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ON THE MINIMAL REDUCTION AND MULTIPLICITY OF $(X^m, Y^n, X^kY^l, X^rY^s)$

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Abstract. An explicite formula to calculate the multiplicity of the ideal $(X^m, Y^n, X^kY^l, X^rY^s) \cdot A$ in $A = K[X,Y]_{(X,Y)}$ is given.

Let $K[X,Y]$ be a polynomial ring over a field K, $A = K[X,Y]_{(X,Y)}$ be a local ring with the maximal ideal $M = (X,Y) \cdot A$ and Q be an M-primary ideal in A. The multiplicity $e_0(Q,A)$ of Q in A is defined to be the leading coefficient of the Hilbert-Samuel polynomial $L_A(A/Q^t)$, $t \gg 0$ (see e.g. [**Z–S,** Vol. II, Chap. VIII, §10]).

Our main result is the following theorem.

Theorem. *Let* $Q = (X^m, Y^n, X^kY^l, X^rY^s) \cdot A$ *be an ideal in the local ring* $A = K[X,Y]_{(X,Y)}$. *Then*

$$e_0(Q,A) = \min\{mn, ml + nk, ms + nr, ml + nr + ks - lr\}\,.$$

(assuming $m \geq n$, $m > k > r$ *and* $n > s > l$, *without loss of generality).*

The idea of counting the multiplicity $e_0(Q,A)$ is based on the notions "minimal reduction" and "analytic spread" introduced by Northcott and Rees, see [**N–R**]. Recall that an ideal $J \subseteq I$ of A is called a **reduction** of I, if $J \cdot I^{t-1} = I^t$ for an integer $t > 1$. If J is a reduction of I and $\dim(I) = 0$ then $\dim(J) = 0$ and $e_0(J,A) = e_0(I,A)$. If Q is an M-primary ideal in a local ring (A,M), then there exists an M-primary ideal $Q' \subseteq Q$ which is a reduction of Q such that Q' is generated by a system of parameters (see [**N–R**, §6, Theorem 2]).

In the following we show how to construct such a parametrical ideal Q'.

Let $Q = (X^m, Y^n, X^kY^l, X^rY^s) \cdot A$ in $A = K[X,Y]_{(X,Y)}$ and $A[X^mt, Y^nt, X^kY^lt, X^rY^st] = R_A(Q)$ be the Rees ring of A with respect to the

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ideal Q, $R_A(Q) = \bigoplus_{N\geq 0} Q^N t^N$. Let a, b, c, d be independent indeterminates over K and

$$\varphi\colon A[a,b,c,d] \twoheadrightarrow A[X^m t, Y^n t, X^k Y^l t, X^r Y^s t] = R_A(Q)$$

be the natural epimorphism which sends a, b, c, d onto $X^m t$, $Y^n t$, $X^k Y^l t$, $X^r Y^s t$ respectively.

Let J be an ideal contained in Ker φ, $J \subseteq \operatorname{Ker} \varphi$. Then there is an epimorphism

$$\varphi^*\colon A[a,b,c,d]/J \twoheadrightarrow A[X^m t, Y^n t, X^k Y^l t, X^r Y^s t]$$

and an epimorphism

$$\varphi'\colon A[a,b,c,d]/(J+M) \twoheadrightarrow R_A(Q)/M \cdot R_A(Q)$$

of factor rings $A[a,b,c,d]/(J+M) = K[a,b,c,d]/J'$ with $J' = (J+M)/M$ and

$$R_A(Q)/M \cdot R_A(Q) = \bigoplus_{N\geq 0} Q^N/M \cdot Q^N\,.$$

Let 's take J such that $\dim(J') = 2$. This is always possible for

$$\dim(R_A(Q)/M \cdot R_A(Q)) = l(Q) = 2$$

(by $l(Q)$ we denote the analytic spread of Q).

Let $\{\alpha, \beta\}$ be a system of parameters for $K[a,b,c,d]/J'$, i.e.

$$\dim(K[a,b,c,d]/(J', \alpha, \beta)) = 0\,.$$

Then there exists an epimorphism Φ induced by φ'

$$\Phi\colon K[a,b,c,d]/(J',\alpha,\beta) \twoheadrightarrow R_A(Q)/(M, \overline{\alpha}t, \overline{\beta}t) \cdot R_A(Q)\,,$$

with $\varphi(\alpha) = \overline{\alpha}t$, and $\varphi(\beta) = \overline{\beta}t$. Then the ring

$$R_A(Q)/(M, \overline{\alpha}t, \overline{\beta}t) \cdot R_A(Q) = \oplus\, Q^N/(M \cdot Q^N, (\overline{\alpha}, \overline{\beta}) \cdot Q^{N-1})$$

is 0-dimensional. For (J', α, β) is (a,b,c,d)-primary and Φ is an epimorphism of graded rings, it follows that there is an integer $N_0 > 0$ such that for all $N > N_0$ it is $Q^N/(M \cdot Q^N, (\overline{\alpha}, \overline{\beta}) \cdot Q^{N-1}) = 0$, i.e.

$$Q^N = M \cdot Q^N + (\overline{\alpha}, \overline{\beta}) \cdot Q^{N-1}\,.$$

Then $Q^N/(\overline{\alpha}, \overline{\beta}) \cdot Q^{N-1} = M \cdot Q^N/(\overline{\alpha}, \overline{\beta}) \cdot Q^{N-1} = M \cdot (Q^N/(\overline{\alpha}, \overline{\beta}) \cdot Q^{N-1})$. Using Nakayama's Lemma we get that $Q^N = (\overline{\alpha}, \overline{\beta}) \cdot Q^{N-1}$.

Summarizing we can formulate a following result.

Proposition. *Let (A, M), Q, α, β be as above. Then the ideal $(\overline{\alpha}, \overline{\beta}) \cdot A$ is a reduction of Q and therefore $e_0(Q, A) = e_0((\overline{\alpha}, \overline{\beta}) \cdot A, A)$.*

Before proving the Theorem we prove an easy but useful lemma.

Lemma. *Let $m \geq n$, $m > k > r$ and $n > s > l$; all m, n, k, l, r, s are positive integers. Then*

(a) $mn = \min\{mn, ml + nk, ms + nr, ml + nr + ks - lr\}$ *if and only if* $mn = \min\{mn, ml + nk, ms + nr\}$.

(b) $ml + nk = \min\{mn, ml + nk, ms + nr, ml + nr + ks - lr\}$ *if and only if* $ml + nk = \min\{mn, ml + nk, ml + nr + ks - lr\}$.

(c) $ml + nr + ks - lr = \min\{mn, ml + nk, ms + nr, ml + nr + ks - lr\}$ *if and only if* $ml + nr + ks - lr = \min\{ml + nk, ms + nr, ml + nr + ks - lr\}$.

Proof.

(a) It is enough to prove the following:
If $mn \leq ml + nk$ and $mn \leq ms + nr$, then $mn \leq ml + nr + ks - lr$.

It is easy to see that the inequality $mn \leq ml + nk$ is equivalent to

$$\frac{m}{n}(n - l) \leq k \tag{1}$$

and $mn \leq ms + nr$ to

$$\frac{n}{m}(m - r) \leq s \tag{2}$$

From (1) and (2) we get $(n - l)(m - r) \leq ks$ and $mn \leq ml + nr + ks - lr$ as required.

(b) It is again enough to prove the implication:
If $ml + nk \leq mn$ and $ml + nk \leq ml + nr + ks - lr$, then $ml + nk \leq ms + nr$.

The first inequality is equivalent to $\frac{k}{n-l} \leq \frac{m}{n}$ and the second one to $\frac{k-r}{s-l} \leq \frac{k}{n-l}$. Then it follows $\frac{k-r}{s-l} \leq \frac{m}{n}$. But this is equivalent to $ml + nk \leq ms + nr$ as wanted.

(c) It is sufficient to prove that $ml+nr+ks-lr \leq ml+nk$ and $ml+nr+ks-lr \leq ms + nr$ imply $ml + nr + ks - lr \leq mn$.

The assumed inequalities are equivalent to

$$k \leq \frac{(k - r)(n - l)}{s - l} \qquad \text{and} \qquad s \leq \frac{(m - r)(s - l)}{k - r}\,.$$

Then $ks \leq (n - l)(m - r)$ and this is equivalent to $ml + nr + ks - lr \leq mn$. The proof of the lemma is complete. □

Now we are ready to prove the Theorem.

Proof of the Theorem. We will make it in 4 steps.

Step 1. Let $\min\{mn, ml+nk, ms+nr, ml+nr+ks-lr\} = mn$.
By the previous lemma this is equivalent to $mn \le ml+nk$ and $mn \le ms+nr$. From the first inequality we get $nk \ge m(n-l)$ and

$$(X^k Y^l)^n \in ((X^m)^{n-l} \cdot (Y^n)^l) \cdot A\,.$$

This implies that $c^n - a^{n-l}b^l \in J'$ (if $nk = m(n-k)$) or $c^n \in J'$. From the second inequality we have $nr \ge m(n-s)$ and

$$(X^r Y^s)^n \in ((X^m)^{n-s} \cdot (Y^n)^s) \cdot A\,.$$

From this it follows that $d^n - a^{n-s}b^s \in J'$ (if $nr = m(n-s)$ or $d^n \in J'$.

Now J' is the ideal generated by u_1, v_1, $J' = (u_1, v_1)$, where u_1 is one of the elements c^n or $c^n - a^{n-l}b^l$ and v_1 equals to either d^n or $d^n - a^{n-s}b^s$. In all these cases the ideal (J', a, b) is (a,b,c,d)-primary and by the Proposition the ideal $(X^m, Y^n)\cdot A$ is a reduction of Q. Therefore $e_0(Q,A) = e_0((X^m,Y^n)\cdot A, A) = mn$.

Step 2. Now let $ml+nk = \min\{mn, ml+nk, ms+nr, ml+nr+ks-lr\}$, i.e. $ml+nk \le mn$ and $ml+nk \le ml+nr+ks-lr$ by lemma. Then it follows $m(n-l) \ge nk$ and $(X^m)^{n-l}\cdot(Y^n)^l \in ((X^kY^l)^n)\cdot A$. Therefore either $a^{n-l}b^l - c^n \in J'$ (in case of equality $ml+nr+ks-lr = ml+nk$) or $a^{n-l}b^l \in J'$ from the first inequality and $r(n-l) \ge k(n-s)$, $(X^rY^s)^{n-l} \in ((Y^n)^{s-l}\cdot(X^kY^l)^{n-s})\cdot A$ and therefore either $d^{n-l} - b^{s-l}c^{n-s} \in J'$ (if $ml+nr+ks-lr = ml+nk$) or $d^{n-l} \in J'$ from the second one. If $J' = (u_2, v_2)$, $u_2 \in \{a^{n-l}b^l - c^n, c^n\}$, $v_2 \in \{d^{n-l} - b^{s-l}c^{n-s}, d^{n-l}\}$, the ideal $J' = (J, a+b, c)$ is again (a,b,c,d)-primary and the ideal $(X^m+Y^n, X^kY^l)\cdot A$ is a reduction of Q by the Proposition. Therefore

$$\begin{aligned} e_0(Q,A) &= e_0((X^m+Y^n, X^kY^l)\cdot A, A) \\ &= e_0((X^m+Y^n, X^k)\cdot A, A) + e_0((X^m+Y^n, Y^l)\cdot A, A) \\ &= nk + ml\,. \end{aligned}$$

Step 3 is equivalent to the second one (changing the roles of $ml+nk$ and $ms+nr$).

Step 4. Let $ml+nr+ks-lr = \min\{mn, ml+nk, ms+nr, ml+nr+ks-lr\}$. This is again equivalent to

$$ml+nr+ks-lr = \min\{ml+nk, ms+nr, ml+nr+ks-lr\}\,.$$

From $ml+nr+ks-lr \le ml+nk$ one gets $k(n-s) \ge r(n-l)$. Then

$$(Y^n)^{s-l}\cdot(X^kY^l)^{n-s} \in ((X^rY^s)^{n-l})\cdot A\,.$$

This implies either $b^{s-l}c^{n-s} - d^{n-l} \in J'$ (in case $ml + nr + ks - lr = ml + nk$) or $b^{s-l}c^{n-s} \in J'$.
From $ml + nr + ks - lr \leq ms + nr$ we have $s(m-k) \geq l(m-r)$ and

$$(X^m)^{k-r} \cdot (X^rY^s)^{m-k} \in ((X^kY^l)^{m-r}) \cdot A\,.$$

But this implies either $a^{k-r}d^{m-k} - c^{m-r} \in J'$ (if $ml + nr + ks - lr = ms + nr$) or $a^{k-r}d^{m-k} \in J'$.

Put $J' = (u_4, v_4)$, with $u_4 \in \{b^{s-l}c^{n-s}, b^{s-l}c^{n-s} - d^{n-l}\}$, $v_4 \in \{a^{k-r}d^{m-k}, a^{k-r}d^{m-k} - c^{m-r}\}$. If $ml + nr + ks - lr = ml + nk = ms + nr$ then $ml + nr + ks - lr = mn$ by the lemma and $e_0(Q, A) = mn = ml + nr + ks - lr$ by the step 1. In the rested cases the ideal $J' = (J, a + d, b + c)$ is (a, b, c, d)-primary, thus $(X^m + X^rY^s, Y^n + X^kY^l) \cdot A$ is a reduction of Q.

Therefore

$$\begin{aligned}
e_0(Q, A) &= e_0((X^m + X^rY^s, Y^n + X^kY^l) \cdot A, A) \\
&= e_0((X^r, Y^l) \cdot A, A) + e_0((X^{m-r} + Y^s, Y^l) \cdot A, A) \\
&\quad + e_0((X^r, Y^{n-l} + X^k) \cdot A, A) + e_0((X^{m-r} + Y^s, Y^{n-l} + X^k) \cdot A, A) \\
&= rl + (m-r)l + r(n-l) + sk = ml + nr + ks - lr
\end{aligned}$$

for $(X^{m-r} + Y^s, Y^{n-l} + X^k) \cdot A = (X^k, Y^s) \cdot A$.

The proof of the theorem is now complete. □

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