)/|a|$ will determine an invariant integral in that direction.

Given $x \in X$ and $a \in \mathbb{Z}^m, a \neq 0$, define $\gamma_{(x,a)} \in \mathcal{C}^*$ by

$$\gamma_{(x,a)}(h) = \frac{h(x,a)}{|a|}.$$

Obviously, $\|\gamma_{(x,a)}\| \le 1$ because $|h(x, a)| \le \|h\|\,|a|$. Using $T = \sum_{j=1}^m \operatorname{sign}(a_j)e_j^* \in \mathcal{L}$, we have $\|T\| = 1$ and $\gamma_{(x,a)}(T) = 1$. Hence $\|\gamma_{(x,a)}\| = \|\gamma_{(x,a)} \mid \mathcal{L}\| = 1$.

Theorem 5.2. *Let $\{x_k\}$ and $\{a_k\}$ be sequences in X and $\mathbb{Z}^m$. If $\gamma_{(x_k,a_k)}$ converges to γ in the weak* topology and $|a_k|$ goes to infinity, then $a_k/|a_k|$ converges to w and $\gamma \in \mathcal{I}(w)$.*

Proof. Observe that

$$\gamma(e_j^*) = \lim_{k\to\infty} \gamma_{(x_k,a_k)}(e_j^*) = \lim_{k\to\infty} \frac{(a_k)_j}{|a_k|},$$

and hence $\lim_{k\to\infty} a_k/|a_k|$ converges to $w = (\gamma(e_1^*), \cdots, \gamma(e_m^*))$. Clearly $|w| = 1$ and $\|\gamma\| \le 1 = \sum_{j=1}^m |\gamma(e_j^*)| = \|\gamma \mid \mathcal{L}\| \le \|\gamma\|$. So $\|\gamma\| = 1$ and it remains to show that γ is invariant.

It suffices to show that $\gamma(e_j h) = \gamma(h)$, which is equivalent to showing that

$$0 = \lim_{k\to\infty} \left(\gamma_{(x_k,a_k)}(e_j h) - \gamma_{(x_k,a_k)}(h)\right) = \lim_{k\to\infty} \frac{h(e_j x_k, a_k) - h(x_k, a_k)}{|a_k|}.$$

This follows from

$$\begin{aligned} |h(e_j x_k, a_k) - h(x_k, a_k)| &= |h(x_k, a_k + e_j) - h(x_k, e_j) - h(x_k, a_k)| \\ &\le |h(x_k, a_k + e_j) - h(x_k, a_k)| + |h(x_k, e_j)| \\ &\le \|h\|\,|a_k + e_j - a_k| + \|h\|\,|e_j| = 2\|h\| \end{aligned}$$

because $|a_k|$ goes to infinity. □

Let $\mathcal{M}_j$ denote the invariant Borel measures for $T_j(x) = e_j x$ and let $\mathcal{E}_j$ denote the ergodic measures for T_j. So $\mathcal{M} = \bigcap_{j=1}^m \mathcal{M}_j$.

Theorem 5.3. *For $j = 1, \cdots, m$, $\mathcal{I}(e_j) = \theta_j^*(\mathcal{M}_j)$. If γ is an extreme point of $\mathcal{I}(e_j)$, there exists $x \in X$ such that $\gamma_{(x,Ne_j)}$ converges to γ in the weak* topology.*

Proof. Because $\mathcal{M} \subset \mathcal{M}_j$ we know $\mathcal{I}(e_j) \subset \theta_j^*(\mathcal{M}_j)$. Since θ_j^* is linear, it suffices to show that $\theta_j^*(\mathcal{E}_j) \subset \mathcal{I}(e_j)$ to prove that $\theta_j^*(\mathcal{M}_j) = \mathcal{I}(e_j)$.

If $\mu \in \mathcal{E}_j$, then because μ is ergodic there are points $x \in X$ such that for all $f \in C(X)$

$$\mu(f) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(T_j^k(x)).$$

Thus for $h \in \mathcal{C}$

$$\theta_j^*(\mu)(h) = \lim_{N\to\infty} \frac{h(x, Ne_j)}{N} = \lim_{N\to\infty} \gamma_{(x,Ne_j)}(h)$$

and by the previous proposition the weak* limit of $\gamma_{(x,Ne_j)}$ is in $\mathcal{I}(e_j)$.

To prove the second part proceed as in the proof of Theorem 5.1. □

6. Uniform Convergence of Ergodic Averages

In this section we will show that for uniform convergence the two types of ergodic averages, $A_N\sigma_x(h)$ and $h(x,a)/|a|$, have the same behavior. The invariant cocycle integrals provide the link between them and clean generalizations of the classical results for $\mathbb{Z}$. These results then lead to characterizations of a unique cocycle integral and of the closure of the coboundaries.

Proposition 6.1. *Let $h \in \mathcal{C}$. The ergodic average*

$$\frac{1}{N^m} \sum_{0 \leq a_i < N} h(ax, e_j)$$

converges to 0 *uniformly in x if and only if $\gamma(h) = 0$ for all $\gamma \in \mathcal{I}(e_j)$.*

Proof. Suppose it does not converge uniformly to 0. Then there exists $\epsilon > 0$ and sequences $\{x_k\}$ and $\{N_k\}$ in X and $\mathbb{Z}$ such that N_k goes to infinity and

$$|A_{N_k} \sigma_{x_k}(h)(e_j)| \geq \epsilon \,.$$

We can assume without loss of generality that $A_{N_k}\sigma_{x_k}$ converges in the weak* topology to $\sigma \in \mathcal{I}$. Then clearly $|\sigma(h)(e_j)| \geq \epsilon$ and $\gamma(h) \neq 0$ for all $\gamma \in \mathcal{I}(e_j)$.

Now suppose the average does converge uniformly to 0. It suffices to show that $\gamma(h) = 0$ for every extreme point of $\mathcal{I}(e_j)$. If γ is an extreme point of $\mathcal{I}(e_j)$, then using Theorem 5.1 it follows from the uniform convergence to 0 that $\gamma(h) = \sigma(h)(e_j) = 0$.

Proposition 6.2. *Let $h \in \mathcal{C}$ and let $w \in \mathbb{R}^m$ with $|w| = 1$. If $\gamma(h) = 0$ for all $\gamma \in \mathcal{I}(w)$, then $h(x, a_k)/|a_k|$ converges to* 0 *uniformly in x whenever $\lim_{k\to\infty} a_k/|a_k| = w$ and $\lim_{k\to\infty} |a_k| = \infty$.*

Proof. Suppose the convergence is not uniform to 0. By choosing a subsequence of a_k we can assume

$$|\gamma_{(x_k,a_k)}(h)| = \frac{|h(x_k, a_k)|}{|a_k|} \geq \epsilon$$

for some $\epsilon > 0$ and a sequence $\{x_k\}$ from X. Then by taking a weak* limit of a subsequence of $\gamma_{(x_k,a_k)}$, we obtain $\gamma \in \mathcal{I}(w)$ such that $|\gamma(h)| \geq \epsilon$. □

The converse holds for $w = e_j$.

Proposition 6.3. *Let $h \in \mathcal{C}$. The sequence $h(x, ne_j)/n$ converges to* 0 *uniformly in x as n goes to infinity if and only if $\gamma(h) = 0$ for all $\gamma \in \mathcal{I}(e_j)$.*

Proof. The "if" part follows from the previous proposition. Assume the convergence is uniform and let γ be an extreme point of $\mathcal{I}(e_j)$. Then by Theorem 5.3 γ is the weak* limit of $\gamma_{(x,Ne_j)}$ with N going to infinity, and

$$\gamma(h) = \lim_{N\to\infty} \frac{h(x, Ne_j)}{N} = 0 \,.$$

□

Propositions 6.1 and 6.3 have the following corollary

Corollary 6.4. *Let $h \in \mathcal{C}$. The ergodic average*

$$\frac{1}{N^m} \sum_{0 \leq a_i < N} h(ax, e_j)$$

converges to 0 *uniformly in x as N goes to infinity if and only if the sequence $h(x, ne_j)/n$ converges to* 0 *uniformly in x as n goes to infinity.*

These three propositions can also be assembled into the following global result.

Theorem 6.5. *Let $h \in \mathcal{C}$. The following are equivalent:*

(1) *The average $A_N \sigma_x(h)$ converges to* 0 *uniformly in x as N goes to infinity.*
(2) *As $|a|$ goes to infinity $h(x, a)/|a|$ converges to* 0 *uniformly in x.*
(3) $\sigma(h) = 0$ *for all $\sigma \in \mathcal{I}$.*

The now obvious proof that (3) implies (2) is precisely where a real advantage has been gained from the use of $\mathcal{I}$. The next theorem is an application that builds on the full strength of Theorem 6.5.

Theorem 6.6. *The following are equivalent:*

(1) *There exists a unique invariant cocycle integral σ_0.*
(2) *For $h \in \mathcal{C}$ there exists $T_h \in \mathcal{L}$ such that $A_N \sigma_x(h)$ converges uniformly in x to T_h as N goes to infinity.*
(3) *For $h \in \mathcal{C}$ there exists $T_h \in \mathcal{L}$ such that*

$$\frac{|h(x, a) - T_h(a)|}{|a|}$$

goes to 0 *uniformly in x as $|a|$ goes to infinity.*

Proof. Assuming (1) holds we can apply Theorem 6.5 to $h - \sigma_0(h)$ because $\sigma_0(h - \sigma_0(h)) = \sigma_0(h) - \sigma_0(h) = 0$ and obtain $A_N \sigma_x(h - \sigma_0(h)) = A_N \sigma_x(h) - \sigma_0(h)$ converging uniformly to 0. Similarly (2) implies (3) because Theorem 6.5 can be applied to $h - T_h$. Finally if (3) holds, Theorem 6.5 can be used again to conclude that $\sigma(h - T_h) = 0$ for all $\sigma \in \mathcal{I}$. It follows that $\sigma(h) = T_h$ for all $\sigma \in \mathcal{I}$ and hence $\mathcal{I}$ contains at most one element. □

If $\sigma \in \mathcal{I}$ is given by a single ergodic measure, that is

$$\sigma(h) = \sum_{j=1}^{m} \int_X h(x, e_j)\, d\mu e_j^*$$

where μ is an extreme point of $\mathcal{M}$, then the results of Boivin and Derriennic [**1**] apply. In particular, their Theorem 1 implies that for $h \in \mathcal{C}$

$$\lim_{|a| \to \infty} \frac{|h(x, a) - \sigma(h)(a)|}{|a|} = 0$$

μ-almost everywhere. Theorem 5.3 can be interpreted as a generalization of this result, but may not be the best possible. Although working with elements of $\mathcal{I}$ given by several invariant measures may seem cumbersome, these cocycle invariant integrals do lead to a characterization of $\overline{\mathcal{B}}$ as the next theorem shows.

Our final result is to use $\mathcal{I}$ to identify the subspace $\overline{\mathcal{B}}$ of $\mathcal{C}$. As in Section 2, define $S\colon C(X) \to \mathcal{C}$ by $Sf(x,a) = f(ax) - f(x)$ and $\mathcal{B}$ is the set of coboundaries of the range of S. Clearly $\sigma(Sf) = 0$ and hence $\sigma(h) = 0$ for all $h \in \overline{\mathcal{B}}$. We will prove the converse.

The following calculation which H. Furstenberg showed us will be essential. Given $h \in \mathcal{C}$, set

$$f_N(x) = -\frac{1}{N^m} \sum_{0 \le a_i < N} h(x,a).$$

Clearly $f_N \in C(X)$. Now set

$$h_N = h - Sf_N$$

so h_N and h differ by a coboundary.

Now using the cocycle formula we have

$$\begin{aligned} h_N(x,e_j) &= h(x,e_j) + \frac{1}{N^m} \sum_{0 \le a_i < N} \{h(e_j x, a) - h(x,a)\} \\ &= h(x,e_j) + \frac{1}{N^m} \sum_{0 \le a_i < N} \{h(ax, e_j) - h(x,e_j)\} \\ &= A_N \sigma_x(h)(e_j). \end{aligned}$$

We can now prove:

Theorem 6.7. $\overline{\mathcal{B}} = \{h : \sigma(h) = 0$ *for all* $\sigma \in \mathcal{I}\}$.

Proof. We have already pointed out that $\sigma(h) = 0$ for all $\sigma \in \mathcal{I}$ when $h \in \mathcal{B}$. Suppose that $\sigma(h) = 0$ for all $\sigma \in \mathcal{I}$. By Proposition 6.1 $A_N \sigma_x(h)(e_j)$ converges to 0 uniformly in x for $j = 1, \cdots, m$. But $h_N(x,e_j) = A_N \sigma_x(h)(e_j)$, so

$$\|h - Sf_N\| = \|h_N\|$$

converges to 0 and $h \in \overline{\mathcal{B}}$. □

Theorem 6.8. *There is a unique invariant cocycle integral if and only if* $\mathcal{C} = \mathcal{L} \oplus \overline{\mathcal{B}}$.

Proof. If $\mathcal{I} = \{\sigma_0\}$, then Theorem 6.7 applies to $h - \sigma_0(h)$. If $\mathcal{C} = \mathcal{L} \oplus \overline{\mathcal{B}}$, then $h \in \mathcal{C}$ can be uniquely expressed as $h = T + g$ with $T \in \mathcal{L}$ and $g \in \overline{\mathcal{B}}$. Hence $\sigma(h) = T$ for $\sigma \in \mathcal{I}$ and there is a unique cocycle integral. □

Clearly unique ergodicity implies there is a unique cocycle integral. The converse is an interesting open question.

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H. B. Keynes, School of Mathematics, University of Minnesota, Minneapolis, MN 55455, U.S.A.

N. G. Markley, Department of Mathematics, University of Maryland, College Park, MD 20742, U.S.A.

M. Sears, Department of Mathematics, University of the Witwatersrand, Johannesburg, South Africa