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DISCRETIZATION AND MORSE–SMALE DYNAMICAL SYSTEMS ON PLANAR DISCS

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Abstract. In a previous paper [**12**], we have shown that locally, in the vicinity of hyperbolic equilibria of autonomous ordinary differential equations, the time-h-map of the induced dynamical system is conjugate to the h-discretized system i.e. to the discrete dynamical system obtained via one-step discretization with stepsize h. The present paper is devoted to Morse-Smale dynamical systems defined on planar discs and having no periodic orbits. Using elementary extension techniques, we point out that local conjugacies about saddle points can be glued together: the time-h-map is globally conjugate to the h-discretized system. This is a discretization analogue of the famous Andronov-Pontryagin theorem [**2**], [**18**] on structural stability. For methods of order p, the conjugacy is $O(h^p)$-near to the identity. The paper ends with some general remarks on similar problems.

0. Introduction and the Main Result

Let $|\cdot|$ denote the Euclidean norm on $\mathbf{R}^2$ and consider the unit disc $D = \{x \in \mathbf{R}^2 \mid |x| \le 1\}$. The boundary of D is denoted by ∂D. Assume that $\mathcal{N}$ is an open neighbourhood of D and that, for some positive integer p, the function $f\colon \mathcal{N} \to \mathbf{R}^2$ is of class C^{p+1} (with all derivatives bounded — the norm in C^{p+1} is defined by

$$|f|_{p+1} = \max\{\sup\{f^{(j)}(x) \mid x \in \mathcal{N}\} \mid j = 0, 1, \dots, p+1\})$$

and satisfies

(i) $x \cdot f(x) < 0$ whenever $x \in \partial D$;
(ii) $\dot{x} = f(x)$ has a finite number of equilibrium points in D, all hyperbolic;
(iii) alpha- and omega- limit sets of trajectories in D are equilibria;
(iv) there are no saddle connections in D.

Actually, conditions (i)–(iv) concern the local dynamical system Φ induced by the differential equation $\dot{x} = f(x)$, $x \in \mathcal{N}$, and not function f itself. Geometrically, (i) means that D is positively invariant for Φ and ∂D is transversally cutted by the trajectories. By (ii), the equilibria of $\dot{x} = f(x)$ in D can be classified as

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sinks, saddles and sources according as the eigenvalues of the Jacobian satisfy $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 < 0$, $\lambda_2 < 0 < \lambda_1$ and $0 < \operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2$, respectively. Outgoing and ingoing trajectories at saddle points are called separatrices. Condition (iv) means that the second endpoint of a separatrix in D is a nonsaddle equilibrium. Separatrices as edges and equilibria as vertices form the so-called separatrix graph and define a cell-decomposition of D. The detailed topological and dynamical description of these cells was given in [**6**], [**24**].

Without going into the details here, we remark that differential equations satisfying (i)–(iv) are exactly those Morse-Smale dynamical systems on D which have no periodic orbits — essentially, Morse-Smale gradient systems [**18**]. There study is/was one of the starting points [**2**], [**24**], [**26**] for the theory of structural stability [**22**], [**25**].

With the differential equation $\dot{x} = f(x)$, $x \in \mathcal{N}$, we consider also its h-discretized system i.e. the discrete local dynamical system induced by a C^{p+1} mapping $\varphi\colon [0,h_0] \times \mathcal{N} \to \mathbf{R}^2$ where $h_0 > 0$ (and, as usual, the C^{p+1} property of φ on $[0,h_0] \times \mathcal{N}$ is understood as the existence of a C^{p+1} extension $\tilde{\varphi}$ defined on an open neighbourhood of $[0,h_0] \times \mathcal{N}$ in $\mathbf{R} \times \mathbf{R}^2$). Besides the differentiability assumption, the only requirement on φ is the existence of a positive constant K and of an open neighbourhood $\mathcal{M}$ of D in $\mathcal{N}$ such that

$$|\Phi(h,x) - \varphi(h,x)| \le Kh^{p+1} \quad \text{for all } h \in [0,h_0],\ x \in \mathcal{M}. \tag{1}$$

The main result of the present paper is that Morse-Smale gradient systems on planar discs — up to a conjugacy $O(h^p)$- near to the identity — are correctly reproduced by numerical methods. In particular, the h-discretized system $\varphi(h,\cdot)\colon D \to \mathbf{R}^2$ embeds in a continuous-time dynamical system.

Assume that $p \ge 2$. (This is a technical assumption we are not able to get rid of: see Remark 1.3. On the other hand, formula (15) in Section 2 is an indication that the Theorem might be false with $p = 1$.)

Theorem. *Let $\mathcal{N}$, p, D, f, Φ and φ, $\mathcal{M}$ be as above. Then there is a positive constant K and, for all h sufficiently small, there exists a homeomorphism $\mathcal{H}_h\colon D \to \mathcal{H}_h(D)$ such that $\mathcal{H}_h(\Phi(h,x)) = \varphi(h,\mathcal{H}_h(x))$ and $|\mathcal{H}_h(x) - x| \le Kh^p$ whenever $x \in D$.*

The proof of the Theorem is presented in the next Section. Section 2 contains some general remarks on whether and how qualitative properties of ordinary differential equations are inherited by one-step numerical methods — an approach [**16**], [**3**], [**4**], [**9**], [**12**] (and the references therein) complementary to detecting qualitative properties of ordinary differential equations by their numerical solutions.

1. The Proof of the Theorem

The proof is subdivided into several steps.

Step 1: **a preliminary globalizing extension**. Recall condition (i) and set $\mathcal{D} = \{x \in \mathbf{R}^2 \mid |x| \leq 2\}$. Modifying f outside a closed neighbourhood of D, we may assume that $\mathcal{N} = \mathcal{M} = \mathbf{R}^2$ and, in Cartesian coordinates, $N = (0, 2)$ is a source, $S = (0, -2)$ is a saddle; $\partial\mathcal{D}$ consists of four trajectories (i.e. of the two equilibria N, S plus of two connecting separatrices); the outgoing separatrices at S tend to sinks (in D and in $\mathbf{R}^2 \backslash \mathcal{D}$, respectively) and; last but not least, all points in $\mathcal{D} \backslash \partial\mathcal{D}$ are attracted by D.

As a slight generalization of the Theorem, we prove in the sequel that there is a positive constant K and, for all h sufficiently small, there is a homeomorphism $\mathcal{J}_h \colon \mathcal{D} \to \mathcal{J}_h(\mathcal{D})$ such that $\mathcal{J}_h(\Phi(h, x)) = \varphi(h, \mathcal{J}_h(x))$ and $|\mathcal{J}_h(x) - x| \leq Kh^p$ whenever $x \in \mathcal{D}$.

The reason for passing to $\mathcal{D}$ from D is to overcome the technical difficulty caused by the fact that some trajectories of $\dot{x} = f(x)$ leave D with finite negative escape time. On the other hand, the domain of nonextendable solutions of $\dot{x} = f(x)$ on $\mathcal{D}$ is $\mathbf{R}$: we have a C^{p+1} dynamical system on $\mathcal{D}$.

Step 2: **some useful properties of the discretization**. As an easy exercise on Taylor expansion formula, inequality (1) plus the differentiability assumption on φ imply (cf. [**12**, Formula (6)]) that, for all $j = 0, 1, \ldots, p + 1$, there holds

$$(2) \qquad |\Phi_x^{(j)}(h, x) - \varphi_x^{(j)}(h, x)| \leq Kh^{p+1-j} \quad \text{whenever } h \in [0, h_0],\ x \in \mathbf{R}^2.$$

By Gronwall lemma applied to the equation $\Phi''_{xt} = f'_x(\Phi)\Phi'_x$, we obtain that

$$(3) \qquad |\Phi'_x(t, x) - Id| \leq \exp(|f'_x|_0 \cdot t) - 1, \quad \text{for all } t \geq 0,\ x \in \mathbf{R}^2.$$

Using the $j = 1$ case of inequality (2) and $\Phi'_x(0, x) = Id$, it follows that $\varphi'_x(h, x)$ is invertible and $|(\varphi'_x(h, x))^{-1}| \leq K$ for all $h \in [0, h_0]$, $x \in \mathbf{R}^2$. (The constants K, h_0 etc. are not necessarily the same at different appearances.) In virtue of the Hadamard-Levy global inverse function theorem [**1**, Thm. 2.5.17], $\varphi(h, \cdot)$ is a diffeomorphism of $\mathbf{R}^2$ onto $\mathbf{R}^2$, $h \in [0, h_0]$. Thus, we may write that $\varphi(h, \psi(h, x)) = x$, $\psi(h, \varphi(h, x)) = x$ where, by the classical implicite function theorem, ψ is C^{p+1} on $[0, h_0] \times \mathbf{R}^2$. Setting $\Psi(h, x) = \Phi(-h, x)$ and comparing $\psi(h, \varphi(h, x)) = x$ with $\Psi(h, \Phi(h, x)) = x$, inequalities (1) and (3) imply that

$$(4) \qquad |\psi(h, x) - \Psi(h, x)| \leq Kh^{p+1} \quad \text{for all } h \in [0, h_0],\ x \in \mathbf{R}^2.$$

For $k \in \mathbf{N}$, define recursively

$$\varphi(0, h, x) = x, \quad \varphi(k + 1, h, x) = \varphi(h, \varphi(k, h, x)), \quad h \in [0, h_0], \quad x \in \mathbf{R}^2.$$

Via a standard application of the discrete Gronwall lemma, a further consequence of (1) is that, given $T > 0$ arbitrarily, there exists a constant $c(T)$ such that (see also formula (14) in Section 2)

(5) $|\Phi(kh, x) - \varphi(k, h, x)| \le c(T)h^p$ whenever $h \in [0, h_0]$, $kh \le T$ and $x \in \mathbf{R}^2$.

For later purposes, we note a trivial consequence of (3). Given $T > 0$ arbitrarily, there exists a constant $c(T)$ such that

(6) $|\Phi(t, z) - \Phi(t, w)| \le c(T)|z - w|$ whenever $0 \le t \le T$ and $z, w \in \mathbf{R}^2$.

It is clear from the previous considerations that, continuing the pairing (1) $\leftrightarrow$ (4), the (ψ, Ψ) – counterparts of inequalities (2), (3), (5), (6) are also valid.

Step 3: recalling a Hartman-Grobman result [12] on discretizations. The saddle points of $\dot{x} = f(x)$, $x \in \mathcal{D}$, are denoted by $P_0 = S, P_1, \ldots, P_n$. The remaining equilibria are denoted by $Q_0 = N$, $Q_1, \ldots, Q_m$. In virtue of the index theorem, $m - n = 1$.

Consider a saddle point P in $\mathcal{D}$ and let $\mathcal{U}$ be a compact neighbourhood of P. For brevity, we say that $\mathcal{U}$ is an isolating block of P — the terminology is borrowed from Conley's theory [**8**] on isolated invariant sets — if there are C^{p+1} curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_8$ and points $A_1, A_2, \ldots, A_8$ such that $\partial\mathcal{U} = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_8$; $\Gamma_s \cap \Gamma_{s+1} = A_{s+1}$, $s = 1, 2, \ldots, 8$, $\Gamma_9 = \Gamma_1$, $A_9 = A_1$; $\Gamma_s \cap \Gamma_t = \emptyset$ whenever $|s - t| > 1$; Γ_s, $s = 1, 3, 5, 7$ is a trajectory segment for Φ; Γ_s, $s = 2, 4, 6, 8$ is a transversal section for Φ; $\Gamma_2 \cup \Gamma_6$ and $\Gamma_4 \cup \Gamma_8$ consist of entry and exit points, respectively; and, last but not least, P is the maximal compact invariant set in $\mathcal{U}$. It is well-known that the collection of isolating blocks forms a neighbourhood basis for P. For $h \in (0, h_0]$, define

$$\mathcal{D}_{s,h} = \{\Phi(-t, x) \in \mathbf{R}^2 \mid 0 \le t \le h, \ x \in \Gamma_s\}, \ s = 4, 8,$$
$$\mathcal{E}_{s,h} = \{\Phi(t, x) \in \mathbf{R}^2 \mid 0 \le t \le h, \ x \in \Gamma_s\}, \ s = 2, 6.$$

Without loss of generality, we may assume that $\mathcal{D}_{4,h_0} \cup \mathcal{D}_{8,h_0}$ and $\mathcal{E}_{2,h_0} \cup \mathcal{E}_{6,h_0}$ are disjoint subsets of $\mathcal{U}$.

Lemma 1.1. *For each i, $i = 0, 1, \ldots, n$, there exists an isolating block $\mathcal{U}_i$ of the saddle point P_i and, for each h, $h \in (0, h_0]$, there exists a homeomorphism $\mathcal{H}_{h,i} \colon \mathcal{U}_i \to \mathcal{H}_{h,i}(\mathcal{U}_i)$ such that $\mathcal{H}_{h,i}(\Phi(h, x)) = \varphi(h, \mathcal{H}_{h,i}(x))$ whenever $x \in \mathrm{cl}\,(\mathcal{U}_i \setminus (\mathcal{D}_{4,h,i} \cup \mathcal{D}_{8,h,i}))$ (i.e. whenever both sides are defined) and, with some constant K (independent of h and i), $|\mathcal{H}_{h,i}(x) - x| \le Kh^p$ whenever $x \in \mathcal{U}_i$, $h \in (0, h_0]$.*

Proof. This is a special case of [**12**, Cor. 2.3]. □

There is no loss of generality in assuming that $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$, $i \ne j$.

Now we are in a position to start with the definition of $\mathcal{J}_h$. By letting

$$\mathcal{J}_h(x) = \mathcal{H}_{h,i}(x) \text{ for } x \in \mathcal{U}_i, \quad i = 0, 1, \dots, n,$$

$\mathcal{J}_h$ is defined on $\cup\{\mathcal{U}_i \mid i = 0, 1, \dots, n\}$.

Step 4: **extending $\mathcal{J}_h$ along the separatrix graph**. In virtue of (ii)–(iv), there is no loss of generality in assuming that

$$\{\Phi(t,x) \in \mathbf{R}^2 \mid t \in \mathbf{R},\ x \in \mathcal{U}_i\} \cap \mathcal{U}_j = \emptyset \text{ for all } i, j, i \neq j. \tag{7}$$

Consider the saddle point $P \in \mathcal{D}$ again and let $\mathcal{U}$ be an isolating block of P. Set

$$\mathcal{V}_s = \{\Phi(t,x) \in \mathbf{R}^2 \mid t \geq 0, x \in \Gamma_s\}, \quad s = 4, 8,$$
$$\mathcal{W}_s = \{\Phi(-t,x) \in \mathbf{R}^2 \mid t \geq 0, x \in \Gamma_s\}, \quad s = 2, 6.$$

Using (ii)–(iv) again, we may assume that

$$\mathrm{cl}\,(\mathcal{V}_s) \backslash \mathcal{V}_s \text{ is a sink, say } R_s, \text{ and } \mathcal{V}_s \text{ is attracted by } R_s, \ s = 4, 8 \tag{8}$$

(it may well happen that $R_4 = R_8$) and similarly,

$$\mathrm{cl}\,(\mathcal{W}_s) \backslash \mathcal{W}_s \text{ is a source, say } S_s, \text{ and } \mathcal{W}_s \text{ is repelled by } S_s, \ s = 2, 6. \tag{9}$$

Recall that $P = P_i$, $\mathcal{U} = \mathcal{U}_i$ for some $i \in \{0, 1, \dots, n\}$ and set $\mathcal{H}_h = \mathcal{H}_{h,i}$. Now we extend $\mathcal{H}_h$ to $\mathcal{U} \cup \mathcal{V}_4 \cup \mathcal{V}_8$. This can be done by a simple recursive argument sometimes (e.g. in [**15**, p. 93]) called *the method of fundamental domains*.

In fact, for $s = 4, 8$, $k \in \mathbf{N}$, define recursively

$$\mathcal{D}^0_{s,h} = \mathcal{D}_{s,h}, \ \mathcal{D}^{k+1}_{s,h} = \{\Phi(h,x) \in \mathbf{R}^2 \mid x \in \mathcal{D}^k_{s,h}\}, \ h \in (0, h_0].$$

Observe that $\cup\{\mathcal{D}^{k+1}_{s,h} \mid k \in \mathbf{N}\} = \mathcal{V}_s$ and $\mathcal{H}_h(x) = \varphi(h, \mathcal{H}_h(\Phi(-h,x)))$ whenever $x \in \mathcal{D}^0_{s,h}$. Since $\mathcal{D}^k_{s,h} = \{\Phi(-h,x) \in \mathbf{R}^2 \mid x \in \mathcal{D}^{k+1}_{s,h}\}$, the recursive formula

$$\mathcal{H}_h(x) = \varphi(h, \mathcal{H}_h(\Phi(-h,x))), \quad x \in \mathcal{D}^{k+1}_{s,h}, \quad k \in \mathbf{N}$$

is well-defined and extends $\mathcal{H}_h$ to $\mathcal{V}_s$. It is easy to check that $\mathcal{H}_h(\Phi(h,x)) = \varphi(h, \mathcal{H}_h(x))$ for all $h \in (0, h_0]$, $x \in \mathcal{U} \cup \mathcal{V}_4 \cup \mathcal{V}_8$.

Next we point out that

$$|\mathcal{H}_h(x) - x| \leq K h^p \text{ whenever } x \in \mathcal{V}_4 \cup \mathcal{V}_8, \ h \in (0, h_0]. \tag{10}$$

Lemma 1.2. *For $s = 4, 8$, there exist a new norm $||\cdot||$ on $\mathbf{R}^2$ and a positive constant η such that, for $h \in (0, h_0]$, the mappings $\varphi(h, \cdot)$ and $\Phi(h, \cdot)$ are contractions on $\mathcal{N}(R_s, \eta) = \{x \in \mathbf{R}^2 \mid ||x - R_s|| \le \eta\}$. Further, with some positive constants κ, K (independent of h), the contraction constant is $1 - \kappa h$ and the unique fixed point $Q_s(h)$ of $\varphi(h, \cdot)$ in $\mathcal{N}(R_s, \eta)$ satisfies $||Q_s(h) - R_s|| \le K h^p$.*

Proof. The contraction properties were proved in [**12**, Prop. 1.2, 1.3 and 2.2] (case $\varepsilon = \varepsilon_0$, sufficiently small). Estimate $||Q_s(h) - R_s|| \le K h^p$ is a reformulation of (the $z = 0$ case of) the last assertion in [**12**, Cor. 2.3]. □

By a simple compactness/uniformity argument, there is a constant $\tau > 0$ such that $\{\Phi(t, y) \in \mathbf{R}^2 | \ t \ge \tau, \ y \in \mathcal{D}^0_{s,h}\} \subset \mathcal{N}(R_s, \eta/2)$.

Pick an $x \in \mathcal{V}_s$. Then $x = \Phi(kh, y)$ for some $k = k(x) \in \mathbf{N}$, $y \in \mathcal{D}^0_{s,h}$. In order to prove (10), we distinguish two cases according as $kh \le \tau$ or not. If $kh \le \tau$, then, applying inequalities (5) and (6), we have that

$$\begin{aligned} |\mathcal{H}_h(x) - x| &= |\mathcal{H}_h(\Phi(kh, y)) - \Phi(kh, y)| = |\varphi(k, h, \mathcal{H}_h(y)) - \Phi(kh, y)| \\ &\le |\varphi(k, h, \mathcal{H}_h(y)) - \Phi(kh, \mathcal{H}_h(y))| + |\Phi(kh, \mathcal{H}_h(y)) - \Phi(kh, y)| \\ &\le c(\tau) h^p + c(\tau) |\mathcal{H}_h(y) - y| \le c(\tau) h^p + c(\tau) K h^p. \end{aligned}$$

On the other hand, if $kh > \tau$ then, with $j = \dot{j}(x, h) = \min\{i \in \mathbf{N} \mid ih \ge \tau\}$, set $w_i = \varphi(i, h, \mathcal{H}_h(y))$, $z_i = \Phi(ih, y)$, $i = j, \, j+1, \ldots, k$. Since $z_j \in \mathcal{N}(R_s, \eta/2)$ — as a matter of fact, $\{z_i\}_j^k \subset \mathcal{N}(R_s, \eta/2)$ — and

$$\begin{aligned} ||w_j - z_j|| &\le ||w_j - \Phi(jh, \mathcal{H}_h(y))|| + ||\Phi(jh, \mathcal{H}_h(y)) - z_j|| \\ &\le \text{const} \left(|\varphi(j, h, \mathcal{H}_h(y)) - \Phi(jh, \mathcal{H}_h(y))| + |\Phi(jh, \mathcal{H}_h(y)) - \Phi(jh, y)| \right) \end{aligned}$$

which in turn, using (5) and (6), is less than

$$\text{const} \left(c(\tau + h_0) h^p + c(\tau + h_0) |\mathcal{H}_h(y) - y| \right) \le K h^p;$$

there is no loss of generality in assuming that $w_j \in \mathcal{N}(R_s, \eta)$. Thus, in virtue of the φ-part of Lemma 1.2, we may assume that $\{w_i\}_j^k \subset \mathcal{N}(R_s, \eta)$. By a repeated application of (1) and the Φ-part of Lemma 1.2, it follows that

$$\begin{aligned} ||\mathcal{H}_h(x) - x|| &= ||\mathcal{H}_h(\Phi(kh, y)) - z_k|| = ||\varphi(k, h, \mathcal{H}_h(y)) - z_k|| \\ &= ||w_k - z_k|| = ||\varphi(h, w_{k-1}) - \Phi(h, z_{k-1})|| \\ &\le ||\varphi(h, w_{k-1}) - \Phi(h, w_{k-1})|| + ||\Phi(h, w_{k-1}) - \Phi(h, z_{k-1})|| \\ &\le K h^{p+1} + (1 - \kappa h) ||w_{k-1} - z_{k-1}|| \le \ldots \\ &\le K h^{p+1} (1 + (1 - \kappa h) + \cdots + (1 - \kappa h)^{k-j-1}) + (1 - \kappa h)^{k-j} ||w_j - z_j|| \\ &\le (K/\kappa) h^p + ||w_j - z_j|| \le (K/\kappa) h^p + K h^p. \end{aligned}$$

Thus, in both cases, $|\mathcal{H}_h(x) - x| \le K h^p$ and this concludes the proof of (10).

By letting $\mathcal{H}_h(R_s) = Q_s(h)$, $s = 4, 8$, we extend $\mathcal{H}_h$ to $\mathrm{cl}\,(\mathcal{U} \cup \mathcal{V}_4 \cup \mathcal{V}_8)$. By the construction, the (extended) $\mathcal{H}_h$ is a homeomorphism.

Now we make use of what we obtained in Step 2. Replacing φ, Φ, $\mathcal{V}_s$, R_s by ψ, Ψ, $\mathcal{W}_{s-2}$, S_{s-2} respectively, and starting from $\mathcal{E}_{s-2,h}$ instead of $\mathcal{D}_{s,h}$, $s = 4, 8$, the whole extension process can be repeated. By (8), (9), the two extensions can be made simultaneously; the resulting map (denoted also by) $\mathcal{H}_h$ is a homeomorphism and satisfies $\mathcal{H}_h(\Phi(h,x)) = \varphi(h, \mathcal{H}_h(x))$, $|\mathcal{H}_h(x) - x| \leq Kh^p$ whenever $h \in (0, h_0]$, $x \in \mathrm{cl}\,(\mathcal{U} \cup \mathcal{V}_4 \cup \mathcal{V}_8 \cup \mathcal{W}_2 \cup \mathcal{W}_6)$.

Returning to our mapping $\mathcal{J}_h$ defined on $\cup\{\mathcal{U}_i \mid i = 0, 1, \ldots, n\}$, the previous extension process can be repeated with $P = P_i$, $\mathcal{U} = \mathcal{U}_i$, $\mathcal{H}_h = \mathcal{J}_h|\mathcal{U}_i$, $i = 0, 1, \ldots, n$. By (7), (8), (9), this can be done simultaneously.

From now on, let $\mathcal{J}_h$, $h \in (0, h_0]$, denote the restriction of this simultaneous extension to the set

$$\mathcal{A} = \cup\{\mathrm{cl}(\mathcal{U}_i \cup \mathcal{V}_{4,i} \cup \mathcal{V}_{8,i} \cup \mathcal{W}_{2,i} \cup \mathcal{W}_{6,i}) \mid i = 0, 1, \ldots, n\} \cap \mathcal{D}.$$

It remains to extend $\mathcal{J}_h$ to the entire $\mathcal{D}$.

Step 5: **the components of** $\mathcal{D}\backslash\mathcal{A}$. Consider now a component $\mathcal{B}$ of $\mathcal{D}\backslash\mathcal{A}$. Since $\partial\mathcal{D} \subset \mathcal{A}$ and $\mathcal{A}$ is closed, $\mathcal{B}$ is open. By the construction, $\partial\mathcal{A}$ consists of a finite number of nonsaddle equilibria plus of a finite number of connecting orbits. The same is true for $\partial\mathcal{B}$. It follows that the number of components of $\mathcal{D}\backslash\mathcal{A}$ is finite. Observe that $\mathcal{G} \cap \mathcal{D} \subset \mathcal{A}$, where $\mathcal{G}$ denotes the separatrix graph. Consequently, $\mathcal{B}$ is contained in a single component of $\mathcal{D}\backslash\mathcal{G}$. It is known [**6**] that each component of $\mathcal{D}\backslash\mathcal{G}$ contains exactly one sink and one source on its boundary. Consequently, $\partial\mathcal{B}$ consists of two nonsaddle equilibria, a sink R and a source S, and of two connecting trajectories (trajectories containing $\Gamma_{s,i}$ for some $s \in \{1, 3, 5, 7\}$ and $i \in \{0, 1, \ldots, n\}$). It follows that $\mathcal{B}$ is a collection of trajectories starting from S and tending to R.

Consider now the set $\mathcal{J}_h(\partial\mathcal{B})$ and apply the Jordan curve theorem. Since $\mathcal{J}_h$ is a homeomorphism, also $\mathcal{J}_h(\partial\mathcal{B})$ is a simple closed curve. Its interior is denoted by $\mathcal{C}_h$. The conjugacy property of $\mathcal{J}_h$ and the Φ-invariance of $\partial\mathcal{B}$ implies that $\mathcal{J}_h(\partial\mathcal{B}) = \partial\mathcal{C}_h$ is invariant under $\varphi(h, \cdot)$. Since $\varphi(h, \cdot)$ is a global homeomorphism (Step 2), it follows from Brouwer's invariance of domain theorem that $\mathcal{C}_h$ is invariant under $\varphi(h, \cdot)$.

By the previous considerations, in extending $\mathcal{J}_h$ from $\mathcal{A}$ to $\mathcal{D}$, we can work on each component of $\mathcal{D}\backslash\mathcal{A}$, separately.

Step 6: **a Schönfliess argument on** $\mathcal{B}$ **combined with the method of fundamental domains**. What also left to prove is that, for $h \in (0, h_0]$, $\mathcal{J}_h|\partial\mathcal{B}$ extends to a homeomorphism $\mathcal{K}_h$ of $\mathrm{cl}\,(\mathcal{B})$ onto $\mathrm{cl}\,(\mathcal{C}_h)$ such that $\mathcal{K}_h(\Phi(h,x)) = \varphi(h, \mathcal{K}_h(x))$ and $|\mathcal{K}_h(x) - x| \leq Kh^p$ whenever $x \in \mathrm{cl}\,(\mathcal{B})$.

With R playing the role of R_s in Lemma 1.2, set $\Gamma = \mathrm{cl}\,(\mathcal{B}) \cap \partial\mathcal{N}(R, \eta)$. Since Φ is transversal to $\partial\mathcal{N}(R, \eta)$, $\Gamma \cap \partial\mathcal{B}$ consists of exactly two points, say B_1, B_2. Set $\gamma_h = \{\Phi(h,x) \in \mathbf{R}^2 \mid x \in \Gamma\}$ and $\alpha_{s,h} = \{\Phi(t, B_s) \in \mathbf{R}^2 \mid 0 \leq t \leq h\}$, $s = 1, 2$.

Let λ_h denote the simple closed curve formed by the four arcs $\alpha_{1,h}$, Γ, $\alpha_{2,h}$, γ_h. The closure of the interior of λ_h is denoted by $\mathcal{F}_h$. We extend $\mathcal{J}_h$ first to λ_h, then to $\mathcal{F}_h$ and, finally, by the method of fundamental domains described in Step 4, to the entire $\mathrm{cl}\,(\mathcal{B})$. (The fundamental domains are $\mathcal{D}_{s,h}$, $\mathcal{E}_{s,h}$ and $\mathcal{F}_h$, respectively.)

By elementary considerations from plane topology, there is an arc Δ_h connecting $\mathcal{J}_h(B_1)$ with $\mathcal{J}_h(B_2)$ and a homeomorphism $\mathcal{H}_h$ of Γ onto Δ_h with the properties that $\Delta_h \subset \mathrm{cl}\,(\mathcal{C}_h)$, $\Delta_h \cap \partial\mathcal{C}_h = \{\mathcal{J}_h(B_1), \mathcal{J}_h(B_2)\}$, $\mathcal{H}_h(B_1) = \mathcal{J}_h(B_1)$, $\mathcal{H}_h(B_2) = \mathcal{J}_h(B_2)$, and, last but not least, $|\mathcal{H}_h(x) - x| < 3Kh^p$ whenever $x \in \Gamma$. (Here of course, K is the fixed constant for which $|\mathcal{J}_h(x) - x| \le Kh^p$ whenever $x \in \partial\mathcal{B}$). Set $\delta_h = \{\varphi(h,x) \in \mathbf{R}^2 \mid x \in \Delta_h\}$ and $\beta_{s,h} = \mathcal{J}_h(\alpha_{s,h})$, $s = 1,2$. Given $x \in \Gamma$ arbitrarily, inequalities (6) and (1) imply that

$$\begin{aligned}|\Phi(h,x) - \varphi(h,\mathcal{H}_h(x))| &\le |\Phi(h,x) - \Phi(h,\mathcal{H}_h(x))| + |\Phi(h,\mathcal{H}_h(x)) - \varphi(h,\mathcal{H}_h(x))|\\ &\le c(h_0)|x - \mathcal{H}_h(x)| + Kh^{p+1} \le 3c(h_0)Kh^p + Kh^{p+1}.\end{aligned}$$

In virtue of the Φ-part of Lemma 1.2, $\gamma_h \subset \mathcal{N}(R,(1-\kappa h)\eta)$. It follows immediately that $\delta_h \subset \mathcal{N}(R,(1-\kappa h)\eta + (3c(h_0)+h_0)Kh^p)$. Since $\Delta_h \cap \mathcal{N}(R, \eta - 3Kh^p) = \emptyset$, assumption $p \ge 2$ implies that, for all $h \in (0,h_0]$, $\Delta_h \cap \delta_h = \emptyset$. Hence, the four arcs $\beta_{1,h}$, Δ_h, $\beta_{2,h}$, δ_h form a simple closed curve in $\mathrm{cl}(\mathcal{C}_h)$, say μ_h. The closure of the interior of μ_h is denoted by $\mathcal{G}_h$.

For $x \in \lambda_h$, set

$$\mathcal{R}_h(x) = \begin{cases} \mathcal{J}_h(x) & \text{if } x \in \alpha_{1,h} \cup \alpha_{2,h},\\ \mathcal{H}_h(x) & \text{if } x \in \Gamma,\\ \varphi(h,\mathcal{H}_h(\Phi(-h,x))) & \text{if } x \in \gamma_h.\end{cases}$$

It is easy to check that $\mathcal{R}_h$ is well-defined, $\mathcal{R}_h$ is a homeomorphism of λ_h onto μ_h, $\mathcal{R}_h(\Phi(h,x)) = \varphi(h,\mathcal{R}_h(x))$ whenever $x \in \Gamma$ and, with $b = \max\{3, 3c(h_0)+h_0\}$, $|\mathcal{R}_h(x) - x| \le bKh^p$ whenever $x \in \lambda_h$.

Using $p \ge 2$ again, the geometric properties of λ_h imply the existence of a simple closed curve $\omega_h \subset (\mathcal{F}_h \backslash \lambda_h) \cap (\mathcal{G}_h \backslash \mu_h)$ and of a homeomorphism $\mathcal{Z}_h$ of ω_h onto λ_h such that, with a suitable constant $c > 1$ (independent of h but depending on the angle between Γ and $\alpha_{s,h}$ at B_s, $s = 1,2$), $|\mathcal{Z}_h(x) - x| \le bcKh^p$ whenever $x \in \omega_h$. The closure of the interior of ω_h is denoted by Ω_h.

Next we construct a homeomorphism $\mathcal{S}_h$ of the annulus $\mathrm{cl}\,(\mathcal{F}_h \backslash \Omega_h)$ onto the annulus $\mathrm{cl}\,(\mathcal{G}_h \backslash \Omega_h)$ with the properties that $\mathcal{S}_h(x) = \mathcal{R}_h(x)$ whenever $x \in \lambda_h$, $\mathcal{S}_h(x) = x$ whenever $x \in \omega_h$ and $|\mathcal{S}_h(x) - x| \le 24bcKh^p$ for all $x \in \mathrm{cl}(\mathcal{F}_h \backslash \Omega_h)$. This will be done by applying Schönfliess theorem to the cells of a suitable cellular decomposition of $\mathrm{cl}\,(\mathcal{F}_h \backslash \Omega_h)$, $h \in (0,h_0]$.

In fact, by elementary considerations from plane topology, there exists a planar graph G_h with vertices $A_h^1, A_h^2, \dots, A_h^N$, $C_h^1, C_h^2, \dots, C_h^N$ and edges $\varphi_h^j = A_h^j A_h^{j+1}$, $\varphi_h^{N+j} = C_h^j C_h^{j+1}$, $\varphi_h^{2N+j} = A_h^j C_h^j$, $j = 1,2,\dots,N$ ($A_h^{N+1} = A_h^1$, $C_h^{N+1} = C_h^1$) such that $\cup\{\varphi_j^h \mid j=1,2,\ldots,N\}=\omega_h$, $\cup\{\varphi_h^{N+j}|j=1,2,\ldots,N\}=\mu_h$ and, last but not least, for $j=1,2,\ldots,N$, the diameter of the curvilinear rectangle $A_h^j A_h^{j+1} C_h^{j+1} C_h^j$ is less than $12bcKh^p$. Similarly, there exists a planar graph F_h with vertices $A_h^1, A_h^2, \ldots, A_h^N$, $B_h^1=\mathcal{R}_h^{-1}(C_h^1)$, $B_h^2=\mathcal{R}_h^{-1}(C_h^2)$, $\ldots$, $B_h^N=\mathcal{R}_h^{-1}(C_h^N)$ and edges $\varepsilon_h^j=\varphi_h^j$, $\varepsilon_h^{N+j}=B_h^j B_h^{j+1}$, $\varepsilon_h^{2N+j}=A_h^j B_h^j$, $j=1,2,\ldots,N$ ($B_h^{N+1}=B_h^1$) such that $\cup\{\varepsilon_h^{N+j} \mid j=1,2,\ldots,N\}=\lambda_h$ and, for $j=1,2,\ldots,N$, the diameter of the curvilinear rectangle $A_h^j A_h^{j+1} B_h^{j+1} B_h^j$ is less than $12bcKh^p$. Finally, there exists a homeomorphism $\mathcal{S}_h$ of F_h onto G_h with the properties that $\mathcal{S}_h(x)=\mathcal{R}_h(x)$ whenever $x\in\lambda_h$, $\mathcal{S}_h(x)=x$ whenever $x\in\omega_h$ (and consequently, $\mathcal{S}_h(\varepsilon_h^j)=\varphi_h^j$, $j=1,2,\ldots,3N$).

By an N-fold application of Schönfliess theorem (applied to the curvilinear rectangles $A_h^j A_h^{j+1} B_h^{j+1} B_h^j$, $j=1,2,\ldots,N$, separately), we may extend $\mathcal{S}_h$ to a homeomorphism of $\mathrm{cl}\,(\mathcal{F}_h\backslash\Omega_h)$ onto $\mathrm{cl}\,(\mathcal{G}_h\backslash\Omega_h)$. Further, by letting $\mathcal{S}_h(x)=x$ for all $x\in\Omega_h$, $\mathcal{S}_h$ is extended to a homeomorphism (denoted also by $\mathcal{S}_h$) of $\mathcal{F}_h$ onto $\mathcal{G}_h$. The estimates on the diameter of the curvilinear rectangles imply that $|\mathcal{S}_h(x)-x|\le 24bcKh^p$ for all $x\in\mathcal{F}_h$.

It is easy to check that the conditions of applying the method of fundamental domains described in Step 4 are satisfied. Starting with $\mathcal{F}_h$ as fundamental domain, the homeomorphism $\mathcal{S}_h\colon\mathcal{F}_h\to\mathcal{G}_h$ can be extended to a homeomorphism $\mathcal{K}_h\colon \mathrm{cl}\,(\mathcal{B})\to\mathrm{cl}\,(\mathcal{C}_h)$ such that $\mathcal{K}_h(\Phi(h,x))=\varphi(h,\mathcal{K}_h(x))$ and $|\mathcal{K}_h(x)-x|\le$ const $\cdot\, h^p$ for all $x\in\mathrm{cl}\,(\mathcal{B})$. This concludes Step 6 as well as the proof of the Theorem.

Remark 1.3. Though condition $p\ge 2$ was made use of only in Step 6, the proof of the Theorem breaks down badly in the case $p=1$. The main difficulty is that the arcs $\beta_{1,h}$ and $\beta_{2,h}$ are only Hölder continuous (like the conjugacy in the classical Hartman-Grobman lemma [**15**, Thm. 5.14]) and so the construction of the simple closed curve μ_h requires delicate considerations. This is the first point where condition $p\ge 2$ was exploited. (As a matter of fact, there exists a C^1 conjugacy — this is a planar case improvement [**13**] of the classical Hartman-Grobman lemma — but this does not imply automatically the validity of the C^1 version of Lemma 1.1. Alternatively, the construction of μ_h might be possibly furnished within the framework [**19**] of Conley's index theory for discrete dynamical systems. Neither this approach seems to be straightforward.) The second point where condition $p\ge 2$ was exploited is the construction of graphs F_h and G_h. (This made the application of the classical Schönfliess extension theorem possible. However, no matter whether $p\ge 2$ or $p=1$, the whole Schönfliess argument could be replaced by an affirmative answer to the following **Conjecture: Given the simple closed planar curves γ_1, γ_2 with respective interiors Ω_1, Ω_2 and a homeomorphism $\mathcal{H}$ of γ_1 onto γ_2, there exists an extended homeomorphism $\tilde{\mathcal{H}}$ of $\gamma_1\cup\Omega_1$ onto $\gamma_2\cup\Omega_2$ with the property that** $\max\{|\tilde{\mathcal{H}}(x)-x| \mid x\in\gamma_1\cup\Omega_1\}=\max\{|\mathcal{H}(x)-x| \mid x\in\gamma_1\}$. The above conjecture relates to a conjecture in [**11**] whose special cases were solved in [**5**] and [**20**]).

Remark 1.4. It is known that hyperbolic periodic orbits of autonomous ordinary differential equations persist under discretization. More precisely, hyperbolic periodic orbits go over into nearby hyperbolic invariant curves of the discretized system [**3**], [**9**]. Since rotation numbers on these hyperbolic invariant curves may depend on the stepsize of the discretization, no conjugacy results can be expected. Instead of, we pose the following conjecture (see Remark 2.4 as well where also the meaning of the expression "for most h" will be specified).

Conjecture. *With* (iii) *replaced by condition*

(iii)' *there are finitely many periodic orbits in D, all hyperbolic; alpha- and omega-limit sets of trajectories in D are equilibria or periodic orbits,*

let $\mathcal{N}$, p, D, f, Φ, φ, $\mathcal{M}$ be as in the Theorem. Then there is a positive constant K and, for most h sufficiently small, there exist a homeomorphism $\mathcal{H}_h\colon D \to \mathcal{H}_h(D)$ and a continuous-time local dynamical system $\tilde{\Phi}_h$ (defined on $\mathcal{N}$) with the properties that $\varphi(h,x) = \tilde{\Phi}_h(h,x)$ and $|\mathcal{H}_h(x) - x| \le Kh^p$ whenever $x \in D$ and, last but not least, preserving time-orientation, the homeomorphism $\mathcal{H}_h$ maps trajectory segments of Φ in D onto trajectory segments of $\tilde{\Phi}_h$ in $\mathcal{H}_h(D)$.

2. Some General Remarks. The Real Problems to Solve

Let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be a C^{p+1} function and consider the differential equation $\dot{x} = f(x)$. The induced dynamical system and its time-T-map are denoted by Φ and $\Phi(T,\cdot)$, respectively. The h-discretized system of order p is defined as a C^{p+1} mapping $\varphi\colon \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^n$ satisfying, with some positive constants K and h_0 (independent of h and x),

$$|\Phi(h,x) - \varphi(h,x)| \le Kh^{p+1} \quad \text{for all} \quad h \in (0,h_0],\ x \in \mathbf{R}^n. \tag{11}$$

As in Step 2 of the proof of the Theorem, it follows easily that, for all $h \in (0,h_0]$,

$$\varphi(h,\cdot) \text{ is a } C^{p+1} \text{ diffeomorphism of } \mathbf{R}^n \text{ onto } \mathbf{R}^n, \tag{12}$$

$$|\Phi_x^{(j)}(h,x) - \varphi_x^{(j)}(h,x)| \le Kh^{p+1-j} \text{ whenever } j = 0,1,\dots,p+1 \text{ and } x \in \mathbf{R}^n \tag{13}$$

and, with $\varphi(k,h,x)$ defined by the recursion $\varphi(0,h,x) = x$, $\varphi(k+1,h,x) = \varphi(h,\varphi(k,h,x))$, $x \in \mathbf{N}$,

$$|\Phi(1,x) - \varphi(N,1/N,x)| \le KN^{-p} \text{ whenever } 1/N < h_0 \text{ and } x \in \mathbf{R}^n \tag{14}$$

(— the constants K, h_0 etc. are not necessarily the same at different appearances).

Proposition 2.1. *There holds also*

$$|\Phi_x'(1,x) - \varphi_x'(N,1/N,x)| \le KN^{-(p-1)} \text{ whenever } 1/N < h_0 \text{ and } x \in \mathbf{R}^n. \tag{15}$$

Proof. For brevity, we write $F(x) = \Phi(1/N, x)$, $G(x) = \varphi(1/N, x)$. Then $\Phi(1,x) = F^N(x)$, $\varphi(N, 1/N, x) = G^N(x)$ and, in virtue of the simple inequalities

$$\begin{gathered}|F'(x)| \le 1 + a/N, \quad |G'(x)| \le 1 + a/N,\\ |F'(F^k(x)) - G'(G^k(x))| \le |F'(F^k(x)) - G'(F^k(x))| + |G'(F^k(x)) - G'(G^k(x))|\\ \le KN^{-p} + b|F^k(x) - G^k(x)| \le cN^{-p}, \quad k = 0,1,\dots,N, \quad x \in \mathbf{R}^n,\end{gathered}$$

(where a, b, c are constants independent of N), we obtain via some elementary “product formulas [**7**]” that

$$\begin{aligned}&|\Phi_x'(1,x) - \varphi_x'(N,1/N,x)| = |(F^N)' - (G^N)'|\\ &= |F'(F^{N-1}) \cdot F'(F^{N-2}) \cdot \ldots \cdot F' - G'(G^{N-1}) \cdot G'(G^{N-2}) \cdot \ldots \cdot G'|\\ &\le \sum_{k=0}^{N-1} |F'(F^{N-1}) \cdot \ldots \cdot F'(F^{k+1}) \cdot [F'(F^k) - G'(G^k)] \cdot G'(G^{k-1}) \cdot \ldots \cdot G'|\\ &\le N(1 + a/N)^{N-1} cN^{-p}\\ &\le \exp(a) \cdot cN^{-(p-1)} \quad \text{whenever } 1/N < h_0 \text{ and } x \in \mathbf{R}^n.\end{aligned}$$

□

Remark 2.2. As a joint generalization of (14) and (15), an inductive application of the arguments we used in proving Proposition 2.1 yields that, for $j = 0, 1, \dots, p$, there holds

$$|\Phi_x^{(j)}(1,x) - \varphi_x^{(j)}(N,1/N,x)| \le KN^{-(p-j)} \text{ whenever } 1/N < h_0,\ x \in \mathbf{R}^n. \tag{16}$$

It is worth to note that (16) is conform to the Hadamard-Landau interpolation inequality (see e.g. [**12**, Thm. 2.6]) valid for $u \in C^p(\mathbf{R}^n, \mathbf{R}^n)$:

$$|u^{(j)}|^p \le \text{const}\,(j,p) \cdot |u|^{p-j} \cdot |u^{(p)}|^j, \quad j = 0, 1, \dots, p\,. \tag{17}$$

Conformity means that, with $u = \Phi(1,\cdot) - \varphi(N, 1/N, \cdot)$, inequality (17) implies that (16) is equivalent to its special cases $j = 0$ and $j = p$. For Runge-Kutta or l-derivative one-step methods of order p, (16) can be slightly improved [**9**, Prop. 1].

Now we turn back to the qualitative properties of discretizations. Assuming $p \ge 2$, inequalities (14) and (15) imply that $\varphi(N, 1/N, \cdot)$ is a small C^1 perturbation of $\Phi(1, \cdot)$. Encouraged by this observation, especially when treating persistence phenomena for qualitative properties under discretizations, one is induced to apply perturbation theory for the limiting process $\varphi(N, 1/N, \cdot) \to \Phi(1, \cdot)$, $N \to \infty$. An immediate result is that, for N sufficiently large, $\varphi(N, 1/N, \cdot)$ and $\Phi(1, \cdot)$ are conjugate providing $\Phi(1, \cdot)$ is structurally stable. This is the case e.g. for Morse-Smale gradient systems [**23**, Cor. 1.3].

Everything is much harder if the relationship between $\Phi(h, \cdot)$ and $\varphi(h, \cdot)$ is investigated where standard perturbation results can not be directly applied. The difficulty is that, with $h \to 0^+$, both $\Phi(h, \cdot)$ and $\varphi(h, \cdot)$ approach the identity, an operator which behaves badly in perturbation theory. The fact that $h^{-1}(\Phi(h, \cdot) - Id) \to f$ and $h^{-1}(\varphi(h, \cdot) - Id) \to f$ is fundamental for applying [**7**] nonlinear semigroup theory in numerical analysis but does not give much help in studying the relationship between $\Phi(h, \cdot)$ and $\varphi(h, \cdot)$. However, from the view-point of introducing stepsize as an additional small parameter, a refinement of classical qualitative theory might help. The proof of the Theorem is nothing else but such an analysis of the proofs in [**6**], [**24**]. Reconsideration of some hard proofs (e.g. in [**14**], [**22**], [**25**]) would also be desirable.

The previous two paragraphs indicate that a natural way for comparing the qualitative properties of $\Phi(h, \cdot)$ and $\varphi(h, \cdot)$ is to subdivide the problem into comparing the pairs $\Phi(1/N, \cdot) \longleftrightarrow \Phi(1, \cdot)$, $\Phi(1, \cdot) \longleftrightarrow \varphi(N, 1/N, \cdot)$ and $\varphi(N, 1/N, \cdot) \longleftrightarrow \varphi(1/N, \cdot)$, respectively — for simplicity, we assumed that $h = 1/N$ for some positive integer N. Roughly speaking, the three steps are "taking Nth power", "applying classical perturbation results" and "taking Nth root". (We feel the last step is the hardest. In general, contrary to the situation considered in [**12**], the conjugacy found for time $T = 1$ (i.e. the conjugacy between $\Phi(1, \cdot)$ and $\varphi(N, 1/N, \cdot)$) does not work automatically for time $T = 1/N$ (i.e. between $\Phi(1/N, \cdot)$ and $\varphi(1/N, \cdot)$). Neither time averages of conjugacies work.) This is the context the first question of Problem 2.3 has to be understood. The second question of Problem 2.3 relates already to Remark 2.4 as well. Of course, we do not expect one-word "yes" or "no" answers but look for examples, necessary and/or sufficient conditions, connections to weakenings of the concept of structural stability etc.

Problem 2.3. Is it true that the qualitative properties of $\varphi(N, 1/N, \cdot)$ and $\varphi(1/N, \cdot)$ are, in some sense, the same **(qualitative root problem)**? Is it possible to embed $\varphi(h, \cdot)$ into a continuous-time dynamical system $\tilde{\Phi}_h$ such that the qualitative properties of $\tilde{\Phi}_h$ and Φ are, in some sense, the same **(qualitative embedding problem)**?

Remark 2.4. The problem of embeddability of discrete dynamical systems into continuous ones is extremely difficult. For a survey, see [**27**]. Besides the line case which is trivial, only the circle case is entirely solved [**17**]. C^2 self-diffeomorphisms of the circle with irrational rotation numbers are conjugate to rotations. In particular, they embed into continuous-time dynamical systems. If the nonzero rotation number is rational, then embeddability implies all points are periodic [**17**]. Thus, $\varphi(h, \cdot)$ restricted to the collection of the hyperbolic invariant curves embeds into a continuous-time dynamical system provided that all rotation numbers are irrational. But rotation numbers are strictly increasing continuous functions of the stepsize [**3**, Thm. 4.1]. This gives an indication that the exceptional set in the Conjecture might be countable. (Or should one take only the "nice" irrational numbers?)

Concluding this paper, we present a simple local conjugacy result which corresponds to the local flow-box/rectification theorem [**15**, Thm. 5.8] and, in its various aspects, is implicitely contained in several papers [**4**], [**10**], [**12**], [**16**], [**21**] on discretizations and/or embeddings but was probably never stated explicitely. The easy proof is omitted.

Proposition 2.5. *Assume that the conditions listed at the beginning of this section are all satisfied and let P be a nonequilibrium point for $\dot{x} = f(x)$. Then there is an open neighbourhood $\mathcal{N}$ of P in $\mathbf{R}^n$ with the properties as follows. There exist positive constants h_0, K and, for all $h \in (0, h_0]$, there exists a C^{p+1} diffeomorphism $\mathcal{H}_h : \mathcal{N} \to \mathcal{H}_h(\mathcal{N})$ such that $\mathcal{H}_h(\Phi(h,x)) = \varphi(h, \mathcal{H}_h(x))$ and $|\mathcal{H}_h(x) - x| \leq Kh^p$ whenever $x \in \mathcal{N}$, $\Phi(h,x) \in \mathcal{N}$ and $h \in (0, h_0]$. In particular, locally, in a neighbourhood of P, $\varphi(h, \cdot)$ embeds into a continuous-time local C^{p+1} dynamical system which is transversal to the codimension one affine hyperplane orthogonal to $f(P)$ at P.*

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