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MULTIPLICATION OPERATORS ON GENERALIZED LORENTZ-ZYGMUND SPACES

S. C. ARORA AND SATISH VERMA

ABSTRACT. The invertible, compact and Fredholm multiplication operators on generalized Lorentz-Zygmund (GLZ) spaces $L_{p,q;\alpha}, 1 < p \leq \infty$, $1 \leq q \leq \infty, \alpha$ in the Euclidean space $\mathbb{R}^m$, are characterized in this paper.

1. INTRODUCTION

Let f be a complex-valued measurable function defined on a σ-finite measure space $(X, \mathcal{A}, \mu)$. For $s \geq 0$, define μ_f the *distribution function* of f as

$$\mu_f(s) = \mu\{x \in X : |f(x)| > s\}.$$

By f^* we mean the *non-increasing rearrangement* of f given as

$$f^*(t) = \inf\{s > 0 : \mu_f(s) \leq t\}, \quad t \geq 0.$$

For $t > 0$, let

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds \quad \text{and} \quad f^{**}(0) = f^*(0).$$

Now, let $m \in \mathbb{N}$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m$. Let us denote v_α^m, real-valued function defined by

$$v_\alpha^m(t) = \prod_{i=1}^m l_i^{\alpha_i}(t), \quad t \in (0, +\infty),$$

where $l_1, l_2, \ldots, l_m$ are non-negative functions defined on $(0, +\infty)$ by

$$l_1(t) = 1 + |\log t|, \; l_i(t) = 1 + \log l_{i-1}(t), \quad i \in 2, \ldots, m$$

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For $1 < p \leq \infty$, $1 \leq q \leq \infty$, and for f measurable function on X define $\|f\|_{p,q;\alpha}$ as

$$\|f\|_{p,q;\alpha} = \begin{cases} \{\int_0^\infty (\, t^{1/p} v_\alpha^m(t) f^{**}(t))^q \, \frac{dt}{t}\}^{1/q}, & 1 < p < \infty, 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{1/p} v_\alpha^m(t) f^{**}(t), & 1 < p \leq \infty, q = \infty. \end{cases}$$

The *generalized Lorentz-Zygmund* (GLZ) space $L_{p,q;\alpha}$, introduced in [10], consists of those measurable functions f on X such that $\|f\|_{p,q;\alpha} < \infty$. Also $\|\cdot\|_{p,q;\alpha}$ is a norm and $L_{p,q;\alpha}$ is a Banach space with respect to this norm. The GLZ-spaces are a particular case of more general spaces, namely the Lorentz-Karamata spaces [20]. The GLZ spaces include many familiar spaces, in particular those of Lebesgue, Lorentz, Lorentz-Zygmund, and some exponential Orlicz spaces by taking special choices of p, q, α and the functions v_α^m, $\alpha \in \mathbb{R}^m$, for more on these spaces one can refer to [5,6,8-12,19-21] and references therein. The above norm can also be put, using the usual L_q norm over the interval $(0, +\infty)$, in the form

$$\big\|f\big\|_{p,q;\alpha} = \big\|t^{\frac{1}{p}-\frac{1}{q}} v_\alpha^m(t) f^{**}(t)\big\|_q.$$

Let $F(X)$ be a vector space of all complex-valued functions on a non-empty set X. Let $u : X \to \mathbb{C}$ be a measurable function on X such that $u \cdot f \in F(X)$ whenever $f \in F(X)$. This gives rise to a linear transformation $M_u : F(X) \to F(X)$ defined as $M_u f = u \cdot f$, where the product of functions is pointwise. In case $F(X)$ is a topological vector space and M_u is continuous, we call it a *multiplication operator* induced by u.

Multiplication operators have been studied on various function spaces in [1-4,7,13,14,16,18,22,24], and references therein. Along the line of their arguments we study the multiplication operators on the generalized Lorentz-Zygmund spaces $L_{p,q;\alpha}$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. First, we prove a characterization of the boundedness of M_u in terms of u, and show that the set of multiplication operators on $L_{p,q;\alpha}$, $1 < p < \infty$, $1 \leq q < \infty$ is a maximal abelian subalgebra of $\mathcal{B}(L_{p,q;\alpha})$, the Banach algebra of all bounded linear operators on $L_{p,q;\alpha}$. We use it to characterize the invertibility of M_u on $L_{p,q;\alpha}$. The compact and Fredholm multiplication operators are also characterized in this paper.

2. Characterizations

In this section boundedness and invertibility of the multiplication operator M_u are characterized in terms of boundedness and invertibility of the complex-valued measurable function u respectively.

Theorem 2.1. *The linear transformation $M_u : f \to u \cdot f$ on the GLZ space $L_{p,q;\alpha}$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ is bounded if and only if u is essentially bounded. Moreover*

$$\|M_u\| = \|u\|_\infty.$$

Proof. Suppose u is essentially bounded. For $f \in L_{p,q;\alpha}$, $t > 0$ we have

$$(u \cdot f)^*(t) = \inf\{s > 0 : \mu_{u \cdot f}(s) \le t\}$$
$$\le \inf\{ \|u\|_\infty \; s > 0 : \mu_f(s) \le t\} = \|u\|_\infty \; f^*(t).$$

Consequently,

$$(u \cdot f)^{**}(t) \le \|u\|_\infty f^{**}(t).$$

Therefore, for $1 < p < \infty$, $1 \le q < \infty$,

$$\big\|M_u f\big\|^q_{p,q;\alpha} = \big\|t^{\frac{1}{p}-\frac{1}{q}} v^m_\alpha(t)(u \cdot f)^{**}(t)\big\|^q_q \le \|u\|^q_\infty \; \big\|f\big\|^q_{p,q;\alpha}. \tag{2.1}$$

For $q = \infty$, $1 < p \le \infty$, we have

$$\|M_u f\|_{p,\infty;\alpha} = \sup_{t>0} t^{1/p} v^m_\alpha(t)(u \cdot f)^{**}(t)$$
$$\le \|u\|_\infty \sup_{t>0} t^{1/p} v^m_\alpha(t) f^{**}(t) = \|u\|_\infty \|f\|_{p,\infty;\alpha}.$$

Conversely, suppose M_u is a bounded operator on $L_{p,q;\alpha}$, $1 < p < \infty$, $1 \le q < \infty$. If u is not χ_{E_n} lies in the GLZ space $L_{p,q;\alpha}$. This follows from the following:

We have for $t > 0$,

$$\chi^*_{E_n}(t) = \chi_{[\;0,\; \mu(E_n))}(t),$$

and so

$$\chi^{**}_{E_n}(t) = \frac{1}{t}\int_0^t \chi^*_{E_n}(s)ds = \begin{cases} 1, & 0 \le t < \mu(E_n), \\ \frac{1}{t}\mu(E_n), & t \ge \mu(E_n). \end{cases}$$

Since $v^m_\alpha(t)$ is a slowly varying function [8, 12, 20], therefore using property of the slowly varying function [12, Proposition 2.2(iv), p. 88], we have

$$\begin{aligned}
\big\|\chi_{E_n}\big\|^q_{p,q;\alpha} &= \big\|t^{\frac{1}{p}-\frac{1}{q}} v^m_\alpha(t)\chi^{**}_{E_n}(t)\big\|^q_q = \int_0^\infty t^{\frac{q}{p}-1}(v^m_\alpha(t))^q(\chi^{**}_{E_n}(t))^q dt \\
&= \int_0^{\mu(E_n)} t^{\frac{q}{p}-1}(v^m_\alpha(t))^q dt \;+\; (\mu(E_n))^q \int_{\mu(E_n)}^\infty t^{-q(1-\frac{1}{p})-1}(v^m_\alpha(t))^q dt \\
&= \big\|t^{\frac{1}{p}-\frac{1}{q}} v^m_\alpha(t)\big\|^q_{q,(0,\mu(E_n))} \;+\; (\mu(E_n))^q \big\|t^{-(1-\frac{1}{p})-\frac{1}{q}} v^m_\alpha(t)\big\|^q_{q,(\mu(E_n),\infty)} \\
&\approx (\mu(E_n))^{q/p}(v^m_\alpha(\mu(E_n)))^q \;+\; (\mu(E_n))^q(\mu(E_n))^{-q(1-1/p)}(v^m_\alpha(\mu(E_n)))^q \\
&= 2(\mu(E_n))^{q/p}(v^m_\alpha(\mu(E_n)))^q < \infty.
\end{aligned}$$

Now,

$$\{x \in X : |u(x)\chi_{E_n}(x)| > s\} \supseteq \{x \in X : |\chi_{E_n}(x)| > s/n\},$$

gives for $t > 0$

$$(u\chi_{E_n})^*(t) \ge \inf\{s > 0 : \mu\{x \in X : |\chi_{E_n}(x)| > s/n\} \le t\}$$
$$= \inf\{ns > 0 : \mu\{x \in X : |\chi_{E_n}(x)| > s\} \le t\} = n\chi^*_{E_n}(t).$$

Thus,

$$\big\|M_u\chi_{E_n}\big\|^q_{p,q;\alpha} \ge n^q \big\|\chi_{E_n}\big\|^q_{p,q;\alpha}$$

This contradicts the boundedness of M_u. Hence, u is essentially bounded.

For $q = \infty$, $1 < p \leq \infty$, the proof is similar.

From (2.1) we have $\|M_u\| \leq \|u\|_\infty$. Now, for any $\epsilon > 0$, let E denote the set

$$\{x \in X : |u(x)| \geq \|u\|_\infty - \epsilon\}.$$

Proceeding as above, on replacing E_n by E and n by $\|u\|_\infty - \epsilon$, we get

$$\|M_u \chi_E\|_{p,q;\alpha} \geq (\|u\|_\infty - \epsilon)\|\chi_E\|_{p,q;\alpha}.$$

Therefore $\|M_u\| \geq \|u\|_\infty - \epsilon$, and hence $\|M_u\| \geq \|u\|_\infty$. □

Theorem 2.2. *The set of all multiplication operators on the GLZ space $L_{p,q;\alpha}$, $1 < p < \infty$, $1 \leq q < \infty$, is a maximal abelian subalgebra of $\mathcal{B}(L_{p,q;\alpha})$, the Banach algebra of all bounded linear operators on $L_{p,q;\alpha}$.*

Proof. Let $\mathcal{M} = \{M_u : u \in L^\infty(\mu)\}$. Clearly, $\mathcal{M}$ is an abelian subalgebra of $\mathcal{B}(L_{p,q;\alpha})$. Let T be any bounded operator on $L_{p,q;\alpha}$ such that $TM_u = M_u T$ for every $u \in L^\infty(\mu)$. We shall prove that $T = M_v$ for some $v \in L^\infty(\mu)$. The proof given below runs on the same lines as in Conway [7, Proposition 12.4, p. 57]. Consider two cases.

Case 1. $\mu(X) < \infty$. Let e denote the unity function $(e(x) = 1, x \in X)$, then $e \in L_{p,q;\alpha}$, as

$$\|e\|^q_{p,q;\alpha} \approx (\mu(X))^{1/p}(v^m_\alpha(\mu(X))) < \infty.$$

Let $v = Te$. Then for each $E \in \mathcal{A}$,

$$T\chi_E = TM_{\chi_E} e = M_{\chi_E} Te = \chi_E v = v\chi_E = M_v \chi_E.$$

Let if possible v be not essentially bounded, then the set

$$F_n = \{x \in X : |v(x)| > n\}$$

has a positive measure for each natural number n. Thus,

$$\|T\chi_{F_n}\|_{p,q;\alpha} = \|M_v \chi_{F_n}\|_{p,q;\alpha} \geq n\|\chi_{F_n}\|_{p,q;\alpha}.$$

This is a contradiction to the fact that T is bounded. Thus, $v \in L^\infty(\mu)$. Since simple functions are dense [15, (2.4), p. 258] in $L_{p,q;\alpha}$, we have $T = M_v$. This proves that $T \in \mathcal{M}$ and so $\mathcal{M}$ is maximal.

Case 2. $\mu(X) = \infty$. Write $X = \bigcup_{n=1}^\infty A_n$, where $A_n \in \mathcal{A}$ and $\mu(A_n) < \infty$, since μ being σ-finite. For $n \geq 1$, let

$$L_{p,q;\alpha}(\mu|A_n) = \{\chi_{A_n} f : f \in L_{p,q;\alpha}\} \text{ and } L^\infty(\mu|A_n) = \{\chi_{A_n} u : u \in L^\infty(\mu)\}.$$

Now, for $f \in L_{p,q;\alpha}(\mu|A_n)$, we have

$$Tf = T\chi_{A_n} f = TM_{\chi_{A_n}} f = M_{\chi_{A_n}} Tf = \chi_{A_n} Tf \in L_{p,q;\alpha}(\mu|A_n).$$

Let T_{A_n} be the restriction of T to $L_{p,q;\alpha}(\mu|A_n)$. It is routine to show that $T_{A_n} M_{u_{A_n}} = M_{u_{A_n}} T_{A_n}$, for all $u_{A_n} \in L^\infty(\mu|A_n)$. Apply *Case 1* to T_{A_n}, there is a function $v_{A_n} \in L^\infty(\mu|A_n)$, such that

$$T_{A_n} = M_{v_{A_n}}. \tag{2.2}$$

For $m \neq n$ and $f \in L_{p,q;\alpha}(\mu|(A_m \cap A_n))$, we have

$$v_{A_m} f = M_{v_{A_m}} f = T_{A_m} f = Tf = T_{A_n} f = v_{A_n} f.$$

Thus, $v_{A_m} = v_{A_n}$ on $A_m \cap A_n$. Therefore, setting $v(x) = v_{A_n}(x)$ when $x \in A_n$ gives a well-defined function on X. It is seen that v is measurable since $v|A_n = v_{A_n}$ is measurable for each $n \geq 1$. Also,

$$\|v_{A_n}\|_\infty = \|M_{v_{A_n}}\| = \|T_{A_n}\| \leq \|T\|$$

implies

$$\mu\{x \in A_n : |v_{A_n}(x)| > \|T\|\} = 0, \quad \forall n \geq 1$$

and so

$$\mu\{x \in X : |v(x)| > \|T\|\} \leq \sum_{n=1}^{\infty} \mu\{x \in A_n : |v_{A_n}(x)| > \|T\|\} = 0$$

gives that $\mu\{x \in X : |v(x)| > \|T\|\} = 0$, hence $\|v\|_\infty \leq \|T\|$ and $v \in L^\infty(\mu)$. Moreover, using (2.2), we have $Tf = M_v f$ whenever $n \geq 1$ and $f \in L_{p,q;\alpha}(\mu|A_n)$. Because $X = \bigcup_{n=1}^{\infty} A_n$, the linear span of the spaces $\{L_{p,q;\alpha}(\mu|A_n)\}$ is dense in $L_{p,q;\alpha}$. Therefore $T = M_v$. □

Corollary 2.3. *The multiplication operator M_u on $L_{p,q;\alpha}$, $1 < p < \infty$, $1 \leq q < \infty$ is invertible if and only if u is invertible in $L^\infty(\mu)$.*

Proof. If M_u is invertible then M_u^{-1} commutes with all multiplication operators on $L_{p,q;\alpha}$ and hence $M_u^{-1} = M_v$ for some $v \in L^\infty(\mu)$. Therefore v is the inverse of u.

Conversely, if u is invertible, then $M_u^{-1} = M_{u^{-1}}$. □

3. Compact Multiplication Operators

In this section we characterize compact multiplication operators.

Theorem 3.1. *A multiplication operator M_u on GLZ space $L_{p,q;\alpha}$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ is compact if and only if $L^{p,q;\alpha}(u,\epsilon)$ is finite dimensional for each $\epsilon > 0$, where*

$$(u,\epsilon) = \{x \in X : |u(x)| \geq \epsilon\} \text{ and } L^{p,q;\alpha}(u,\epsilon) = \{f\chi_{(u,\epsilon)} : f \in L_{p,q;\alpha}\}.$$

Proof. If M_u is a compact operator, then $L^{p,q;\alpha}(u,\epsilon)$ is a closed invariant subspace of M_u and hence $M_u|_{L^{p,q;\alpha}(u,\epsilon)}$ is a compact operator. Also for $f \in L_{p,q;\alpha}$, $t > 0$

$$\begin{aligned}(uf\chi_{(u,\epsilon)})^*(t) &\geq \epsilon \inf\left\{s > 0 : \mu\{x \in X : |f(x)\chi_{(u,\epsilon)}(x)| > s\} \leq t\right\} \\ &= \epsilon(f\chi_{(u,\epsilon)})^*(t).\end{aligned}$$

This gives

$$\|M_u f\chi_{(u,\epsilon)}\|_{p,q;\alpha} \geq \epsilon \|f\chi_{(u,\epsilon)}\|_{p,q;\alpha}.$$

Thus, $M_u|_{L^{p,q;\alpha}(u,\epsilon)}$ has closed range in $L^{p,q;\alpha}(u,\epsilon)$ and hence invertible. Being compact, $L^{p,q;\alpha}(u,\epsilon)$ is finite dimensional.

Conversely, suppose that $L^{p,q;\alpha}(u,\epsilon)$ is finite dimensional for each $\epsilon > 0$. In particular for each natural number n, $L^{p,q;\alpha}(u,1/n)$ is finite dimensional. For each n, define

$$u_n(x) = \begin{cases} u(x), & x \in (u,1/n), \\ 0, & \text{otherwise.} \end{cases}$$

Then $u_n \in L^\infty(\mu)$ as $u \in L^\infty(\mu)$. Moreover, for any $f \in L_{p,q;\alpha}$, $t > 0$

$$\begin{aligned}\Big((u_n - u)\cdot f\Big)^*(t) &= \inf\Big\{s > 0 : \mu_{(u_n-u)\cdot f}(s) \le t\Big\} \\ &\le \inf\Big\{s > 0 : \mu\{x \in X : |f(x)| > ns\} \le t\Big\} = \frac{1}{n} f^*(t).\end{aligned}$$

Consequently

$$\|(M_{u_n} - M_u)f\|_{p,q;\alpha} \le \frac{1}{n}\ \|f\|_{p,q;\alpha}.$$

This implies that M_{u_n} converges to M_u uniformly. As $L^{p,q;\alpha}(u,1/n)$ is finite dimensional so M_{u_n} is a finite rank operator. Therefore M_{u_n} is a compact operator and hence M_u is a compact operator. □

Corollary 3.2. *If μ is a non-atomic measure, then the only compact multiplication operator on the GLZ space $L_{p,q;\alpha}$ is the zero operator.*

Corollary 3.3. *If for each $\epsilon > 0$, the set (u,ϵ) contains only finitely many atoms, then M_u is a compact multiplication operator on the GLZ space $L_{p,q;\alpha}$.*

4. Fredholm Multiplication Operators

In this section we first establish a condition for a multiplication operator to have a closed range and then we make use of it to characterize Fredholm multiplication operators on $L_{p,q;\alpha}$, $1 < p < \infty$, $1 < q < \infty$, where μ is a non-atomic measure. Here the operator M_u is Fredholm if its range $R(M_u)$ is closed, $\dim N(M_u) < \infty$ and $\operatorname{codim} R(M_u) < \infty$.

Theorem 4.1. *A multiplication operator M_u on the GLZ space $L_{p,q;\alpha}$, $1 < p \le \infty$, $1 \le q \le \infty$ has closed range if and only if there exists a $\delta > 0$ such that $|u(x)| \ge \delta$ a.e. on $S = \{x \in X : u(x) \ne 0\}$, the support of u.*

Proof. If $|u(x)| \ge \delta$ a.e. on S, then

$$\|M_u f\chi_S\|_{p,q;\alpha} \ge \delta\|f\chi_S\|,$$

for all $f \in L_{p,q;\alpha}$. Hence M_u has closed range.

Conversely if M_u has closed range, then there exists an $\epsilon > 0$ such that

$$\|M_u f\|_{p,q;\alpha} \ge \epsilon\|f\|_{p,q;\alpha},$$

for all $f \in L^{p,q;\alpha}(S)$, where

$$L^{p,q;\alpha}(S) = \{f\chi_S : f \in L_{p,q;\alpha}\}.$$

Let

$$E = \{x \in S : |u(x)| < \epsilon/2\}.$$

If $\mu(E) > 0$, then we can find a measurable set $F \subseteq E$ such that $\chi_F \in L^{p,q;\alpha}(S)$. Then we have

$$\left\{\frac{\epsilon}{2}\, s > 0 : \mu_{\chi_F}(s) \leq t\right\} \subseteq \left\{s > 0 : \mu_{u\cdot\chi_F}(s) \leq t\right\}$$

so that

$$(u \cdot \chi_F)^*(t) \leq \frac{\epsilon}{2}\, \chi_F^*(t).$$

Hence,

$$\|M_u \chi_F\|_{p,q;\alpha} \leq \frac{\epsilon}{2}\, \|\chi_F\|_{p,q;\alpha}.$$

This is a contradiction. Therefore, $\mu(E) = 0$. This completes the proof. □

Theorem 4.2. *Suppose that μ is a non-atomic measure. Let M_u be a multiplication operator on the GLZ space $L_{p,q;\alpha}$, $1 < p < \infty$, $1 < q < \infty$, where $u \in L_\infty(\mu)$. Then the following conditions are equivalent :*

(1) *M_u is an invertible operator.*
(2) *M_u is a Fredholm operator.*
(3) *$R(M_u)$ is closed and* codim $R(M_u) < \infty$*.*
(4) *$|u(x)| \geq \delta$ a.e. on X for some $\delta > 0$.*

Proof. We here show that (3) implies (4), because other implications are obvious. Suppose that $R(M_u)$ is closed and codim $R(M_u) < \infty$. Then there exists a $\delta > 0$ (Theorem 4.1) such that $|u| \geq \delta$ a.e. on S, the support of u. Hence, it is enough to show that $\mu(S^c) = 0$, where $S^c = \{x \in X : u(x) = 0\}$. First, we claim that M_u is onto. If possible M_u be not onto and let $f_0 \in L_{p,q;\alpha} \setminus R(M_u)$. Since, $R(M_u)$ is closed, we can find a function $g_0 \in L_{p',q';-\alpha}$, the conjugate space, where $1/p + 1/p' = 1/q + 1/q' = 1$ such that

$$\int f_0 g_0 d\mu = 1, \text{and} \int (M_u f) g_0 d\mu = 0, \text{ for all } f \in L_{p,q;\alpha}.$$

From the first equality, $\int Re(f_0 g_0) d\mu = 1$. Hence, the set

$$E_\epsilon = \{x \in X : Re(f_0 g_0)(x) \geq \epsilon\}$$

must have positive measure for some $\epsilon > 0$. Since μ is non-atomic, we can choose a sequence $\{E_n\}$ of subsets of $E_\epsilon$with $0 < \mu(E_n) < \infty$ and $E_m \cap E_n = \emptyset (m \neq n)$. Let $g_n = \chi_{E_n} g_0$. Then $g_n \in L_{p',q';-\alpha}$, and is nonzero because

$$Re \int f_0 g_n d\mu = Re \int_{E_n} f_0 g_0 d\mu \geq \epsilon \mu(E_n) > 0.$$

Furthermore, for each $f \in L_{p,q;\alpha}$, $\chi_{E_n} f$ is in $L_{p,q;\alpha}$, and so

$$(M_u^* g_n)(f) = g_n(M_u f) = \int (M_u f) g_n d\mu = \int (M_u f \chi_{E_n}) g_0 d\mu = 0,$$

where M_u^* is the conjugate operator of M_u. This implies $g_n \in N(M_u^*)$. Thus, the sequence $\{g_n\}$ forms a linearly independent subset of $N(M_u^*)$. This contradicts the fact that $\dim N(M_u^*) = \operatorname{codim} R(M_u) < \infty$. Hence M_u is onto. Now, it is easily seen that $\mu(S^c) = 0$. In fact, if $\mu(S^c) > 0$, then there exists a subset A of S^c with $0 < \mu(A) < \infty$. Then $\chi_A \in L_{p,q;\alpha} \setminus R(M_u)$, which contradicts the fact that M_u is onto. Therefore $\mu(S^c) = 0$, and so we have $|u| \geq \delta$ a.e. on X. □

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S. C. Arora,
Department of Mathematics,
University of Delhi,
Delhi-110007,
India
E-mail address: scarora@maths.du.ac.in; scaroradu@gmail.com

Satish Verma,
Department of Mathematics,
SGTB Khalsa College,
University of Delhi,
Delhi-110007,
India
E-mail address: vermas@maths.du.ac.in