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SOME L_p INEQUALITIES FOR THE FAMILY OF B-OPERATORS

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ABSTRACT. Let $\mathcal{P}_n$ be the class of polynomials of degree at most n and $\mathcal{B}_n$ be a class of operators that map $\mathcal{P}_n$ into itself. For every $P \in \mathcal{P}_n$ and $B \in \mathcal{B}_n$, we investigate on $|z| = 1$, the dependence of $||B[P(R\,\cdot)] - B[P(r\,\cdot)]||_q$ on $||P||_q$, for every $R > r \geq 1$ and $q \geq 1$.

1. INTRODUCTION

Let $\mathcal{P}_n$ be the class of polynomials $P(z) := \sum\limits_{j=0}^{n} a_j z^j$ of degree at most n with complex coefficients. For $P \in \mathcal{P}_n$, define

$$\|P\|_q := \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| P(e^{i\theta}) \right|^q d\theta \right\}^{1/q} \text{ and } \|P\|_\infty := \max_{|z|=1} |P(z)|.$$

Rahman [6] (see also Rahman and Schmeisser [8, p. 538]) introduced a class $\mathcal{B}_n$ of operators B that map $P \in \mathcal{P}_n$ into itself. That is, the operator B carries $P \in \mathcal{P}_n$ into

$$B\left[P\right](z) := \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) \frac{P'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!}, \tag{1}$$

where λ_0, λ_1 and λ_2 are real or complex numbers such that all the zeros of

$$\mathcal{U}(z) := \lambda_0 + C(n,1)\lambda_1 z + C(n,2)\lambda_2 z^2, \quad C(n,r) = \frac{n!}{r!(n-r)!}, \tag{2}$$

lie in the half plane

$$|z| \leq \left| z - \frac{n}{2} \right| \tag{3}$$

and observed:

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Theorem 1.1. *If $P \in P_n$, then for $|z| \geq 1$,*

$$||B[P]||_\infty \leq |B[E_n]|||P||_\infty, \tag{4}$$

where $E_n(z) := z^n$.

As an improvement of (4), Shah and Liman [9] proved the following:

Theorem 1.2. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $|z| \geq 1$*

$$||B\,[P]\,||_\infty \leq \frac{1}{2}\Big\{\,|B\,[E_n]| + |\lambda_0|\Big\}||P||_\infty. \tag{5}$$

The result is sharp and equality holds for a polynomial whose all zeros lie on the unit disk.

Recently, Shah and Liman [10] extended the above results to the L_p norm by proving the following more general results:

Theorem 1.3. *If $P \in \mathcal{P}_n$, then for every $R \geq 1$, $q \geq 1$ and $|z| = 1$,*

$$\|B[P(R\,\cdot)]\|_q \leq |B[E_n(R\,\cdot)]|\|P\|_q, \tag{6}$$

where $B \in \mathcal{B}_n$ and $E_n(z) := z^n$. The result is best possible and equality holds for $P(z) = \alpha z^n$, $\alpha \neq 0$.

Theorem 1.4. *Let $P \in \mathcal{P}_n$ be such that $P(z) \neq 0$ in $|z| < 1$, then for every $R \geq 1$, $q \geq 1$ and $|z| = 1$,*

$$\|B\,[P(R\,\cdot)]\,\|_q \leq \frac{|B[E_n(R\,\cdot)]| + |\lambda_0|}{\|1 + E_n\|_q}||P||_q, \tag{7}$$

where $B \in \mathcal{B}_n$ and $E_n(z) := z^n$. The result is best possible and equality holds for the polynomial $P(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$.

2. Statement and proof of results

For the proofs of these theorems, we need the following lemmas. The first lemma is a special case of a result due to Aziz and Zargar [2].

Lemma 2.1. *If $P(z)$ is a polynomial of degree n having all zeros in $|z| \leq 1$, then for $R > r \geq 1$ and $|z| = 1$*

$$|P(Rz)| > |P(rz)|.$$

The next lemma follows from Corollary 18.3 of [5, p.86].

Lemma 2.2. *If all the zeros of a polynomial $P(z)$ of degree n lie in a circle $|z| \leq 1$, then all the zeros of the polynomial $B[P](z)$ also lie in the circle $|z| \leq 1$.*

Lemma 2.3. *If $P(z)$ is a polynomial of degree n having all zeros in $|z| \geq 1$ and $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, then for every real or complex number α with $|\alpha| \leq 1$ and $R > r \geq 1$,*

$$|B[P(R\,\cdot)] - \alpha B[P(r\,\cdot)]| \leq |B[Q(R\,\cdot)] - \alpha B[Q(r\,\cdot)]|\,. \tag{8}$$

The result is sharp and equality holds for $P(z) = z^n + 1$.

Proof. The result is trivial if $R = r$. So we assume that $R > r$. Since $P(z)$ has all zero in $|z| \geq 1$, therefore all the zeros of $Q(z) = z^n \overline{P(1/\bar{z})}$ lie in $|z| \leq 1$. By maximum modulus principle, $|Q(z)| \leq |P(z)|$ for $|z| \leq 1$, and in particular, $|P(z)| \leq |Q(z)|$ for $|z| \geq 1$. By Rouches theorem, it follows that for α with $|\alpha| \leq 1$ all the zeros of $F(z) = P(z) - \beta Q(z)$ lie in $|z| \leq 1$, for every β with $|\beta| > 1$. Applying Lemma 2.1 to $F(z)$, we get for $|z| = 1, R > r \geq 1$

$$|F(rz)| < |F(Rz)|.$$

Since all the zeros of $F(Rz)$ lie in $|z| \leq \frac{1}{R} < 1$, by Rouches theorem it follows that all the zeros of

$$F(Rz) - \alpha F(rz)$$

lie in $|z| < 1$. Since B is a linear operator (see [6, sec. 5]), it follows by Lemma 2.2, that all zeros of

$$\begin{aligned} H(z) &:= B[F(Rz) - \alpha F(rz)] \\ &= \{B[P(Rz)] - \alpha B[P(rz)]\} - \beta\{B[Q(Rz)] - \alpha B[Q(rz)]\} \end{aligned}$$

lie in $|z| < 1$. This gives, for $|z| \geq 1$,

$$|B[P(Rz)] - \alpha B[P(rz)]| \leq |B[Q(Rz)] - \alpha B[Q(rz)]| \,.$$

For, if this is not true, then there exists a point $z = z_0$ with $|z_0| \geq 1$, such that

$$|B[P(Rz)] - \alpha B[P(rz)]|_{z=z_0} > |B[Q(Rz)] - \alpha B[Q(rz)]|_{z=z_0} \,.$$

We take

$$\beta = \frac{\{B[P(Rz)] - \alpha B[P(rz)]\}_{z=z_0}}{\{B[Q(Rz)] - \alpha B[Q(rz)]\}_{z=z_0}},$$

so that $|\beta| > 1$. With this value of z, $H(z) = 0$, for $|z| \geq 1$. This is a contradiction to the fact that all the zeros of $H(z)$ lie in $|z| < 1$. Hence the proof of lemma is complete. □

Lemma 2.4. *If* $P \in \mathcal{P}_n$*, then for every* α *with* $|\alpha| \leq 1$*,* $R > r \geq 1$ *and* $|z| \geq 1$*,*

$$|B[P(R\,\cdot)] - \alpha B[P(r\,\cdot)]| \leq |B[E_n(R\,\cdot) - \alpha B[E_n(r\,\cdot)]|\,||P||_\infty. \tag{9}$$

Proof. Let $M = \max_{|z|=1}|P(z)|$, then $|P(z)| \leq M$ for $|z| = 1$. By Rouches theorem, it follows that all the zeros of polynomial $F(z) = P(z) - \zeta z^n M$ lie in $|z| < 1$ for every real or complex number ζ with $|\zeta| > 1$. Therefore, it follows from Lemma 2.1, that for $R > r \geq 1$, and $|z| = 1$,

$$|F(rz)| < |F(Rz)|.$$

Since all the zeros of polynomial $F(Rz)$ lie in $|z| \leq \frac{1}{R} < 1$, again making use of Rouches theorem, we conclude that all the zeros of the polynomial $F(Rz) - \alpha F(rz)$ lie in $|z| < 1$, for every real or complex number α with $|\alpha| \leq 1$. By Lemma 2.2, the polynomial

$$B[F(Rz) - \alpha F(rz)] = (B[P(Rz)] - \alpha B[P(rz)]) - \zeta(R^n - \alpha r^n)B[z^n]M, \tag{10}$$

has all the zeros in open unit disc for every real or complex number ζ with $|\zeta| > 1$. This implies similarly, as in the case of Lemma 2.3, for $|z| \geq 1$ and $R > r \geq 1$,

$$|B[P(Rz)] - \alpha B[P(rz)]| \leq |R^n - \alpha r^n||B[z^n]|M. \tag{11}$$

This gives the desired result. □

Lemma 2.5. *Let $\mathcal{P}_n$ denote the linear space of polynomials*

$$P(z) = a_0 + \cdots + a_n z^n$$

of degree n with complex coefficients, normed by $\|P\| = \max|P(e^{i\theta})|$, $0 < \theta \leq 2\pi$. Define the linear functional $\mathcal{L}$ on $\mathcal{P}_n$ as

$$\mathcal{L}\colon P \to l_0 a_0 + l_1 a_1 + \cdots + l_n a_n,$$

where l_j's are complex numbers. If the norm of the functional is $\mathcal{N}$, then

$$\int_0^{2\pi} \Theta\left(\frac{\left|\sum_{k=0}^n l_k a_k e^{ik\theta}\right|}{\mathcal{N}}\right) d\theta \leq \int_0^{2\pi} \Theta\left(\left|\sum_{k=0}^n a_k e^{ik\theta}\right|\right) d\theta, \tag{12}$$

where $\Theta(t)$ is a non-decreasing convex function of t.

The above lemma is due to Rahman [6].

In this paper, we prove some results which generalize the above theorems and there by obtain compact generalizations of many polynomial inequalities as well. In fact, we prove:

Theorem 2.6. *If $P \in \mathcal{P}_n$, then for every α, with $|\alpha| \leq 1$, $R > r \geq 1$, $q \geq 1$ and $|z| = 1$,*

$$||B[P(R\,\cdot)] - \alpha B[P(r\,\cdot)]||_q \leq |B[E_n(R\,\cdot) - \alpha B[E_n(r\,\cdot)]|\,||P||_q \tag{13}$$

The result is best possible and equality holds for $P(z) = az^n$, $a \neq 0$.

Proof. Let $M = \max_{|z|=1}|P(z)|$, then by Lemma 2.4, we have, for $|z| \geq 1$ and $R > r \geq 1$,

$$|B[P(Rz)] - \alpha B[P(rz)]| \leq |R^n - \alpha r^n||B[z^n]|M. \tag{14}$$

This in particular gives for every θ, $0 \leq \theta < 2\pi$ and $R > r \geq 1$,

$$|B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]| \leq |R^n - \alpha r^n|\left|\lambda_0 + \frac{n^2}{2}\lambda_1 + \frac{n^3(n-1)}{8}\lambda_2\right|M. \tag{15}$$

Since B is a linear operator (see [6, sec. 5]), therefore

$$\Lambda = B[P(Rz)] - \alpha B[P(rz)]$$

is a bounded linear operator on $\mathcal{P}_n$. Thus in view of (15), the norm of the bounded linear functional

$$\mathcal{L}\colon P \to \{B[P(Rz)] - \alpha B[P(rz)]\}_{\theta=0}$$

is

$$|R^n - \alpha r^n| \left| \lambda_0 + \frac{n^2}{2}\lambda_1 + \frac{n^3(n-1)}{8}\lambda_2 \right|.$$

Hence by Lemma 2.5, for every $q \geq 1$, we have,

$$\int_0^{2\pi} |B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]|^q d\theta$$
$$\leq \left| (R^n - \alpha r^n) \left(\lambda_0 + \frac{n^2}{2}\lambda_1 + \frac{n^3(n-1)}{8}\lambda_2 \right) \right|^q \int_0^{2\pi} \left| P(e^{i\theta}) \right|^q d\theta.$$

From this inequality, (13) follows immediately and this completes the proof of Theorem 2.6. □

Remark 2.7. For $\alpha = 0$, Theorem 2.6 reduces to Theorem 1.3.

The following corollary immediately follows from Theorem 2.6, when we let $q \to \infty$.

Corollary 2.8. *If $P \in \mathcal{P}_n$, then for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| = 1$,*

$$||B[P(R\ \cdot)] - \alpha B[P(r\ \cdot)]||_\infty \leq |B[E_n(R\ \cdot) - \alpha B[E_n(r\ \cdot)]|\ ||P||_\infty.$$

Or, equivalently,

$$(16)\quad ||B[P(R\ \cdot)] - \alpha B[P(r\ \cdot)]||_\infty$$
$$\leq |R^n - \alpha r^n| \left| \lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8} \right| ||P||_\infty.$$

The result is best possible and equality holds for $P(z) = az^n$, $a \neq 0$.

Remark 2.9. Theorem 1.1 is a special case of Corollary 2.8, when we take $\alpha = 0$.

Also, If we choose $\alpha = 0$ and $\lambda_0 = 0 = \lambda_2$ in (16), which is possible, as it can be easily verified that in this case all the zeros of $\mathcal{U}(z)$ defined by (2) lie in (3), we get,

Corollary 2.10. *If $P \in \mathcal{P}_n$, then for every $R \geq 1$, $q \geq 1$ and $|z| = 1$,*

$$||P'||_q \leq nR^{n-1}||P||_q.$$

This in particular for $R = 1$, gives,

$$\|P'\|_q \leq n\|P\|_q, \ \textit{for } q \geq 1.$$

which is an inequality due to Zygmund [11].

Lemma 2.11. *If $P \in \mathcal{P}_n$, then for every α with $|\alpha| \le 1$, $R > r \ge 1$, $q \ge 1$ and $0 \le \theta, \beta < 2\pi$,*

$$\int_0^{2\pi}\int_0^{2\pi} \big|\big(B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]\big) + e^{in\beta}\big(B[Q(Re^{i\theta})] - \alpha B[Q(re^{i\theta})]\big)\big|^q \, d\theta d\beta \le 2\pi\left[|B[E_n(R\ \cdot) - \alpha B[E_n(r\ \cdot)]| + |1-\alpha||\lambda_0|\right]^q \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta. \tag{17}$$

Proof. Let $M = \max_{|z|=1} |P(z)|$, then $|P(z)| \le M$ for $|z| = 1$. By Rouches theorem, it follows that all the zeros of polynomial $F(z) = P(z) - \zeta M$ lie in $|z| \ge 1$, for every real or complex number ζ with $|\zeta| > 1$. Applying Lemma 2.3 to the polynomial $F(z)$ and using the fact that B is a linear operator, it follows that for every real or complex number α with $|\alpha| \le 1$, $R > r \ge 1$,

$$|B[F(Rz)] - \alpha B[F(rz)]| \le |B[G(Rz)] - \alpha B[G(rz)]| \tag{18}$$

for $|z| \ge 1$, where

$$G(z) = z^n\overline{F(1/\bar{z})} = Q(z) - z^n\bar{\zeta}M.$$

Using the fact that$B[1] = \lambda_0$, we get from (18),

$$|B[P(Rz)] - \alpha B[P(rz)] - \zeta(1-\alpha)\lambda_0 M| \le |B[Q(Rz)] - \alpha B[Q(rz)] - \bar{\zeta}(R^n - \alpha r^n)B[z^n]M|. \tag{19}$$

Now choosing argument of ζ such that

$$|B[Q(Rz)] - \alpha B[Q(rz)] - \bar{\zeta}(R^n - \alpha r^n)B[z^n]M| = |\zeta||(R^n - \alpha r^n)||B[z^n]|M - |B[Q(Rz)] - \alpha B[Q(rz)]|,$$

which is possible by (9), we get from (19), for $|\zeta| > 1$ and $|z| \ge 1$,

$$|B[P(Rz)] - \alpha B[P(rz)]| + |B[Q(Rz)] - \alpha B[Q(rz)]| \le |\zeta|(|R^n - \alpha r^n||B[z^n]| + |1-\alpha||\lambda_0|) \max_{|z|=1} |P(z)|.$$

Letting $|\zeta| \to 1$, we obtain

$$|B[P(Rz)] - \alpha B[P(rz)]| + |B[Q(Rz)] - \alpha B[Q(rz)]| \le (|R^n - \alpha r^n||B[z^n]| + |1-\alpha||\lambda_0|) \max_{|z|=1} |P(z)|.$$

This in particular gives for every θ, $0 \le \theta < 2\pi$ and $R > r \ge 1$,

$$|B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]| + |B[Q(Re^{i\theta})] - \alpha B[Q(re^{i\theta})]| \le (|R^n - \alpha r^n||B[e^{in\theta}]| + |1-\alpha||\lambda_0|) \max_{|z|=1} |P(z)|.$$

Thus for every β with $0 \le \beta < 2\pi$, we have

$$|(B[P(Re^{i\theta})]-\alpha B[P(re^{i\theta})])+e^{i\beta}(B[Q(Re^{i\theta})]-\alpha B[Q(re^{i\theta})])|$$
$$\leq\left(|R^n-\alpha r^n|\left|\lambda_0+\frac{n^2}{2}\lambda_1+\frac{n^3(n-1)}{8}\lambda_2\right|+|1-\alpha||\lambda_0|\right)\max_{|z|=1}|P(z)|.$$

This shows that

$$\Lambda:=\left(B[P(Re^{i\theta})]-\alpha B[P(re^{i\theta})]\right)+e^{i\beta}\left(B[Q(Re^{i\theta})]-\alpha B[Q(re^{i\theta})]\right)$$

is a bounded linear operator on $\mathcal{P}_n$. Thus in view of (10), the norm of the bounded linear functional

$$\mathcal{L}\colon P\to\left\{\left(B[P(Re^{i\theta})]-\alpha B[P(re^{i\theta})]\right)+e^{i\beta}\left(B[Q(Re^{i\theta})]-\alpha B[Q(re^{i\theta})]\right)\right\}_{\theta=0}$$

is

$$|R^n-\alpha r^n|\left|\lambda_0+\frac{n^2}{2}\lambda_1+\frac{n^3(n-1)}{8}\lambda_2\right|+|1-\alpha||\lambda_0|.$$

Therefore, by Lemma 2.5, it follows that

$$\int_0^{2\pi}\left|\left(B[P(Re^{i\theta})]-\alpha B[P(re^{i\theta})]\right)+e^{i\beta}\left(B[Q(Re^{i\theta})]-\alpha B[Q(re^{i\theta})]\right)\right|^q d\theta \tag{20}$$
$$\leq\left[|R^n-\alpha r^n|\left|\lambda_0+\frac{n^2}{2}\lambda_1+\frac{n^3(n-1)}{8}\lambda_2\right|+|1-\alpha||\lambda_0|\right]^q\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta.$$

Integrating the two sides of (20) with respect to β, we get,

$$\int_0^{2\pi}\int_0^{2\pi}\left|(B[P(Re^{i\theta})]-\alpha B[P(re^{i\theta})])+e^{i\beta}(B[Q(Re^{i\theta})]-\alpha B[Q(re^{i\theta})])\right|^q d\beta d\theta$$
$$\leq\int_0^{2\pi}\left[|R^n-\alpha r^n|\left|\lambda_0+\frac{n^2}{2}\lambda_1+\frac{n^3(n-1)}{8}\lambda_2\right|+|1-\alpha||\lambda_0|\right]^q d\beta\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta$$
$$=2\pi\left[|R^n-\alpha r^n|\left|\lambda_0+\frac{n^2}{2}\lambda_1+\frac{n^3(n-1)}{8}\lambda_2\right|+|1-\alpha||\lambda_0|\right]^q\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta.$$

From this the desired result follows. □

Next we prove:

Theorem 2.12. *If $P\in\mathcal{P}_n$ and $P(z)\neq 0$ in $|z|<1$, then for every real or complex number α with $|\alpha|\leq 1$, $q\geq 1$, $R>r\geq 1$ and $|z|=1$,*

$$\|B[P(R\,\cdot)]-\alpha B[P(r\,\cdot)]\|_q\leq\frac{|B[E_n(R\,\cdot)-\alpha B[E_n(r\,\cdot)]|+|1-\alpha||\lambda_0|}{\|1+E_n\|_q}\|P\|_q.$$

Or, equivalently,

$$\|B[P(R\,\cdot)]-\alpha B[P(r\,\cdot)]\|_q \tag{21}$$
$$\leq\frac{|R^n-\alpha r^n|\left|\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}\right|+|1-\alpha||\lambda_0|}{\|1+E_n\|_q}\|P\|_q,$$

where $B\in\mathcal{B}_n$. The result is sharp and equality holds for a polynomial $P(z)=az^n+b, |a|=|b|$.

Proof. Since $P(z) \neq 0$ in $|z| < 1$, by Lemma 2.3, we have for each θ, $0 \leq \theta < 2\pi$ and $R > r \geq 1$,

$$\left|B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]\right| \leq \left|B[Q(Re^{i\theta})] - \alpha B[Q(re^{i\theta})]\right|.$$

Also for every real θ and $t \geq 1$, it can be easily verified that $|1+te^{i\theta}| \geq |1+e^{i\theta}|$ and therefore for every $q \geq 1$,

$$\int_0^{2\pi} |1 + te^{i\theta}|^q d\theta \geq \int_0^{2\pi} |1 + e^{i\theta}|^q d\theta. \tag{22}$$

Now, taking $t = \frac{\left|B[Q(Re^{i\theta})] - \alpha B[Q(re^{i\theta})]\right|}{\left|B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]\right|} \geq 1$ and using inequality (22), we have

$$\begin{aligned}
\int_0^{2\pi}\int_0^{2\pi} \quad & |(B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]) \\
& + e^{in\beta}(B[Q(Re^{i\theta})] - \alpha B[Q(re^{i\theta})])|^q d\beta d\theta \\
&= \int_0^{2\pi}\int_0^{2\pi} |B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]|^q \times \\
&\quad \times \left|1 + e^{in\beta}\frac{B[Q(Re^{i\theta})] - \alpha B[Q(re^{i\theta})]}{B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]}\right|^q d\beta d\theta \\
&= \int_0^{2\pi} \{\left|B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]\right|^q \times \\
&\quad \times \int_0^{2\pi} \left|1 + e^{in\beta}\left|\frac{B[Q(Re^{i\theta})] - \alpha B[Q(re^{i\theta})]}{B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]}\right|\right|^q d\beta\Bigg\} d\theta \\
&\geq \int_0^{2\pi} \left\{|B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]|^q \int_0^{2\pi} \left|1 + e^{in\beta}\right|^q d\beta\right\} d\theta \\
&= \int_0^{2\pi} |B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]|^q d\theta \int_0^{2\pi} \left|1 + e^{in\beta}\right|^q d\beta.
\end{aligned} \tag{23}$$

Inequality (23) in conjunction with Lemma 2.11, gives

$$\begin{aligned}
&\int_0^{2\pi} \left|B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]\right|^q d\theta \\
&\leq \frac{2\pi\left[|R^n - \alpha r^n|\left|\lambda_0 + \frac{n^2}{2}\lambda_1 + \frac{n^3(n-1)}{8}\lambda_2\right| + |1-\alpha||\lambda_0|\right]^q}{\int_0^{2\pi}|1 + e^{in\beta}|^q d\beta} \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta.
\end{aligned}$$

Or, equivalently,

$$\|B[P(R\,\cdot)] - \alpha B[P(r\,\cdot)]\|_q \leq \frac{|B[E_n(R\,\cdot) - \alpha B[E_n(r\,\cdot)]| + |1-\alpha||\lambda_0|}{\|1 + E_n\|_q}\|P\|_q.$$

This completes the proof of Theorem 2.12. □

Remark 2.13. If we choose $\alpha = 0$ in (21), we obtain Theorem 1.4. Also Theorem 1.2 easily follows from Theorem 2.12, if we make $\alpha = 0$ and $q \to \infty$.

Further, if we choose $\alpha = 0$ and $\lambda_0 = 0 = \lambda_2$, $R = 1$ in (21) which is possible, we get the following inequality:

$$\|P'\|_q \leq \frac{n}{\|1+z^n\|_q}\|P\|_q,$$

for every $q \geq 1$, which is a result due to de Brujin [3]. On the other hand, for $\alpha = 0$ and $\lambda_1 = \lambda_2 = 0$, we have the following:

If $P \in \mathcal{P}_n$ be such that $P(z) \neq 0$ in $|z| < 1$, then for every $R > 1$, $q \geq 1$ and $|z| = 1$,

$$\|P(R\,\cdot)\|_q \leq \frac{R^n+1}{\|1+z^n\|_q}\|P\|_q.$$

An inequality proved by Ankeny and Rivlin [1] is a special case of this inequality when we let $q \to \infty$.

Also for $q \to \infty$, Theorem 2.12 yields the following:

Corollary 2.14. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| = 1$*

$$||B[P(R\,\cdot)] - \alpha B[P(r\,\cdot)]||_\infty \leq [\frac{|B[E_n(R\,\cdot) - \alpha B[E_n(r\,\cdot)]| + |1-\alpha||\lambda_0|}{2}]||P||_\infty,$$

where $B \in \mathcal{B}_n$. The result is sharp and equality holds for a polynomial $P(z) = az^n + b, |a| = |b|$.

If we choose $r = 1, \lambda_1 = 0 = \lambda_2$ in (21), we get the following:

Corollary 2.15. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number α with $|\alpha| \leq 1$, $R > 1$ and $|z| = 1$,*

$$||P(R\,\cdot) - \alpha P||_q \leq \frac{|R^n - \alpha| + |1-\alpha|}{\|1+z^n\|_q} \quad \|P\|_q. \tag{24}$$

This is a compact generalization of a result of Shah and Liman [10, Corollary 1].

A polynomial $P(z)$ is said to be self-inversive if $P(z) = uQ(z)$, $|u| = 1$, where $Q(z) = z^n\overline{P(1/\bar{z})}$. It is known [4] that, if $P \in \mathcal{P}_n$ is a self inversive polynomial, then for every $q \geq 1$,

$$\|P'\|_q \leq \frac{n}{\|1+z^n\|_q}\|P\|_q.$$

We next present the following result for the class of self inversive polynomials:

Theorem 2.16. *If $P \in \mathcal{P}_n$ is self inversive, then for every $q \geq 1$, $R > r \geq 1$ and $|z| = 1$,*

$$\|B[P(R\,\cdot)] - \alpha B[P(r\,\cdot)]\|_q \leq \frac{|B[E_n(R\,\cdot) - \alpha B[E_n(r\,\cdot)]| + |1-\alpha||\lambda_0|}{\|1+E_n\|_q}\|P\|_q. \tag{25}$$

The result is sharp and equality holds for $P(z) = z^n + 1$.

Proof. Since $P(z)$ is a self inversive polynomial, therefore for all $z \in C, |z| \geq 1$, we have

$$|B[P(Rz)] - \alpha B[P(rz)]| = |B[Q(Rz)] - \alpha B[Q(rz)]| .$$

This in particular gives, for $0 \leq \theta < 2\pi$,

$$\left|B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]\right| = \left|B[Q(Re^{i\theta})] - \alpha B[Q(re^{i\theta})]\right| . \tag{26}$$

Proceeding similarly as in the case of Theorem 2.12, we get

$$\int_0^{2\pi} \left|B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]\right|^q d\theta$$
$$\leq \frac{2\pi \left[|R^n - \alpha r^n| \left|\lambda_0 + \frac{n^2}{2}\lambda_1 + \frac{n^3(n-1)}{8}\lambda_2\right| + |1-\alpha||\lambda_0|\right]^q}{\int_0^{2\pi} |1 + e^{in\beta}|^q \, d\beta} \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta.$$

Or, equivalently,

$$\|B[P(R\,\cdot)] - \alpha B[P(r\,\cdot)]\|_q \leq \frac{|B[E_n(R\,\cdot) - \alpha B[E_n(r\,\cdot)]| + |1-\alpha||\lambda_0|}{\|1 + E_n\|_q} \|P\|_q.$$

Hence the result is proved. □

The above inequality of Dewan and Govil [4] and many such results follow as special cases from Theorem 2.16.

Further, if we make $q \to \infty$ in inequality (25), we get the following:

Corollary 2.17. *If* $P \in \mathcal{P}_n$ *is self inversive, then for every* $R > r \geq 1$, *and* $|z| = 1$,

$$||B[P(R\,\cdot)] - \alpha B[P(r\,\cdot)]||_\infty \leq [\frac{|B[E_n(R\,\cdot) - \alpha B[E_n(r\,\cdot)]| + |1-\alpha||\lambda_0|}{2}]||P||_\infty,$$

where $B \in \mathcal{B}_n$. *The result is sharp and equality holds for a polynomial* $P(z) = z^n + 1$.

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