Acta Math. Univ. Comenianae
Vol. LXII, 2(1993), pp. 221–227

ON THE QUASI–UNIFORM CONVERGENCE OF TRANSFINITE SEQUENCES OF FUNCTIONS

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Transfinite sequences of functions form some special type of nets. For instance, under some simple assumptions on spaces, the pointwise convergence of such nets suffices to the preservation of continuity, quasi-continuity and other generalized forms of them [**3, 7, 8**]. In this note we investigate the quasi-uniform convergence of transfinite sequences of functions. We formulate certain sufficient conditions for equality of various types of convergence, connections between convergence of functions f_ξ to f and convergence in some sense of sets of continuity points $C(f_\xi)$ to $C(f)$ and cluster sets $L(f_\xi, x)$ to $L(f, x)$. In the last part it is shown that the quasi-uniform convergence is preserved under superpositions.

Let X be a topological space. For a net $\{A_s : s \in S\}$ of sets $A_s \subset X$ by $\liminf A_s$ and $\limsup A_s$ we denote the sets given by

$$\liminf A_s = \bigcup_{p \in S} \bigcap_{s \geq p} A_s \,,$$
$$\limsup A_s = \bigcap_{p \in S} \bigcup_{s \geq p} A_s \,.$$

We will use the symbol $\lim A_s$ if $\liminf A_s = \limsup A_s$. Moreover, Li A_s and Ls A_s are sets consisting of all points $x \in X$ each neighbourhood of which meets $\{A_s : s \in S\}$ eventually or frequently, respectively. If Li $A_s =$ Ls A_s, then this set is denoted as Lt A_s, [**5, 6**].

Now let (Y, d) be a metric space. For any $y \in Y$, a set $A \subset Y$ and $r > 0$ we will write $B(y, r) = \{z \in Y : d(y, z) < r\}$ and $B(A, r) = \cup\{B(y, r) : y \in A\}$. A net $\{f_s : s \in S\}$ of functions $f_s \colon X \to Y$ is called quasi-uniformly convergent to a function $f \colon X \to Y$ if for each $x \in X$ and $\varepsilon > 0$ there exists $s_0 \in S$ such that for each $s \in S$, $s \geq s_0$ there is a neighbourhood U of x with the property $d(f_s(z), f(z)) < \varepsilon$ for $z \in U$, [**9**].

Received November 13, 1992.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 54A20.
Supported by KBN research grant No. 2 1144 91 01(1992-94).

In the sequel we will use the following properties (they are proved in [**9**] for functions with values in uniform spaces):

(1) If functions $f_s\colon X \to Y$, $s \in S$, are continuous at a point $x \in X$ and the net $\{f_s : s \in S\}$ converges to f quasi-uniformly, then f is continuous at x.

(2) If $f_s, f\colon X \to Y$, $s \in S$, are continuous functions and the net $\{f_s : s \in S\}$ is pointwise convergent to f, then the convergence is quasi-uniform.

Through this paper the smallest uncountable ordinal number is denoted by ω_1 and a net $\{f_\xi : \xi < \omega_1\}$ is called a transfinite sequence of functions. If (Y, d) is a metric space and $f, f_\xi : X \to Y$, $\xi < \omega_1$ are any functions, then the transfinite sequence $\{f_\xi : \xi < \omega_1\}$ converges to f:

(3) pointwise if and only if for each $x \in X$ there exists α_x such that $f_\xi(x) = f(x)$ for any ξ, $\alpha_x \leq \xi < \omega_1$;

(4) uniformly if and only if there exists $\alpha < \omega_1$ such that $f_\xi(x) = f(x)$ for each $x \in X$ and ξ, $\alpha \leq \xi < \omega_1$, [**4**].

Let X be a topological space and (Y, d) a metric one. A net $\{f_s : s \in S\}$ of functions $f_s\colon X \to Y$ is called almost uniformly convergent to a function $f\colon X \to Y$ if for each $x \in X$, $\varepsilon > 0$ there exists a neighbourhood U of x and $s_o \in S$ with $d(f_s(z), f(z)) < \varepsilon$ for any $z \in U$, $s \geq s_0$, [**1**].

If X is a compact space, then the almost uniform convergence coincides with the uniform one, [**1**]. Thus we have

$$\begin{array}{ccccc}
\text{uniform convergence} & \Longrightarrow & \begin{array}{c}\text{almost uniform}\\ \text{convergence}\end{array} & \Longrightarrow & \begin{array}{c}\text{quasi-uniform}\\ \text{convergence}\end{array} \\
 & & \Downarrow & & \Downarrow \\
 & & \begin{array}{c}\text{uniform on compact sets}\\ \text{convergence}\end{array} & \Longrightarrow & \begin{array}{c}\text{pointwise}\\ \text{convergence}\end{array}
\end{array}$$

and none of implications in this diagram is invertible.

Theorem 1. *Let X be a separable topological space and (Y, d) a metric one. If $f_\xi\colon X \to Y$, $\xi < \omega_1$, are continuous functions and the sequence $\{f_\xi : \xi < \omega_1\}$is quasi-uniformly convergent to a function $f\colon X \to Y$, then this sequence converges to f uniformly.*

Proof. Let $\{x_n : n \geq 1\}$ be a dense subset of X. According to (3) and from the properties of ordinal numbers we can choose an $\alpha < \omega_1$ such that $f_\xi(x_n) = f(x_n)$ for each $n \geq 1$ and ξ with $\alpha \leq \xi < \omega_1$. The quasi-uniform convergence implies the continuity of f. Continuous functions with values in a metric space which are equal on a dense subset are equal; so we have $f_\xi(x) = f(x)$ for each ξ, $\alpha \leq \xi < \omega_1$ and $x \in X$. In virtue of (4) it means the uniform convergence. $\square$

Using analogous arguments can be shown the following:

Theorem 2. *Let X be a locally compact metric space and Y a metric one. If $f_\xi \colon X \to Y$, $\xi < \omega_1$, are continuous functions and the sequence $\{f_\xi \colon \xi < \omega_1\}$ quasi-uniformly converges to a function $f \colon X \to Y$, then it converges to f almost uniformly.*

Corollary 1. *Let X be a first countable separable space and Y a metric one. If $\{f_\xi : \xi < \omega_1\}$ is a transfinite sequence of continuous functions $f_\xi : X \to Y$ which is pointwise convergent to a function $f \colon X \to Y$, then this sequence uniformly converges to f.*

Proof. Since X is first countable, it follows from [**8,** Th. 1] that f is continuous. Thus the convergence is quasi-uniform and the conclusion is an immediate consequence of Theorem 1. □

Denoting by $C(X,Y)$ the set of all continuous functions from X into Y our results can be expressed as the following:

Corollary 2. *If X is a first countable separable space and Y is a metric one, then for transfinite sequences in $C(X,Y)$ all forms of convergence: uniform, almost uniform, quasi-uniform, uniform on compact sets and pointwise are equivalent.*

For a function f the set of all points at which f is continuous is denoted by $C(f)$. Then we have:

Theorem 3. *Let X be a topological space and (Y,d) a metric one. If $\{f_\xi : \xi < \omega_1\}$ is a transfinite sequence of functions $f_\xi \colon X \to Y$ which is quasi-uniformly convergent to a function $f \colon X \to Y$, then $C(f) = \lim C(f_\xi)$.*

Proof. For a point $x_0 \in \limsup C(f_\xi)$ we put $S_1 = \{\xi < \omega_1 : x_0 \in C(f_\xi)\}$. Then $\{f_\xi : \xi \in S_1\}$ is a net quasi-uniformly convergent to f. Now, applying (1) we have $x_0 \in C(f)$. Thus it is shown that $\limsup C(f_\xi) \subseteq C(f)$.

Conversely, let $x_0 \in C(f)$. From the quasi-uniform convergence for each $n \geq 1$ there exists $\xi_n < \omega_1$ such that for any ξ with $\xi_n \leq \xi < \omega_1$ there is a neighbourhood $U = U(\xi, n)$ of x_0 with the property $d(f_\xi(x), f(x)) < \frac{1}{n}$ for $x \in U(\xi, n)$. We choose $\alpha < \omega_1$ satisfying $\xi_n \leq \alpha$ for each $n \geq 1$. Now we establish ξ with $\alpha \leq \xi < \omega_1$, $\varepsilon > 0$ and a natural number $m \geq 1$ for which $\frac{3}{m} < \varepsilon$. Then using the fact $x_0 \in C(f)$ we take a neighbourhood W of x_0 such that

$$d(f(x), f(x_0)) < \frac{1}{m} \quad \text{and} \quad d(f_\xi(x), f(x)) < \frac{1}{m} \quad \text{for } x \in W.$$

Hence we obtain $d(f_\xi(x), f_\xi(x_0) < \varepsilon$ for $x \in W$. It implies $x_0 \in C(f_\xi)$ for each ξ with $\alpha \leq \xi < \omega_1$: so we have shown $C(f) \subset \liminf C(f_\xi)$ which completes the proof. □

Corollary 3 [**4,** Th. 2.1]**.** *Let X be a locally compact space and (Y,d) a metric one. If a transfinite sequence $\{f_\xi : \xi < \omega_1\}$ of functions $f_\xi \colon X \to Y$ converges to a function $f \colon X \to Y$ uniformly on compact sets, then $C(f) = \lim C(f_\xi)$.*

Proof. If X is a locally compact space, then the uniform convergence on compact sets coincides with the almost uniform convergence [**1,** Th. 2.5]. Thus the conclusion follows from Theorem 3. □

Let us remark that Theorem 3 is not true for usual sequences. For instance, let R be the space of real numbers with the natural topology and Q the set od rationals. We take functions $f_n, f \colon R \to R$ given by $f(x) = 0$ for each $x \in R$,

$$f_n(x) = \begin{cases} \frac{1}{n}, & \text{if } x \in Q; \\ -\frac{1}{n}, & \text{if } x \in R \setminus Q. \end{cases}$$

Then the sequence $\{f_n : n \geq 1\}$ uniformly converges to f but $\lim C(f_n) = \emptyset \neq R = C(f)$.

Let $f \colon X \to Y$ be a function and $x \in X$. The cluster set of f at x, denoted by $L(f,x)$, is defined as the set of all points $y \in Y$ such that there exists a net $\{x_\sigma : \sigma \in \Sigma\}$ in X with $x_\sigma \to x$ and $f(x_\sigma) \to y$. Equivalently, $L(f,x) = \cap\{\overline{f(U)} : U$ is a neighbourhood of $x\}$. The inverse cluster set of f at $y \in Y$ is the set $L^{-1}(f,y)$ of all $x \in X$ such that $y \in L(f,x)$. It can be expressed also as $L^{-1}(f,y) = \cap\{\overline{f^{-1}(V)} : V$ is a neighbourhood of $y\}$, [**2**]. The graph of f we denote by $G(f)$. Then

(5) The graph $G(f)$ of a function f is closed if and only if $L(f,x) = \{f(x)\}$ for each $x \in X$, [**2,** Th. I.1.3].

Theorem 4. *Let X be a topological space and (Y,d) a metric one. If a transfinite sequence $\{f_\xi : \xi < \omega_1\}$ of functions $f_\xi \colon X \to Y$ is quasi-uniformly convergent to f, then:*

$$\begin{aligned} & L(f,x) = \lim L(f_\xi,x) = \mathrm{Lt}\ L(f_\xi,x) \quad \textit{for each } x \in X, \\ \textit{and} \quad & L^{-1}(f,y) = \lim L^{-1}(f_\xi,y) \quad \textit{for each } y \in Y. \end{aligned}$$

Moreover, if the convergence is almost uniform, then $L^{-1}(f,y) = \mathrm{Lt}\ L^{-1}(f_\xi,y)$ for each $y \in Y$.

Proof. Let $x_0 \in X$, $\varepsilon > 0$ and let $k \geq 1$ be such that $\frac{3}{k} < \varepsilon$. For a point $y_0 \in L(f,x_0)$ there exists a net $\{x_\sigma : \sigma \in \Sigma\}$ in X such that $x_\sigma \to x_0$ and $f(x_\sigma) \to y_0$. Using the quasi-uniform convergence and properties of ordinal numbers we can choose $\alpha < \omega_1$ such that for each $n \geq 1$ and each ξ with $\alpha \leq \xi < \omega_1$ there exists a neighbourhood U of x_0 with $d(f_\xi(x), f(x)) < \frac{1}{n}$ for $x \in U$. Let ξ be established with $\alpha \leq \xi < \omega_1$ and let U be a neighbourhood of x_0 for which we have $d(f_\xi(x), f(x)) < \frac{1}{k}$ if $x \in U$. Then $\sigma_0 \in \Sigma$ can be taken such that $x_\sigma \in U$

and $d(f(x_\sigma), y_0) < \frac{1}{k}$ for $\sigma \geq \sigma_0$. It implies $d(f_\xi(x_\sigma), y_0) \leq \frac{2}{k} < \varepsilon$, so $f_\xi(x_\sigma) \to y_0$ for each ξ, $\alpha \leq \xi < \omega_1$. Thus we have shown

$$(*) \qquad L(f, x_0) \subset \liminf L(f_\xi, x_0) \subset \mathrm{Li}\ L(f_\xi, x_0)\,.$$

Now, if $y_0 \in \mathrm{Ls}\ L(f_\xi, x_0)$, then we can choose $\xi < \omega_1$ and a neighbourhood W of x_0 such that $d(f_\xi(x), f(x)) < \frac{1}{k}$ for $x \in W$ and $B\left(y_0, \frac{1}{k}\right) \cap L(f_\xi, x_0) \neq \emptyset$.

Let $y_1 \in B\left(y_0, \frac{1}{k}\right) \cap L(f_\xi, x_0)$; then there exists a net $\{x_\sigma : \sigma \in \Sigma\}$ in X such that $x_\sigma \to x_0$ and $f_\xi(x_\sigma) \to y_1$. So we can take $\sigma_0 \in \Sigma$ with $x_\sigma \in W$ and $d(f_\xi(x_\sigma), y_1) < \frac{1}{k}$ for every $\sigma \geq \sigma_0$. Hence

$$d(f(x_\sigma), y_0) \leq d(f(x_\sigma), f_\xi(x_\sigma)) + d(f_\xi(x_\sigma), y_1) + d(y_1, y_0) < \varepsilon \quad \text{for } \sigma \geq \sigma_0.$$

It implies $f(x_\sigma) \to y_0$ and in the consequence $y_0 \in L(f, x_0)$. Thus we obtain

$$\limsup L(f_\xi, x_0) \subset \mathrm{Ls}\ L(f_\xi, x_0) \subset L(f, x_0)\,.$$

Assume $x \in L^{-1}(f, y)$; it is equivalent to the condition $y \in L(f, x)$. Following to $(*)$ there exists $\beta < \omega_1$ such that $y \in L(f_\xi, x)$ for each ξ with $\beta \leq \xi < \omega_1$. But then $x \in L^{-1}(f_\xi, y)$ for each ξ, $\beta \leq \xi < \omega_1$ which gives

$$L^{-1}(f, y) \subset \liminf L^{-1}(f_\xi, y)\,.$$

Let now $x \in \limsup L^{-1}(f_\xi, y)$. Then for each $\xi < \omega_1$ there is γ, $\xi \leq \gamma < \omega_1$ with $x \in L^{-1}(f_\gamma, y)$ or equivalently $y \in L(f_\gamma, x)$. Then $y \in \limsup L(f_\xi, x) \subset L(f, x)$, so $x \in L^{-1}(f, y)$ and we obtain

$$\limsup L^{-1}(f_\xi, y) \subset L^{-1}(f, y)\,.$$

Finally we suppose that the transfinite sequence $\{f_\xi : \xi < \omega_1\}$ converges to f almost uniformly; to complete the proof it remains to show $\mathrm{Ls}\ L^{-1}(f_\xi, y) \subset L^{-1}(f, y)$. To contrary, if $x \notin L^{-1}(f, y)$, then there exists $r > 0$ and a neighbourhood U of x such that $B(y, 2r) \cap \overline{f(U)} = \emptyset$. According to the almost uniform convergence there exists $\xi_0 < \omega_1$ and a neighbourhood W of x, $W \subset U$ such that $d(f_\xi(x'), f(x')) < r$ for $x' \in W$. Thus for each ξ, $\xi_0 \leq \xi < \omega_1$ we have

$$f_\xi(W) \subset B(f(W), r) \subset B(f(U), r)\,.$$

From this it follows $f_\xi(W) \cap B(y, r) = \emptyset$, so

$$W \cap \overline{f_\xi^{-1}(B(y, r))} = \emptyset\,.$$

This leads to the condition $W \cap L^{-1}(f_\xi, y) = \emptyset$ for each ξ, $\xi_0 \leq \xi < \omega_1$, and then $x \notin \mathrm{Ls}\ L^{-1}(f_\xi, y)$, which finishes the proof. $\square$

Theorem 5. *Let X be a topological space and Y a metric one. If for each $\xi < \omega_1$, $f_\xi \colon X \to Y$ is a function with closed graph and the transfinite sequence $\{f_\xi : \xi < \omega_1\}$ is quasi-uniformly convergent to a function $f \colon X \to Y$, then the graph of f is closed.*

Proof. According to (5) and Theorem 4 we have $L(f, x) = \mathrm{Lt}\, L(f_\xi, x) = \mathrm{Lt}\,\{f_\xi(x)\} = \{f(x)\}$ for each $x \in X$, so $G(f)$ is closed. □

In the last part of this note we will show that for two quasi-uniformly convergent transfinite sequences the net of superpositions is quasi-uniformly convergent. To begin with we consider the following example showing that in a general case this is not true.

Example 1. Let R be the space of real numbers with the usual metric. We define functions $f_n, f, g_n, g \colon R \to R$ assuming

$$\begin{aligned} f(x) &= 0 = g(x) && \text{for each } x \in R, \\ f_n(x) &= \frac{1}{n} && \text{for } x \in R,\ n \geq 1, \\ g_n(x) &= \begin{cases} 1\,, & \text{if } x = \frac{1}{n}; \\ 0\,, & \text{if } x \in R \setminus \{\frac{1}{n}\}. \end{cases} \end{aligned}$$

Then the sequences $\{f_n : n \geq 1\}$ and $\{g_n : n \geq 1\}$ quasi-uniformly converge to f and g respectively. But $gf(x) = 0$ for each $x \in R$ and $g_n f_n(x) = 1$ for each $x \in R$, $n \geq 1$, so $\{g_n f_n : n \geq 1\}$ does not converge to gf even pointwise.

Given two directed sets $(S_1, \leq_{(1)})$ and $(S_2, \leq_{(2)})$ we will consider $S_1 \times S_2$ with the relation "$\leq$" defined by: $(s_1, s_2) \leq (p_1, p_2)$ if and only if $s_1 \leq_{(1)} p_1$ and $s_2 \leq_{(2)} p_2$. For transfinite sequences $\{f_\xi : \xi < \omega_1\}$, $\{g_\xi : \xi < \omega_1\}$ of functions $f_\xi : X \to Y$ and $g_\xi \colon Y \to Z$ we have the net of superpositions $\{g_\xi f_\alpha : (\xi, \alpha) < (\omega_1, \omega_1)\}$.

Theorem 6. *Let X be a topological space, (Y, d), (Z, ρ) metric ones and let $\{f_\xi \colon \xi < \omega_1\}$, $\{g_\xi \colon \xi < \omega_1\}$ be transfinite sequences of functions $f_\xi : X \to Y$ and $g_\xi \colon Y \to Z$. If these sequences quasi-uniformly converge to continuous functions $f \colon X \to Y$ and $g \colon Y \to Z$ respectively, then the net $\{g_\xi f_\beta : (\xi, \beta) < (\omega_1, \omega_1)\}$ is quasi-uniformly convergent to the function gf.*

Proof. The statement (3) and Theorem 3 imply that for each $x \in X$ there is $\alpha < \omega_1$ such that for any ξ, β with $(\alpha, \alpha) \leq (\xi, \beta) < (\omega_1, \omega_1)$ we have $x \in C(f_\beta)$, $f(x) \in C(g_\xi)$ and $g_\xi f_\beta(x) = gf(x)$. Thus $x \in C(g_\xi f_\beta)$ for any ξ, β with $(\alpha, \alpha) \leq (\xi, \beta) < (\omega_1, \omega_1)$. Now, using (1) we obtain the quasi-uniform convergence of $\{g_\xi f_\beta : (\xi, \beta) < (\omega_1, \omega_1)\}$ to gf. □

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