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A COUNTEREXAMPLE TO A FEDORENKO STATEMENT

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Abstract. We present a counterexample to the following statement of Fedorenko: For a continuous map of a real interval these two conditions are equivalent:

(i) $f|\mathrm{Rec}\,(f)$ is a homeomorphism
(ii) every minimal set, which is not an orbit of a periodic point, has an exhausting sequence of periodic decompositions.

The main aim of this paper is to present a counterexample to a proposition due to Fedorenko [**F**], namely to that one which claims that for any continuous function $I \to I$ (I is a real compact interval) $f|\mathrm{Rec}\,(f)$ is a homeomorphism if and only if every minimal set, which is not an orbit of a periodic point, has an exhaustive sequence of periodic decompositions.

Let us recall the corresponding definitions:

Definition 1. A point $x \in I$ is a periodic point of a continuous function $f\colon I \to I$ (denoted by $f \in C(I,I)$), if there exists n such that $f^n(x) = x$, where f denotes the n-th iterate of f. A point $x \in I$ is asymptotically periodic, if the sequence $f^n(x)$ converges to the orbit of some periodic point $y \in I$, when $n \to \infty$. We denote by $\mathrm{Per}(f)$ the set of all periodic points of f.

Definition 2. Let M be a closed set, $M \subset I$. Then we will call the family of sets $\{\, M_i; i = 1, \dots, n \,\}$ satisfying

1. $M_i \cap M_j = \emptyset \qquad$ for $i \neq j$
2. $\cup_{i=1}^{n} M_i = M$

a decomposition of the set M.

We will say that a decomposition $\{M_i\}$ of the set M refines a decomposition $\{N_i\}$ of M if for every M_i there is an N_j such that $M_i \subset N_j$.

A sequence of decompositions $\{M_i^n, i = 1, \dots, i_n\}$, $n = 1, 2, \dots$ of M is called a refining sequence if $\{M_i^{n+1}\}$ refines $\{M_i^n\}$ for all n. Refining sequence of decompositions is exhaustive, if $\lim_{n\to\infty} \sup_i \ \mathrm{diam}\, M_i^n = 0$.

A decomposition $\{\, M_i, i = 1, \dots, k \,\}$ is periodic if its members are subsets of closed pairwise disjoint intervals and

$$f(M_i) = M_{i+1}, \quad i = 1, \dots, k-1, \quad f(M_k) = M_1.$$

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Definition 3. An interval J is called wandering, if J, $f(J)$, $f(J)^2$, $\dots$ are disjoint and no point $x \in J$ is asymptotically periodic.

Definition 4. A minimal set M is an invariant, closed set which has no proper subset of the same type.

Definition 5. The set of recurrent points is the set

$$\mathrm{Rec}\,(f)\colon = \{x \in I,\ \forall\, \varepsilon > 0\ \exists\, N \geq 0\ \forall\, i \geq 0\ \exists\, i+1 \leq n \leq i+N\, |f^n(x) - x| < \varepsilon\}.$$

The following theorem is useful not only in our particular case but also in general.

Theorem 1. *Denote $\mathcal{M} = \cup M$ the union of all minimal sets of a map f. Then* $\mathrm{Rec}\,(f) = \mathcal{M}$.

Proof. Our proof will follow the original proof of Birkhoff [**B**], where this theorem is proved for the smooth dynamical systems.

1. $\mathrm{Rec}\,(f) \subset \mathcal{M}$.

Let $x \in \mathrm{Rec}\,(f)$. Define $N = \overline{\{f^n(x)\}}$ ($\overline{A}$ denotes the closure of the set A). Since $x \in \mathrm{Rec}\,(f)$, N is a minimal set and $x \in N$.

2. $\mathcal{M} \subset \mathrm{Rec}\,(f)$.

Let $x \in \mathcal{M}$, then $x \in M$ for some minimal set M. If M is a periodic orbit then clearly $x \in \mathrm{Rec}\,(f)$.

So assume that M is not a periodic orbit and $x \notin \mathrm{Rec}\,(f)$. Then,

$$\exists\, \varepsilon > 0\ \forall\, N > 0\ \exists\, i\ \forall\, i+1 \leq n \leq i+N\, |f^n(x) - x| > \varepsilon. \tag{1}$$

Take $\varepsilon > 0$ from (1) and a sequence $\{N_j\}_{j=1}^{\infty}$ tending to infinity which with the corresponding sequence $\{i_j\}_{j=1}^{\infty}$ satisfies (1). Define a sequence $J = \{f^{i_1}(x), f^{i_2}(x), \dots\}$ and let y be its limit point.

By construction of J we see that if $z \in U = \overline{\{f^n(y)\}}$, then $|z - x| \geq \varepsilon$.

Since U is a closed invariant set, $U \subseteq M$, $U \neq M$, and M is a minimal set, we have a contradiction. □

Theorem 2. *There exists a continuous function $g\colon I \to I$ such that $g|\mathrm{Rec}\,(f)$ is a homeomorphism and there is a minimal set M of g, which has no exhausting sequence of periodic decompositions.*

Proof. Proof will be divided into several lemmas.

Take a function $f(x) = \lambda^* x(1-x)$ for $\lambda^* = 3,569\dots$. It is known [**SKSF**], that such a function has cycles of orders 1,2,4, $\dots$, and no odd cycle, and such that the set $K = \overline{\{f^n(\frac{1}{2})\}}$ is homeomorphic to the Cantor set ($\frac{1}{2}$ is the critical point c of f) and $\mathrm{Rec}\,(f) = K \cup \mathrm{Per}(f)$ (K is the infinite ω-limit set).

It is also known, that our f has no wandering interval (cf. [**vS**], therefore $K = \cap_{n=1}^{\infty} \cup_{i=1}^{2^n} I_i^n$, where I_i^n for fixed n, are closed periodic intervals of the period 2^n, has the empty interior and K has an exhausting sequence of periodic decompositions. In such a case any two points $u, v \in K$ are f-separable (two points u, v are f-separable if there exist disjoint closed periodic intervals J_u, J_v, containing u and v, respectively) and according to [**JaSm**] f is non-chaotic.

Then Theorem 1 of [**MiSm**] implies that $f|K$ is a homeomorphism. □

Definition 6. A point x of the Cantor set is of the type i (inside) if it is a two-side limit point of the Cantor set and it is of the type o (outside) if it is only one-side limit point of that set.

It is easy to see (cf. renormalization process [**vS**]), that $x \in \{f^n(c)\}_{n=1}^{\infty}$ if and only if x is of the type o.

Definition 7. Denote by A-orb(y) the set $\{x \in A,\ \exists\ m, n > 0\colon f^m(x) = f^n(y)\}$ i.e. A-orb(y) is the full orbit of y within the set A.

Since $f|K$ is a homeomorphism and $f(K) = K$, for every n there exists precisely one $x \in K$, such that $f^n(x) = c$. Therefore card(K-orb$(c)) = \aleph_0$.

If we denote $C = K \backslash K - \text{orb}(c)$ then C is uncountable and

$$\text{every } x \in C \text{ is of the type } i. \tag{2}$$

Now we will use the technic of blowing-up the orbits, which was introduced by Denjoy [**D**]:

Take an arbitrary sequence of compact intervals such that the sum of their lenghts will be less than, say, $\frac{1}{4}$.

Now take some $z \in C$ and construct a new function g in the following way:

We replace every $v \in I$-orb(z) by a compact interval I_v from our sequence in such a way, that

$$g(I_v) = I_{f(v)}; \quad g|I_v \text{ is linear;}$$

and the trajectories of other points remain unchanged.

In other words we define a continuous nondecreasing (and outside, intervals I_v increasing) function $\tau \in C(I, I)$ such that $\tau(u) = v$ for all $u \in I_v$ and then we define g by

$$f \circ \tau = \tau \circ g. \tag{3}$$

Let c^* be the critical point of g and let $K^* = \overline{\{g^n(c^*)\}}$.

Remark. By our construction $\tau(\text{Per}(g)) = \text{Per}(f)$,
$\tau(K^*) = K$ and so $\tau(\text{Rec}\,(g)) = \text{Rec}\,(f)$ (we "add" only the interiors of wandering intervals to the dynamics, which doesn't affect the above mentioned sets). Since $\text{Rec}\,(f)$ is closed (Theorem 3.11 of [**SMR**]) and τ is continuous and nondecreasing, $\text{Rec}\,(g)$ is also closed.

Observation. Since $\text{int}(I_v)$ is a wandering interval for all v, $\text{int}(I_v) \cap \text{Rec}\,(g) = \emptyset$ for all v.

Lemma 1. *$g|\mathrm{Rec}\,(g)$ is a homeomorphism.*

Proof. Since $\mathrm{Rec}\,(g)$ is a compact set it is sufficient to show that g is one to one (g is continuous) on $\mathrm{Rec}\,(g)$.

We see that τ is one to one on $\mathrm{Rec}\,(g)$ except the end points of $I_v = [v_1, v_2]$, $v_1, v_2 \in K^*$ where $\tau(v_1) = \tau(v_2) = v$.

Since for all $v \in I$-orb(z) $v \neq c$ holds, we have $c^* \notin \cup_v I_v$ and $g|I_v$ is one to one for all v. Thus $g(v_1) \neq g(v_2)$ for all v.

This and (3) imply that $g|\mathrm{Rec}\,(g)$ is one to one and thus a homeomorphism. □

Lemma 2. *There exists a minimal set M for the map g, which is not a periodic orbit and which has no exhausting sequence of periodic decompositions.*

Proof. Take $M\colon = K^*$. It is easy to see that K^* is the minimal set for g, and that it is not a periodic orbit. Further, since $K = \cap_{n=1}^{\infty} \cup_{i=1}^{2^n} I_i^n$, we have $K^* \subset \cap_{n=1}^{\infty} \cup_{i=1}^{2^n} J_i^n$, where $\tau(J_i^n) = I_i^n$.

But now $\lim_{n\to\infty}$ diam $J_{i(n)}^n > 0$ for those sequences of the intervals $J_{i(n)}^n$ for which $\cap_{n=1}^{\infty} I_{i(n)}^n = a$, where $a \in C$-orb(z).
(For every a such a sequence of intervals $I_{i(n)}^n$ exists, see (2)). Thus $\cap_{n=1}^{\infty} J_{i(n)}^n = I_a$.

Since for every sequence of periodic decompositions $\{S^n\}_{n,i}$ of K^* there is the corresponding sequence of periodic decompositions $\{T_i^n\}_{n,i}$ of K, by the argument above there is no exhausting sequence of periodic decompositions for K^*. □

Now putting together Lemma 1 and Lemma 2 we are done.

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