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AN EXAMPLE OF INFINITELY MANY SINKS FOR SMOOTH INTERVAL MAPS

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Abstract. We show, for arbitrary $\epsilon > 0$, the existence of a $C^{2-\epsilon}$ unimodal interval map with infinitely many sinks outside a neighbourhood of the critical point. It is known that such C^2 maps do not exist.

1. Introduction

Let a continuous map f of a closed interval I be given. Recall that an attracting cycle of f is called a **sink**, and an interval $J \subset I$ such that $f^n(J) \bigcap f^m(J) = \emptyset$, $n \neq m$, and J is not attracted by a sink, is called a **wandering interval**.

C^∞ interval maps having wandering intervals or infinitely many sinks can be constructed by using similar procedures [**6**]. Note that C^∞ circle map with wandering intervals was first constructed by Hall [**2**] as an improvement of the classical Denjoy example. Using a procedure suggested by Coven and Nitecki [**1**] it can be easily transformed into C^∞ interval map with wandering intervals. Some other examples were given by de Melo [**5**].

In [**3**] Mañé proved that C^2 interval maps cannot possess infinitely many sinks or wandering intervals outside a neighbourhood of the critical set (see also [**7**]). Recall that the critical set K for a smooth interval map f is defined by $K = \{x \in I | f'(x) = 0\}$. Martens, de Melo, and van Strien have shown [**4**] that C^2 interval maps which are C^3 in some neighbourhood of the critical set[1] cannot have wandering intervals or infinitely many sinks provided all critical points are nonflat. Given $f(x)$, a critical point $c \in K$ is called nonflat if there exists an integer $k \geq 2$ such that $f(x) \in C^k$ in some neighbourhood of $x = c$ and $f^{(k)}(c) \neq 0$. This means that typical (in C^3 topology) interval maps have finitely many sinks and do not have wandering intervals.

We construct, for any given $0 < \epsilon < 1$, an example of a unimodal interval map which has infinitely many sinks outside a neighbourhood of the critical point and which is C^∞ everywhere except at one point where it is $C^{2-\epsilon}$. A similar example

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[1]The exact statement in [**4**] is slightly stronger

of $C^{2-\epsilon}$ map with wandering intervals can be constructed. However, the proof is different (unlike [**6**] where both examples are treated along the same line), and we plan to discuss it in a forthcoming paper. This shows, in particular, that C^2 smoothness for the above results by Mañé and Martens et.al. can not be decreased.

Recall that given $0 < \epsilon < 1$ and a set $M \subset \mathbb{R}$, $f(x)$ is said to belong to C^ϵ on M if $\sup_{x,y\in M} |f(x) - f(y)|/|x-y|^\epsilon < \infty$. Given $k \in \mathbb{N}$ and $0 < \epsilon < 1$ it is said that $f(x) \in C^{k+\epsilon}$ iff $f^{(k)}(x) \in C^\epsilon$. We say that $f(x) \in C^{k+\epsilon}$ at a point if it is of this class in some neighbourhood of the point.

2. Auxiliary Functions

Consider on the interval $[0, 1]$ the following function

$$\phi(x) = \int_0^x \exp\{1/t(t-1)\}\,dt \Big/ \int_0^1 \exp\{1/t(t-1)\}\,dt, x \in (0,1),$$
$$\phi(0) = 0,\ \phi(1) = 1.$$

It is an easy exercise to verify that $\phi(x)$ has the following properties:

— $\phi(x)$ strictly increases for $x \in [0, 1]$;
— $\phi(x) \in C^\infty[0, 1],\ \phi^{(k)}(0) = \phi^{(k)}(1) = 0,\ k \in \mathbb{N}$;
— for every $k \in \mathbb{N},\ \sup\{|\phi^{(k)}(x)|, x \in [0, 1]\} = c_k,\ c_k$ is a constant depending on k only.

Given an interval $J = [a, b]$ and prescribed values $g_1 \neq g_2$, suppose it is required to construct a smooth function $g(x), x \in [a, b]$, such that $g(a) = g_1, g(b) = g_2$. For this purpose the above $\phi(x)$ can be used. We set

$$g(x) = \phi(x, J) = g_1 + (g_2 - g_1)\phi(\frac{x-a}{b-a})$$

The properties of $\phi(x)$ imply that $\phi(a, J) = g_1, \phi(b, J) = g_2, \phi(x, J)$ is strictly monotone and infinitely differentiable on $J = [a, b]$ with $\phi^{(k)}(a, J) = \phi^{(k)}(b, J) = 0$ for all $k \geq 1$, and

$$\sup\{|\phi^{(k)}(x, J)|, x \in J\} = c_k|g_2 - g_1|/|b-a|^k \tag{1}$$

where c_k are the above constants depending on k only.

Let $a_n, n \in \mathbb{N}$, be a monotone sequence such that $\lim_{n\to\infty} a_n = a_0$ exists. Define J_n to be intervals with endpoints a_n and a_{n+1}. The function $\phi(x, J)$ will be used on the sequence of intervals J_n with given values g_n at the endpoints. Suppose in addition that $\lim_{n\to\infty} g_n = g_0$ exists. Then a function $g(x)$ is defined on the whole interval $[a_1, a_0]$ (or $[a_0, a_1]$) by

$$g(x) = \phi(x, J_n),\ x \in J_n,\ g(a_0) = g_0$$

Proposition 1. *Suppose given* J_n *and* g_n *satisfy* $\lim_{n\to\infty} |g_{n+1} - g_n|/|a_{n+1} - a_n|^k = 0$ *for every* $k \in \mathbb{N}$. *Then* $g(x)$ *is of* C^∞ *class on* $[a_1, a_0]$ ($[a_0, a_1]$).

Proof is an immediate consequence of the equality (1).

We shall need also the following

Proposition 2. *Suppose* $g(x) \in C^{k+1}, k \geq 1$, *in some neighborhood of* $x = a$ *and* $g^{(i)}(a) = 0, i = 1, \dots, k$. *Then* $\lim_{b\to a}(g(b) - g(a))/(b - a)^i = 0$ *for all* $i = 1, \dots, k$.

Proof follows from the Taylor expansion of $g(x)$ in the neighborhood of $x = a$.

Proposition 2 includes also the case $g(x) \in C^\infty,\ g^{(k)}(a) = 0,\ k \in \mathbb{N}$, in which $\lim_{b\to a}(g(b) - g(a))/(b - a)^i = 0$ for every $i \in \mathbb{N}$.

Let $z_m, m \in \mathbb{N}$, be an increasing sequence with $0 < z_1$, and $\lim_{m\to\infty} z_m = 1$. With $z_0 = 0$ denote $[z_{m-1}, z_m] = J_m$, and consider $\phi(x, J_m)$ with prescribed values $\phi(z_{m-1}), \phi(z_m)$ at the endpoints. Define $\eta(x) = \phi(x, J_m), x \in J_m$, and $\eta(1) = 1$. The function $\eta(x)$ depends on $\phi(x)$ and particular choice of $z_m, m \in \mathbb{N}$.

Proposition 3. *Let* $k \in \mathbb{N}$ *be given. There exist constants* $L > 0$ *and* $0 < \delta < 1$ *which depend on* k *only and such that*

$$\sup\{|\eta^{(i)}(x)|, x \in [0, 1]\} \leq Lc_i,\ i = 1, \dots, k \tag{2}$$

provided $|1 - z_2| \leq \delta$.

Proof. Let $\alpha \in (0, 1)$ be given. Consider $\phi(x, [0, \alpha])$ and $\phi(x, [\alpha, 1])$ with prescribed values $\phi(0), \phi(\alpha)$, and $\phi(1)$ at the endpoints. Define $\psi(x) = \phi(x, [0, \alpha]), x \in [0, \alpha],\ \psi(x) = \phi(x, [\alpha, 1]), x \in [\alpha, 1]$. Then there exists a positive constant L' independent of α such that

$$\sup\{|\psi^{(i)}(x)|, x \in [0, 1]\} \leq L' \sup\{|\phi^{(i)}(x)|, x \in [0, 1]\} = L'c_i,\ i = 1, \dots, k.$$

To prove this assume $i = 1$ (case $i \geq 2$ is anologous). For $\alpha \in (0, 1)$, $\sup\{|\phi'(x, [0, \alpha])|,\ x \in [0, \alpha]\} = c_1\phi(\alpha)/\alpha$. Since $\phi(\alpha)/\alpha \to 0$ as $\alpha \to 0$ one has $\phi(\alpha)/\alpha \leq L'$ for some $L' > 0$.

Consider now $\eta(x)$ and $\psi(x)$ with $\alpha = z_1$. For every $k \in \mathbb{N}$ the functions $\psi(x)$ and $\eta(x)$ are close in uniform C^k metric provided z_2 is close to 1. This follows from Proposition 2 and equality (1). Therefore, for every $L > L'$ there exists $\delta > 0$ such that inequality (2) holds provided $|1 - z_2| \leq \delta$. □

Let an interval $J = [a, b]$ and a sequence $a < u_1 < u_2 < \cdots < u_n < u_{n+1} < \cdots \to b$ be given with prescribed values $g_1 \neq g_2$ at the endpoints a and b respectively. Define $\eta(x, J) = g_1 + (g_2 - g_1)\eta(\frac{x-a}{b-a}), x \in J$, where $\eta(t), t \in [0, 1]$, is constructed as above with $z_i = (u_i - a)/(b - a)$.

Proposition 4. *Let $k \in \mathbb{N}$ be given. There exist constants $L > 0$ and $0 < \delta < 1$ which depend on k only and such that*

$$\sup\{|\eta^{(i)}(x, J)|, x \in J\} \leq \frac{|g_2 - g_1|}{|b-a|^i} Lc_i,\ i = 1, \dots, k.$$

provided $|b - z_2| \leq \delta(b - a)$.

Proof. By differentiation $\eta(x, J)$ and using (2) the proof follows. □

3. Example

3.1 Construction.

Take a sequence $x_n, n \geq 0$, such that $x_0 = 1$ and x_n monotonically approaches zero (particular choices of x_n will be specified later). Define $f(x_n) = x_{n-1}, n \in \mathbb{N}$. Set $g(x_n) = x_{n-1} - x_n = g_n$ and consider intervals $I_n = [x_{n+1}, x_n]$ with given values g_{n+1}, g_n at the endpoints. Define $g(x) = \phi(x, I_n), x \in I_n, n \in \mathbb{N}$, and $g(0) = 0$.

Take arbitrary (but fixed) $z \in (x_1, 1), \lambda \in (0, 1)$, and define $f(x)$ in the following way:

— $f(x) = x + g(x),\ x \in [0, x_1]$;
— $f(x) = \lambda(1 - x),\ x \geq z$;
— $f(x)$ is an arbitrary unimodal C^∞ function on $[x_1, z]$ such that $f(x_1) = 1,\ f'(x_1) = 1,\ f(z) = \lambda(1 - z),\ f'(z) = -\lambda,\ f^{(i)}(x_1) = f^{(i)}(z) = 0,\ i = 2, 3, \dots$;
— $f(x) \equiv x,\ x \leq 0$.

We note that given interval $[\alpha, \beta]$ it is always possible to construct $C^\infty[\alpha, \beta]$ function $f(x)$ which takes prescribed values $f^{(i)}(\alpha) = f_i^\alpha,\ f^{(i)}(\beta) = f_i^\beta,\ i = 0, 1, \dots, N$, at the endpoints, and $f^{(i)}(\alpha) = f^{(i)}(\beta) = 0,\ i > N$. In our case such $f(x)$ may be chosen as follows. Take arbitrary $x_* \in (x_1, z)$ and define

$$f(x) = 1 + (x - x_1) - (1 + \lambda) \int_{x_1}^{x} \phi(\frac{t - x_1}{x_* - x_1})\, dt, \quad x \in [x_1, x_*].$$

Set $f_* = f(x_*)$ and define next

$$f(x) = \lambda(1 - x) + [f_* - \lambda(1 - x_*)]\phi(\frac{z - x}{z - x_*}), \quad x \in [x_*, z].$$

It is an easy exercise to verify that f is unimodal for $x_* - x_1$ small enough and is C^∞ in $[x_1, z]$.

If we define $b = \max\{f(x), x \in [x_1, z]\}$ and set $a = \lambda(1 - b)$, the interval $I = [a, b]$ is mapped by f onto itself.

Since $\phi(x, I_n) \in C^\infty$ it is straightforward that $f(x) \in C^\infty$ everywhere on $[a, b]$ except possibly the point $x = 0$.

Next we are going to choose a sequence of cycles of $f(x)$ with unbounded periods. These cycles will be transformed, by a further change of $f(x)$, into attracting ones. To guarantee required smoothness of the resulting map we choose the sequence in a special way.

Due to the construction we have: $f(I_n) = I_{n-1}, n \in N$, and $f(I_0) \supset [0, 1]$ (here $I_0 = [x_1, 1]$). Therefore, for every $n \in N$ there exists a cycle $\beta = \beta(n) = \{z_0^{(n)}, \dots, z_{n-1}^{(n)}\}$ of period n such that $z_0^{(n)} \in I_0, z_1^{(n)} \in I_1, \dots, z_{n-1}^{(n)} \in I_{n-1}$. Since $\lim_{n\to\infty} x_n = 0$ there exists $n_0 \in N$ with $z_0^{(n)} \in [z, 1]$ for all $n \geq n_0$.

Take some $n_1 \geq n_0$ and consider the cycle $\beta_1 = \{z_0^{(n_1)}, z_1^{(n_1)}, \dots, z_{n_1-1}^{(n_1)}\}$ of period n_1 as the first chosen one. Let $L > 1$ and $0 < \delta < 1$ be fixed constants (their further choice is specified in subsection 3.2). Take next $n_2 > n_1$ in such a way that the cycle $\beta_2 = \{z_0^{(n_2)}, z_1^{(n_2)}, \dots, z_{n_2-1}^{(n_2)}\}$ has the following property:

$$x_i - z_i^{(n_2)} \leq \delta \operatorname{diam} I_i,$$

$$\frac{|g(z_i^{(n_2)}) - g(z_i^{(n_1)})|}{|z_i^{(n_2)} - z_i^{(n_1)}|} \leq L \frac{|g(x_i) - g(z_i^{(n_1)})|}{|x_i - z_i^{(n_1)}|},$$

$i = 1, 2, \dots, n_1 - 1$. In view of Proposition 2 such a choice is always possible, since point $z_0^{(n)}$ of the cycle $\beta(n)$ satisfies $\lim_{n\to\infty} z_0^{(n)} = 1$. This implies $z_i^{(n)} \to x_i$ as $n \to \infty$ for arbitrary $i \in \mathbb{N}$.

In the next step choose $n_3 > n_2$ in such a way that the cycle $\beta_3 = \{z_0^{(n_3)}, \dots, z_{n_3-1}^{(n_3)}\}$ has the property:

$$x_i - z_i^{(n_3)} \leq \delta \operatorname{diam} I_i,$$

$$\frac{|g(z_i^{(n_3)}) - g(z_i^{(n_2)})|}{|z_i^{(n_3)} - z_i^{(n_2)}|^2} \leq L \frac{|g(x_i) - g(z_i^{(n_2)})|}{|x_i - z_i^{(n_2)}|^2},$$

$i = 1, 2, \dots, n_2 - 1$. Note that we have to care about inequalities $x_i - z_i^{(n_3)} \leq \delta \operatorname{diam} I_i$ for $i = n_1, \dots, n_2 - 1$ only since they are satisfied for $i = 1, 2, \dots, n_1 - 1$ because $x_i > z_i^{(n_3)} > z_i^{(n_2)}$.

In the k^{th} step we choose $n_k > n_{k-1}$ in such a way that the cycle $\beta_k = \{z_0^{(n_k)}, \dots, z_{n_k-1}^{(n_k)}\}$ has the property:

$$x_i - z_i^{(n_k)} \leq \delta \operatorname{diam} I_i, \tag{3}$$

$$\frac{|g(z_i^{(n_k)}) - g(z_i^{(n_{k-1})})|}{|z_i^{(n_k)} - z_i^{(n_{k-1})}|^{k-1}} \leq L \frac{|g(x_i) - g(z_i^{(n_{k-1})})|}{|x_i - z_i^{(n_{k-1})}|^{k-1}}, \tag{4}$$

$i = 1, 2, \dots, n_{k-1} - 1$.

Proceeding in this way we obtain on each of the intervals $I_n, n \in \mathbb{N}$, a sequence $z_n^{(n_m)}, m \geq l$, for some $l = l(n)$ satisfying $x_{n+1} < z_n^{(n_l)} < z_n^{(n_{l+1})} < z_n^{(n_{l+2})} < \ldots < z_n^{(n_m)} < \ldots \to x_n$. Denote $[x_{n+1}, z_n^{(n_l)}] = J_{nl}, [z_n^{(n_l)}, z_n^{(n_{l+1})}] = J_{nl+1}, \ldots , [z_n^{(n_{m-1})}, z_n^{(n_m)}] = J_{nm}, \ldots$, with prescribed values of given $g(x)$ at the endpoints. Redefine $g(x)$ on I_n by setting:

$$g^*(x) = \phi(x, J_{ni}), x \in J_{ni},\ i \geq l(n), \text{ and, } g^*(x_n) = g(x_n).$$

Consider now new $f(x)$ as defined above with $g(x)$ replaced by $g^*(x)$. Since

$$\frac{d}{dx} f(x)|_{x=z_0^{(n_k)}} = -\lambda, \quad \frac{d}{dx} f(x)|_{x=z_i^{(n_k)}} = 1, \quad i = 1, 2, \ldots , n_k - 1,$$

the multiplicator of the cycle $\beta_k = \{z_0^{(n_k)}, z_1^{(n_k)}, \ldots , z_{n_k-1}^{(n_k)}\}$ equals $-\lambda$. This shows that every cycle $\beta_k, k \in N$, is attracting one.

3.2 Smoothness of $f(x)$.

Theorem. *For arbitrary $0 < \epsilon < 1$ the sequence x_n and cycles $\beta_n, n \in \mathbb{N}$, may be chosen in such a way that $f(x) \in C^\infty$ for $x \in [a, b] \setminus \{0\}$, and $f(x) \in C^{2-\epsilon}$ for $x = 0$.*

Proof. Choose $L > 1$ and $0 < \delta < 1$ for inequalities (3), (4) and Propositions 3, and 4 with $J = I_n$ to hold.

Claim 1. $g^*(x)$ is of C^∞ class on every interval $I_n, n \geq 1$.

Proof. It is enough to show that $\lim_{i\to\infty} \sup\{|\phi^{(k)}(x, J_{ni})|\} = 0$ for every $k \geq 1$. Using (1) and (4) we have $\sup\{|\phi^{(k)}(x, J_{ni})|, x \in J_{ni}\} = c_k|g(z_n^{(n_i)}) - g(z_n^{(n_{i-1})})|/|z_n^{(n_i)} - z_n^{(n_{i-1})}|^k \leq c_k L|g(x_n) - g(z_n^{(n_{i-1})})|/|x_n - z_n^{(n_{i-1})}|^k$ for i sufficiently large. Proposition 2 gives $|g(x_n) - g(z_n^{(n_{i-1})})|/|x_n - z_n^{(n_{i-1})}|^k \to 0$ as $i \to \infty$. Therefore, Proposition 1 applies to conclude $g^*(x) \in C^\infty$ for $x \in I_n$ for every finite n.

Choose the sequence $x_n, n \in \mathbb{N}$, in such a way that $\Delta x_n = x_n - x_{n+1} = d/n^{1+\tau}$ for large N where $0 < d$ and $\tau > 0$ are constants. Then $x_n = \sum_{i=n}^{\infty} \Delta x_i \sim d_1/n^\tau$ as $n \to \infty$ for some $d_1 > 0$.

Claim 2. For every $\tau > 0$ $f'(x)$ is continuous for all $x \in [a, b]$, moreover, $\sup\{|g^{*\prime}(x)|, x \in I_n\} \leq d_2/n$ for some $d_2 > 0$.

Proof. It is enough to show the continuity at $x = 0$ only. With constants L and $0 < \delta < 1$ chosen above one has $\sup\{|g^{*\prime}(x)|, x \in I_n\} \leq L \sup\{|g'(x)|, x \in I_n\}$ provided $|x_n - z_n^{(n_l)}| \leq \delta \operatorname{diam} I_n$. This follows from Proposition 4. Using this and (1) we obtain $\sup\{|g^{*\prime}(x)|, x \in I_n\} \leq c_1 L|\Delta x_{n-1} - \Delta x_n|/\Delta x_n \sim d_2/n$ as $n \to \infty$ for some $d_2 > 0$. Therefore, $\lim_{x\to 0} f'(x) = 1 = f'(0)$.

Claim 3. For arbitrary $0 < \epsilon < 1$ there exists $\tau = \tau(\epsilon)$ such that $f'(x) \in C^{1-\epsilon}$ at $x = 0$.

Proof. We have to show that for every $\epsilon > 0$ there exists $\tau > 0$ such that

$$\limsup_{x,y\to+0} \frac{|f'(x)-f'(y)|}{|x-y|^{1-\epsilon}} < \infty. \tag{5}$$

Suppose first $x, y \in I_n$. Then $|x-y| \le \Delta x_n, g^*(x) = \eta(x, I_n)$ for a respective η. Using intermediate value theorem and Proposition 3 we obtain

$$\frac{|f'(x)-f'(y)|}{|x-y|^{1-\epsilon}} = \frac{|g_n - g_{n+1}|}{|x_n - x_{n+1}|}|\eta'(\frac{x-x_{n+1}}{x_n-x_{n+1}}) - \eta'(\frac{y-x_{n+1}}{x_n-x_{n+1}})|/|x-y|^{1-\epsilon} \le$$

$$\le d_4 \frac{|g_n - g_{n+1}|}{|x_n - x_{n+1}|^2}|x-y|^{\epsilon} \le d_4 n^{\tau-\epsilon(1+\tau)},$$

for some $d_4 > 0$ and large n. By choosing $0 < \tau < \epsilon$ (5) follows.

Suppose next $x \in I_n, y \in I_{n+1}$. Then

$$\frac{|f'(x)-f'(y)|}{|x-y|^{1-\epsilon}} \le \frac{|f'(x)-f'(x_{n+1})|}{|x-x_{n+1}|^{1-\epsilon}} + \frac{|f'(y)-f'(x_{n+1})|}{|y-x_{n+1}|^{1-\epsilon}},$$

and by the first case considered, (4) follows.

Suppose finally $x \in I_n, y \in I_m, n+1 < m$. Then $|x-y| \ge \Delta x_{n+1}$, and in view of Claim 2 we have

$$\frac{|f'(x)-f'(y)|}{|x-y|^{1-\epsilon}} \le d_2 \frac{|1/n+1/m|}{1/(n+1)^{(1+\tau)(1-\epsilon)}} \le d_5 n^{\tau-\epsilon(1+\tau)},$$

for some $d_5 > 0$ and large n. By choosing $0 < \tau < \epsilon$, (4) follows.

The remaining case $x \in I_n, y = 0$ follows from the above with $1/m = 0$. This completes the proof. □

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