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RICCI SOLITONS IN LORENTZIAN α-SASAKIAN MANIFOLDS

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Abstract. We study Ricci solitons in Lorentzian α-Sasakian manifolds. It is shown that a symmetric parallel second order covariant tensor in a Lorentzian α-Sasakian manifold is a constant multiple of the metric tensor. Using this it is shown that if $\mathcal{L}_V g + 2S$ is parallel, V is a given vector field then (g, V) is Ricci soliton. Further, by virtue of this result Ricci solitons for $(2n+1)$-dimensional Lorentzian α-Sasakian manifolds are obtained. Next, Ricci solitons for 3-dimensional Lorentzian α-Sasakian manifold whose scalar curvature is constant are obtained.

1. Introduction

Ricci flow is an excellent tool for simplifying the structure of a manifold and smooth out the topology of that manifold to make it look more symmetric. It is defined for Riemannian manifolds of any dimension. It is a process which deforms the metric of a Riemannian manifold analogous to the diffusion of heat there by smoothing out the regularity in the metric. It is given by

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}\,g.$$

For example, if $ds^2 = e^{2p(x,y)}(dx^2 + dy^2)$, then to compute the Ricci tensor and Laplace-Beltrami operator for two dimensional Riemannian manifold we use the differential forms method of Elie Cartan. We obtain an expression for the Ricci flow:

$$\frac{\partial p}{\partial t} = \triangle p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}.$$

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This is manifestly analogous to the best known of all diffusion equations, the heat equation that is,

$$\frac{\partial T}{\partial t} = \triangle T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2},$$

where now $\triangle = D_x^2 + D_y^2$ is the usual Laplacian on the Euclidean plane.

Let $X(t)$ be a time dependent family of smooth vector fields on M generated by a family of diffeomorphisms $\{\phi_t : t \in R\}$ that is one parameter group of transformations, then the relation between $f\colon M \to R$ and $\{\phi_t : t \in R\}$ is

$$X(\phi_t(p))f = \frac{df \circ \phi_t}{dt}(p).$$

Let $\sigma(t)$ be a smooth function of time. Since $\phi_t\colon M \to M$ is a diffeomorphism and $g(t)$ is a Riemannian metric on M (codomain) then by definition of pull back $\phi_t^* g(t)$ is a metric on M (domain).

Set $\tilde{g}(t) = \sigma(t)\phi_t^*(g(t))$ then we have [21]

$$\frac{\partial \tilde{g}}{\partial t} = \sigma'(t)\phi_t^*(g(t)) + \sigma(t)\phi_t^* \frac{\partial g}{\partial t} + \sigma(t)\phi_t^*(L_X g). \tag{1.1}$$

Suppose we have a metric g_0, a vector field Y and $\lambda \in R$ (all independent of time) such that

$$\mathcal{L}_Y g_0 + 2\,\mathrm{Ric}\, g_0 + 2\lambda g_0 = 0. \tag{1.2}$$

If we choose $g(t) = g_0$, $\sigma(t) = 1 - 2\lambda t$ and $X(t) = \frac{1}{\sigma(t)} Y$ which gives a family of diffeomorphisms ϕ_t with ϕ_0 identity then using (1.2) in (1.1) $\tilde{g}$ defined above is a Ricci flow with $g(0) = g_0$ that is

$$\frac{\partial \tilde{g}}{\partial t} = -2\,\mathrm{Ric}\,\tilde{g}. \tag{1.3}$$

Hence $\mathcal{L}_X g_0 + 2\,\mathrm{Ric}\, g_0 + 2\lambda g_0 = 0$ is a solution of the Ricci flow and is known as Ricci soliton.

Hereafter, we use the notation S instead of Ric for Ricci tensor.

Thus a Ricci soliton on a Riemannian manifold is defined by

$$\mathcal{L}_X g + 2S + 2\lambda g = 0. \tag{1.4}$$

It is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.

An η-Ricci soliton introduced in the paper [3] as a data (g, V, λ, μ) :

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0. \tag{1.5}$$

1.1. **Example (Hamilton Cigar Soliton).** Let $M = R^2$ and $\phi_t\colon R^2 \to R^2$ defined by $\phi_t(x, y) = (e^{-2t}x, e^{-2t}y)$ forms a family of one parameter group of diffeomorphisms. The vector field X generated by $\{\phi_t\}$ is $X = -2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$. The metric g_0 is obtained as $g_0 = \frac{dx^2+dy^2}{1+x^2+y^2}$, $\tilde{g}(t) = \phi_t^*(g_0) = \frac{dx^2+dy^2}{e^{4t}+x^2+y^2}$, $\mathrm{Ric}\, g_0 = \frac{2}{1+x^2+y^2} g_0$, $\mathcal{L}_X g_0 = \frac{4}{1+x^2+y^2} g_0$. Using (1.4) we have $\lambda = 0$. Hence this Ricci soliton is steady and is called cigar soliton because it is a asymptotic to a flat cylinder at infinity.

In 1923, Eisenhart [7] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. In 1925, Levy [12] has obtained the necessary and sufficient conditions for the existence of such tensors. Recently Sharma [9] and [19] has generalized Levy's result by showing that a second order parallel(not necessarily symmetric and non singular) tensor on an n-dimensional $(n > 2)$ space of constant curvature is a constant multiple of the metric tensor. Sharma has also proved in [16] that on a Sasakian manifold there is no nonzero parallel 2-form. In 1964, Y. Wong [23] proved that the existence of linear connections w.r.t which given tensor fields are parallel or recurrent. Also the parallelism of h is involved and appears in his paper as the theory of totally geodesic maps, and $\nabla h = 0$ is equivalent with the fact that $I\colon (M, g) \to (M, h)$ is a totally geodesic map. In 2007, Lovejoy Das [5] in his paper proved that a second order symmetric parallel tensor on an α-K-contact $(\alpha \in R_0)$ manifold is a constant multiple of the associated metric tensor and he also proved that there is no nonzero skew symmetric second order parallel tensor on an α-Sasakian manifold.

Constantin Calin and Mircea Crasmareanu [2] have extended the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds. They have studied the case of f-Kenmotsu manifolds satisfying a special condition called regular and show that a symmetric parallel tensor field of second order is a constant multiple of the Riemannian metric. Using this result they have obtained results on Ricci solitons concerned to f-Kenmotsu manifolds and 3-dimensional β-Kenmotsu manifolds.

2. Basic concepts of Lorentzian α-Sasakian manifolds

A differentiable manifold of dimension $(2n + 1)$ is called Lorentzian α-Sasakian manifold if it admits a $(1, 1)$ tensor field ϕ, a vector field ξ and 1-form η and Lorentzian metric g which satisfy on M respectively such that,

$$\phi^2 = I + \eta \otimes \xi,\ \eta(\xi) = -1,\ \eta \circ \phi = 0,\ \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),\ g(X, \xi) = \eta(X), \tag{2.2}$$

$$\nabla_X \xi = \alpha\phi X,\ (\nabla_X \eta)Y = \alpha g(\phi X, Y), \tag{2.3}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g on M.

Further, on an Lorentzian α-Sasakian manifold M the following relations hold:

$$R(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{2.4}$$

$$R(\xi, X)Y = \alpha^2[g(X, Y)\xi - \eta(Y)X], \tag{2.5}$$

$$S(X, \xi) = 2n\alpha^2\eta(X), \tag{2.6}$$

$$Q\xi = 2n\alpha^2\xi, \tag{2.7}$$

$$S(\xi,\xi) = -2n\alpha^2, \tag{2.8}$$

where α is some constant, R is the Riemannian curvature, S is the Ricci curvature and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

2.1. **Example.** We consider the 3-dimensional manifold $M = \{(x,y,z) \in R^3\}$, where (x,y,z) are the standard co-ordinates in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame field on M given by

$$E_1 = e^z\frac{\partial}{\partial y},\ E_2 = e^z(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}),\ E_3 = k\frac{\partial}{\partial z}. \tag{2.9}$$

Let g be the Riemannian metric defined by

$$g(E_1,E_2) = g(E_2,E_3) = g(E_1,E_3) = 0,$$
$$g(E_1,E_1) = g(E_2,E_2) = 1, g(E_3,E_3) = -1,$$

where g is given by

$$g = \frac{1}{e^{2z}}[dx \otimes dx + dy \otimes dy] - \frac{1}{k^2}dz \otimes dz.$$

The (ϕ,ξ,η) is given by

$$\eta = \frac{1}{k}dz,\ \xi = E_3 = k\frac{\partial}{\partial z}, \phi E_1 = -E_1,\ \phi E_2 = -E_2,\ \phi E_3 = 0.$$

The linearity property of ϕ and g yields that

$$\eta(E_3) = -1,\ \phi^2 U = U + \eta(U)E_3,\ g(\phi U, \phi W) = g(U,W) + \eta(U)\eta(W),$$

for any vector fields U, W on M. By definition of Lie bracket, we have

$$[E_1,E_2] = 0, \quad [E_1,E_3] = -kE_1, \quad [E_2,E_3] = -kE_2.$$

Let ∇ be Levi-Civita connection with respect to the above metric g given by Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned} \tag{2.10}$$

Then

$$\begin{aligned} &\nabla_{E_1}E_1 = -kE_3,\ \nabla_{E_1}E_2 = 0,\ \nabla_{E_1}E_3 = -kE_1, \\ &\nabla_{E_2}E_1 = 0,\ \nabla_{E_2}E_2 = -kE_3,\ \nabla_{E_2}E_3 = -kE_2, \\ &\nabla_{E_3}E_1 = 0,\ \nabla_{E_3}E_2 = 0,\ \nabla_{E_3}E_3 = 0. \end{aligned} \tag{2.11}$$

The tangent vectors X and Y to M are expressed as linear combination of E_1, E_2, E_3, that is $X = \sum_{i=1}^{3} a_i E_i$ and $Y = \sum_{i=1}^{3} b_i E_i$ where $a_i, b_i (i = 1,2,3)$ are scalars. Clearly (ϕ,ξ,η,g) and X, Y satisfy equations (2.1), (2.2) and (2.3) with $\alpha = k$. Thus M is a Lorentzian α-Sasakian manifold.

Definition 1. Let M be a Riemannian manifold with metric g, ξ an unitary vector field, η the 1-form dual to ξ. Further, let h a symmetric tensor field of $(0,2)$-type on M which we suppose to be parallel with respect to ∇ that is $\nabla h = 0$. Applying the Ricci identity [16]

$$\nabla^2 h(X,Y;Z,W) - \nabla^2 h(X,Y;W,Z) = 0, \tag{2.12}$$

we obtain the relation [16]:

$$h(R(X,Y)Z,W) + h(Z,R(X,Y)W) = 0, \tag{2.13}$$

then by taking $Z = W = \xi$ in (2.13) it reduces to

$$A[\eta(Y)h(X,\xi) - \eta(X)h(Y,\xi)] = 0, \tag{2.14}$$

where $A \neq 0$ is some scalar function then M is called regular (that is $M_A^{(2n+1)}(\xi)$ is called regular if $A \neq 0$).

3. Parallel symmetric second order tensors and Ricci solitons in Lorentzian α-Sasakian manifolds

Fix h a symmetric tensor field of $(0,2)$-type which we suppose to be parallel with respect to ∇ that is $\nabla h = 0$. Applying the Ricci identity [16] in (2.12) we obtain (2.13). Replacing $Z = W = \xi$ in (2.13) and using (2.4) and by the symmetry of h, we have

$$2\alpha^2[\eta(Y)h(X,\xi) - \eta(X)h(Y,\xi)] = 0. \tag{3.1}$$

Put $X = \xi$ in (3.1), we have

$$2\alpha^2[\eta(Y)h(\xi,\xi) + h(Y,\xi)] = 0. \tag{3.2}$$

Since $2\alpha^2 \neq 0$, by definition (1) Lorentzian α-Sasakian manifold is regular.

By (3.2), we have

$$h(Y,\xi) = -\eta(Y)h(\xi,\xi). \tag{3.3}$$

Differentiating (3.3) covariantly with respect to X, we have

$$\begin{aligned}(\nabla_X h)(Y,\xi) + h(\nabla_X Y,\xi) + h(Y,\nabla_X \xi) = \\ -[(\nabla_X \eta)(Y) + \eta(\nabla_X Y)]h(\xi,\xi) \\ -\eta(Y)[(\nabla_X h)(\xi,\xi) + 2h(\nabla_X \xi,\xi)].\end{aligned} \tag{3.4}$$

By using (2.2), (2.3) and (3.3), we have

$$-h(Y,\phi X) = g(Y,\phi X)h(\xi,\xi), \tag{3.5}$$

we deduce the above equation then we have

$$h(X,Y) = -g(X,Y)h(\xi,\xi), \tag{3.6}$$

which together with the standard fact that the parallelism of h implies the $h(\xi,\xi)$ is a constant and via (3.3) yields the following:

Theorem 3.1. *A symmetric parallel second order covariant tensor in a regular Lorentzian α-Sasakian manifolds is a constant multiple of the metric tensor.*

Corollary 1. *A locally Ricci symmetric* ($\nabla S = 0$) *regular Lorentzian α-Sasakian manifolds is an Einstein manifold.*

Remark: The following statements for Lorentzian α-Sasakian manifolds are equivalent. The manifold is

(i) Einstein
(ii) locally Ricci symmetric
(iii) Ricci semi-symmetric that is $R \cdot S = 0$.

The implication (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is trivial. Now we prove the implication (iii) $\Longrightarrow$ (i) and $R \cdot S = 0$ means exactly (2.13) with replaced h by S that is

$$(R(X,Y)\cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V). \tag{3.7}$$

Considering $R \cdot S = 0$ and putting $X = \xi$ in equation (3.7), we have

$$S(R(\xi,Y)U,V) + S(U,R(\xi,Y)V) = 0. \tag{3.8}$$

By using (2.5) and (2.6), we obtain

$$2n\alpha^4 g(Y,U)\eta(V) - \alpha^2\eta(U)S(Y,V) + 2n\alpha^4 g(Y,V)\eta(U) - \alpha^2\eta(V)S(U,Y) = 0. \tag{3.9}$$

Again by putting $U = \xi$ in the above equation and by using (2.1), (2.2) and (2.6), we obtain

$$S(Y,V) = 2n\alpha^2 g(Y,V). \tag{3.10}$$

In conclusion:

Proposition 1. *A Ricci semi-symmetric regular Lorentzian α-Sasakian manifolds is Einstein.*

We close this section with applications of our Theorem to Ricci solitons:

Corollary 2. *Suppose that on a regular Lorentzian α-Sasakian manifolds the $(0,2)$-type field $\mathcal{L}_V g + 2S$ is parallel where V is a given vector field. Then (g,V) yield a Ricci soliton. In particular, if the given regular Lorentzian α-Sasakian manifold is Ricci-semi symmetric with $\mathcal{L}_V g$ parallel, we have the same conclusion.*

Proof. Follows from Theorem 3.1 and Corollary 1. □

Naturally, two situations appear regarding the vector field V: $V \in Span\,\xi$ and $V \perp \xi$ but the second class seems far too complex to analyse in practice. For this reason it is appropriate to investigate only the case $V = \xi$.

We are interested in expressions for $\mathcal{L}_\xi g + 2S$. A straightforward computation gives

$$(\mathcal{L}_\xi g)(X,Y) = 2\alpha g(\phi X,Y). \tag{3.11}$$

The metric g is called η-Einstein if there exists two real functions a and b such that the Ricci tensor of g is

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y). \tag{3.12}$$

Let $e_i = 1, 2, \ldots, (2n+1)$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i$ in (3.12) and taking summation over i then we get

$$r = (2n+1)a - b. \tag{3.13}$$

Again putting $X = Y = \xi$ in (3.12) then by using (2.1), (2.2) and (2.8), we have

$$-a + b = -2n\alpha^2, \tag{3.14}$$

from (3.13) and (3.14), we obtain the values of a and b

$$a = \frac{r}{2n} - \alpha^2,\ b = \frac{r}{2n} - (2n+1)\alpha^2.$$

Substituting the values of a and b in (3.12), we have

$$S(X,Y) = \left[\frac{r}{2n} - \alpha^2\right] g(X,Y) + \left[\frac{r}{2n} - (2n+1)\alpha^2\right] \eta(X)\eta(Y). \tag{3.15}$$

The above equation shows that Lorentzian α-Sasakian manifold is η-Einstein.

For $(2n+1)$-dimensional Lorentzian α-Sasakian manifolds, we have

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y). \tag{3.16}$$

Then in (3.16) substituting the values of (3.11) and (3.15), we have

$$\begin{aligned} h(X,Y) = 2\alpha g(\phi X,Y) &+ \left[\frac{r}{n} - 2\alpha^2\right] g(X,Y) \\ &+ \left[\frac{r}{n} - 2(2n+1)\alpha^2\right] \eta(X)\eta(Y). \end{aligned} \tag{3.17}$$

Differentiating the above equation (3.17) with respect to Z then we have

$$\begin{aligned} (\nabla_Z h)(X,Y) = &\\ 2(Z\alpha)g(\phi X,Y) &+ \left[\frac{\nabla_Z r}{n} - 4\alpha(Z\alpha)\right] g(X,Y) \\ &+ \left[\frac{\nabla_Z r}{n} - 4(2n+1)\alpha(Z\alpha))\right] \eta(X)\eta(Y) + 2\alpha g((\nabla_Z\phi)X,Y) \\ &+ \left[\frac{r}{n} - 2(2n+1)\alpha^2\right] \{\alpha g(X,\phi Z)\eta(Y) + \alpha g(Y,\phi Z)\eta(X)\}, \end{aligned} \tag{3.18}$$

by substituting $Z = \xi$ and $X = Y \in (Span\,\xi)^\perp$ in the above equation, we have

$$\nabla_\xi r = 0, \tag{3.19}$$

provided h is parallel. Thus r is constant scalar, then we state that:

Proposition 2. *An η-Einstein Lorentzian α-Sasakian Ricci soliton (g,ξ,λ) with constant scalar curvature r is shrinking.*

Proof. From equation (1.4) and (3.16), we have

$$h(X,Y) = -2\lambda g(X,Y).$$

Putting $X = Y = \xi$ in the above equation, we have

$$h(\xi,\xi) = 2\lambda. \tag{3.20}$$

Now considering (3.17), that is

$$h(X,Y) = 2\alpha g(\phi X,Y) + \left[\frac{r}{n} - 2\alpha^2\right] g(X,Y) + \left[\frac{r}{n} - 2(2n+1)\alpha^2\right] \eta(X)\eta(Y).$$

Putting $X = Y = \xi$ in the above equation, we have

$$h(\xi,\xi) = -4n\alpha^2. \tag{3.21}$$

By equating (3.20) and (3.21), we have

$$\lambda = -2n\alpha^2. \tag{3.22}$$

This shows that $\lambda < 0$ that is the Ricci soliton in $(2n+1)$-dimensional Lorentzian α-Sasakian is shrinking. □

We compute an expression for Ricci tensor for 3-dimensional Lorentzian α-Sasakian manifold as follows: The curvature tensor for 3-dimensional Riemannian manifold is given by

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ & - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.23}$$

put $Z = \xi$ in the above equation that is in (3.23) and by using (2.2), (2.4) and (2.6), we obtain

$$\left[\frac{r}{2} - \alpha^2\right] [\eta(Y)X - \eta(X)Y] = \eta(Y)QX - \eta(X)QY. \tag{3.24}$$

Again put $Y = \xi$ in the equation (3.24) and by using (2.1) and (2.7), we have

$$QX = \left[\frac{r}{2} - \alpha^2\right] X + \left[\frac{r}{2} - 3\alpha^2\right] \eta(X)\xi \tag{3.25}$$

and

$$S(X,Y) = \left[\frac{r}{2} - \alpha^2\right] g(X,Y) + \left[\frac{r}{2} - 3\alpha^2\right] \eta(X)\eta(Y), \tag{3.26}$$

where r is the scalar curvature and α is a constant.

For a 3-dimensional Lorentzian α-Sasakian manifolds, we obtain

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y). \tag{3.27}$$

By using (3.11) and (3.26) in (3.27), we have

$$h(X,Y) = 2\alpha g(\phi X,Y) + [r - 2\alpha^2]g(X,Y) + [r - 6\alpha^2]\eta(X)\eta(Y). \tag{3.28}$$

Differentiating the above equation with respect to Z then we have

$$\begin{aligned} (\nabla_Z h)(X,Y) = {} & \\ & 2(Z\alpha)g(\phi X,Y) + 2\alpha g((\nabla_Z \phi)X,Y) + [\nabla_Z r - 4\alpha(Z\alpha)]g(X,Y) \end{aligned} \tag{3.29}$$

$$+ [\nabla_Z r - 6(2\alpha(Z\alpha))]\eta(X)\eta(Y) + (r - 6\alpha^2)[\alpha g(X, \phi Z)\eta(Y) + \alpha g(Y, \phi Z)\eta(X)].$$

Substituting $Z = \xi$, $X = Y \in (\text{Span } \xi)^\perp$ in (3.29) then we have

$$\nabla_\xi r = 0, \tag{3.30}$$

provided h is parallel. Thus r is constant scalar, then we state that:

Proposition 3. *A Ricci soliton (g, ξ, λ) in 3-dimensional Lorentzian α-Sasakian manifold with constant scalar curvature r is shrinking.*

Proof. From equation (1.4) and (3.27), we have

$$h(X, Y) = -2\lambda g(X, Y).$$

Putting $X = Y = \xi$ in the above equation, we have

$$h(\xi, \xi) = 2\lambda. \tag{3.31}$$

Now considering (3.28), that is

$$h(X, Y) = 2\alpha g(\phi X, Y) + [r - 2\alpha^2]g(X, Y) + [r - 6\alpha^2]\eta(X)\eta(Y).$$

Putting $X = Y = \xi$ in the above equation, we have

$$h(\xi, \xi) = -4\alpha^2. \tag{3.32}$$

By equating (3.31) and (3.32), we have

$$\lambda = -2\alpha^2. \tag{3.33}$$

This shows that $\lambda < 0$ that is the Ricci soliton in 3-dimensional Lorentzian α-Sasakian is shrinking. □

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