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MAPS OF THE INTERVAL LJAPUNOV STABLE ON THE SET OF NONWANDERING POINTS

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ABSTRACT. Any dynamical system generated by a continuous map of the compact unit interval I, is Ljapunov stable on the set of ω-limit points iff it is Ljapunov stable on the set of non-wandering points. This and recent known results imply that Ljapunov stability on the set of non-wandering points characterizes maps non-chaotic in the sense of Li and Yorke.

We consider the class $C(I, I)$ of continuous maps $I \to I$, where I is a compact real interval. For any $f \in C(I, I)$ and any $x \in I$, $\{f^n(x)\}_{n=0}^{\infty}$ is the trajectory of x, $\omega_f(x)$ is its ω-limit set, and $\omega(f) = \cup\{\omega_f(x); x \in I\}$. We use symbols $\mathrm{Per}(f), \Omega(f)$ and $\mathrm{CR}(f)$ for the set of periodic points, non-wandering points, and chain recurrent points, respectively. Clearly,

$$\mathrm{Per}(f) \subseteq \omega(f) \subseteq \Omega(f) \subseteq \mathrm{CR}(f) \tag{1}$$

Recall that a map f is Ljapunov stable on a set A if for any $x \in A$ and any $\varepsilon > 0$ there is a neighbourhood $U(x)$ of x such that $|f^i(x) - f^i(y)| < \varepsilon$ whenever $i \geq 0$ and $y \in U(x) \cap A$.

Our main result reads as follows.

Theorem 1. *Let $f \in C(I, I)$. Then $f|\omega(f)$ is Ljapunov stable iff $f|\Omega(f)$ is Ljapunov stable.*

Since Ljapunov stability of $f|\omega(f)$ characterizes maps f non-chaotic in the sense of Li and Yorke [**FŠS**], we get the following

Corollary. *A map $f \in C(I, I)$ is chaotic in the sense of Li and Yorke iff $f|\Omega(f)$ is Ljapunov unstable.*

Other conditions equivalent to the Ljapunov stability of $f|\omega(f)$ can be found in [**FŠS**]. Recall that a map f is chaotic in the sense of Li and Yorke [**S**] iff, e.g., there is an $\varepsilon > 0$ and a perfect set $S \neq \emptyset$ such that, for any $x, y \in S, x \neq y$, $\limsup |f^n(x) - f^n(y)| > \varepsilon$ and $\liminf |f^n(x) - f^n(y)| = 0$, for $n \to \infty$.

Proof of the theorem is divided into a sequence of lemmas, and is based on the following few known facts:

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If f is Ljapunov stable on some of the sets from (1) then the topological entropy $h(f)$ of f is zero (or equivalently, any periodic orbit of f has period 2^n, for some n). To see this note that if $h(f) > 0$ then for some integer $n > 0$, f^n is topologically semiconjugated to the shift τ on the space X of sequences of binary symbols 0 and 1 [**B**], and clearly, $\tau|\mathrm{Per}\,(\tau)$ is unstable. Therefore, in the sequel we will consider only maps f with zero topological entropy.

If $\omega_f(x)$ is infinite then for any integer $n > 0$ there is a compact periodic interval I_n of period 2^n (i.e., $f^i(I_n) \cap I_n = \emptyset$ for $0 < i < 2^n$, and $f^i(I_n) = I_n$ for $i = 2^n$) such that $I_n \supset I_{n+1}$ and $\omega_f(x) \subset \mathrm{Orb}\,(I_n)$. Denote $M_f(x) = \cap_{n=0}^{\infty} \mathrm{Orb}\,(I_n)$.

If $\omega_f(x)$ is finite, and hence a periodic orbit, let $M_f(x) = \omega(f)$. Clearly, $f(M_f(x)) = M_f(x)$ is invariant, and it turns out that

$$\mathrm{CR}(f) = \cup\{M_f(x); x \in I\}. \tag{2}$$

Consequently, by (1), $\Omega(f) = \cup\{\Omega_f(x); x \in I\}$ where $\Omega_f(x) = \Omega(f) \cap M_f(x)$. For more details see [**FŠS**] or [**ŠKSF**] (cf. also [**S**]).

Lemma 1 (cf. [**Š1**]). *Let $a \in \Omega(f) \setminus \omega(f)$. Then a is an isolated point of $\Omega(f)$.*

Lemma 2. *Every $a \in \Omega(f) \setminus \omega(f)$ is an end-point of a compact wandering interval J_a, which is a connected component of some $M_f(x)$.*

Proof. By (1) and (2), $a \in J_a$, where J_a is a connected component of some $M_f(x)$. Since J_a is wandering (for any integers i, j, n with $0 < i < j < 2^n$, $f^i(J_a)$ and $f^j(J_a)$ lie in disjoint intervals from $\mathrm{Orb}\,(I_n)$), $a \notin \mathrm{int}\,(J_a)$. □

Lemma 3. *Let $f|\omega(f)$ be Ljapunov stable. Let $a \in \Omega_f(x) \setminus \omega(f)$ and let I_n be a compact periodic interval of period 2^n, with $a \in I_{n+1} \subset I_n$ for every n. Then a is an end-point of some I_m.*

Proof. Assume the contrary. For simplicity, let $J_a = [b, a]$ (cf. Lemma 2). Since $J_a = \cap_{n=1}^{\infty} I_n$ and a is isolated (cf. Lemma 1) there is an m such that $I_m = [u, v]\,, a < v$, and $(a, v] \cap \Omega(f) = \emptyset$. Let U be a neighbourhood of a, $U \subset I_m$. Then for some $r > 0, f^r(U) \cap U \neq \emptyset$. By the periodicity of I_m, $f^r(U) \subset I_m$. Since $J_a \cap f^r(J_a) = \emptyset$ and $f^r(a) \in f^r(\Omega(f)) \subset \Omega(f)$, we have $f^r(a) < b$ and $f^r(U) \supset [f^r(a), b]$. Since $J_a = \cap I_n$, there is $k > m$ such that $I_k \subset [f^r(a), v]$. Now $I_{k+1} \cup f^{2^k}(I_{k+1}) \subset I_k$ and $a \in I_{k+1}$, so $f^{2^k}(I_{k+1})$ is to the left of I_{k+1}. Thus $f^{2^k}(I_{k+1}) \subset f^r(U)$ and for $i = r + 2^k, f^i(U) \supset I_{k+1}$ is a neighbourhood of a. By induction we can construct a sequence $\{U_n\}_{n=1}^{\infty}$ of compact neighbourhoods of a with $\lim_{n\to\infty} \mathrm{diam}\, U_n = 0$, and a sequence $\{k(n)\}_{n=1}^{\infty}$ of positive integers such that $f^{k(n)}(U_n) \supset U_{n+1}$ for any n. It is easy to see that for some $y \in U_1, a \in \omega_f(y)$ – a contradiction. □

Lemma 4. *Let $f|\omega(f)$ be Ljapunov stable, let $\{a, f(a)\} \subset \Omega_f(x) \setminus \omega(f)$, and let U be a neighbourhood of a. Then $f(U) \cup J_{f(a)}$ is a neighbourhood of $f(a)$.*

Proof. By Lemmas 2 and 3 we may assume that U is an open interval containing a and so small that $U \cap M_f(x) \subset J_a$ and $\mathrm{diam}\, f(U) < \mathrm{diam}\, J_{f(a)}$. Let $V = f(U) \cup$

$J_{f(a)}$ fail to be a neighbourhood of $f(a)$. Since V is an interval, we have $V = J_{f(a)}$, and consequently, for any $i > 0$, $f^i(U) \cap U \subset f^{i-1}(J_{f(a)}) \cap U \subset f^i(J_a) \cap J_a = \emptyset$. Thus, $a \notin \Omega(f)$ – a contradiction. □

Lemma 5. *Let $f|\omega(f)$ be Ljapunov stable. Then for no $x \in I$ there is a sequence $\{b_n\}_{n=0}^{\infty}$ of points from $\Omega_f(x) \setminus \omega(f)$ such that $f(b_{n+1}) = b_n$ for any n.*

Proof. Assume the contrary. Let for each n, I_n be a compact periodic intreval of period 2^n, with $b_0 \in I_n$ and $\mathrm{Orb}\,(I_n) \supset \omega_f(x)$. By Lemma 3, b_0 is an end-point of some I_m. Clearly there is the least integer $k > 0$ such that, for some j, $b_k \in \mathrm{int}\, f^j(I_m)$. Then $f^j(I_m)$ is a neighbourhood of b_k, hence by Lemma 4, $I_m \cup J_{b_0} = I_m = f^k(f^j(I_m))$ is a neighbourhood of b_0 – a contradiction. □

Lemma 6. *Let $J = [u, v]$ be a periodic interval, and let $u \notin \mathrm{Per}\,(f)$. Then there is an $\varepsilon > 0$ such that $(u - \varepsilon, u) \cap \Omega(f) = \emptyset$.*

Proof. For simplicity we assume that $f(J) = J$. Then for $i = 1$ or 2, $f^i(u) \in \mathrm{int}\, J$, hence for a small ε, $f^i(u-\varepsilon, u) \subset \mathrm{int}\, J$, and so $f^n((u-\varepsilon, u)) \cap (u-\varepsilon, u) = \emptyset$ for any $n > 0$. (Note that if $f(u) = v$, then $f(v) \neq v$, cf., e.g, Lemma 3.5 in [**PS**].) □

Proof of Theorem. Since $\omega(f) \subset \Omega(f)$, we may assume that $f|\omega(f)$ is stable. Let $\varepsilon > 0$ and $a \in \Omega(f)$. Then for some x, $a \in \Omega_f(x)$. For any n and any $j = 1, \ldots, 2^n$, let J_n^j be the convex hull of $f^j(I_n) \cap \omega(f)$, where I_n is a compact periodic interval of period 2^n, with $a \in I_n$ and $\mathrm{Orb}\,(I_n) \supset M_f(x)$. Choose n so large that $\mathrm{diam}\, J_n^j < \varepsilon$ for every j. This is always possible since otherwise $f|\omega(f)$ would not be stable.

By Lemma 2, $A = \Omega_f(x) \setminus \cup\{J_n^j; j = 1, \ldots, 2^n\}$ is a finite set. Let B be the set of preimages of A in $\Omega(f)$. Clearly, $B \subset \Omega_f(x)$. By Lemma 5, B is finite, and since A is isolated in $\Omega(f)$ (cf. Lemma 1), B must be isolated in $\Omega(f)$, too. Thus if $a \in B$ then $f|\Omega(f)$ is stable at a. So let $a \notin B$.

If $a \in \mathrm{int}\, I_n$, let $U \subset I_n$ be a neighbourhood of a with $U \cap B = \emptyset$. Then $f^i(U) \cap \Omega(f) \subset J_n^i$ for any $i \geq 1 \pmod{2^n}$, and $f|\Omega(f)$ is stable at a since ε is arbitrary.

If a is an end-point of I_n then we can apply Lemma 6. There is a neighbourhood U of a such that $U \cap \Omega(f) \subset I_n$ and again $f|\Omega(f)$ is stable at a. □

Remark. Assume that $f|\Omega(f)$ is Ljapunov stable. The above quoted results enable to describe the dynamics of f on $\Omega(f)$. First, it is easy to see that f restricted to any infinite $\omega_f(x)$ acts as the well-known "adding machine", and the representation of points from $\omega_f(x)$ by sequences of binary symbols 0 and 1 is one to one, cf. [**N**], [**S**]. If $A = \Omega(f) \setminus \omega(f)$ is non-empty, then for any $a \in A$ put $A_f(a) = \mathrm{Orb}\,(a) \cap \Omega(f) \setminus \omega(f)$. Since $\omega(f)$ is invariant and $f(\Omega_f(x)) \subset \Omega_f(x)$ for any x, $A_f(a) \subset \Omega_f(y) \setminus \omega(f)$, for some y. Then either $A_f(a)$ is finite and $A_f(a) = \{a_0, \ldots, a_n\}$ such that $f(a_i) = a_{i+1}$ for any $i < n$ (and $f(a_n) \in \omega(f)$), or $A_f(a)$ is infinite and by Lemma 5, $A_f(a) = \{a_i\}_{i=0}^{\infty}$ such that $f(a_i) = a_{i+1}$ for any i. Both these types of behaviour are possible and corresponding examples can be obtained by a slight modification of a map from [**VŠ**].

References

[B] Block L., *Homoclinic points of mappings of the interval*, Proc. Amer. Math. Soc. **72** (1978), 576–580.

[FŠS] Fedorenko V. V., Šarkovskii A. N. and Smítal J., *Characterizations of weakly chaotic maps of the interval*, Proc. Amer. Math. Soc. **110** (1990), 141–148.

[N] Nitecki Z., *Topological dynamics on the interval*, Ergodic Theory and Dynamical Systems II, Proceedings, Birkhäuser, Boston, 1982, pp. 1–73.

[PS] Preiss D. and Smítal J., *A characterization of non-chaotic maps of the interval stable under small perturbations*, Trans. Amer. Math. Soc. **313** (1989), 687–696.

[Š1] Šarkovskii A. N., *Nonwandering points and the center of a continuous map of the line into itself*, Dopovidi AN **7** (1964), 865–868, USSR. (Ukrainian)

[ŠKCF] Šarkovskii A. N., Koljada S. F., Sivak A. G. and Fedorenko V. V., *Dynamics of one-dimensional maps*, Naukova Dumka, Kiev, 1989. (Russian)

[S] Smítal J., *Chaotic maps with zero topological entropy*, Trans. Amer. Math. Soc. **297** (1986), 269–282.

[VŠ] Verejkina M.B. and Šarkovskii A.N., *Recurrence in one-dimensional dynamical systems*, Approx. and Qualitative Methods of the Theory of Differential – Functional Equations, Inst. Math. AN USSR, Kiev, 1983, pp. 35–46. (Russian)

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