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SOME APPLICATIONS OF DIFFERENTIAL SUBORDINATION OF MULTIVALENT FUNCTIONS ASSOCIATED WITH THE WRIGHT GENERALIZED HYPERGEOMETRIC FUNCTION

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Abstract. Making use of the principle of differential subordination, we investigate some inclusion relationships of certain subclasses of multivalent analytic functions associated with the Wright generalized hypergeometric function.

1. Introduction

Let $A_n(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (p, n \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

For convenience, we write $A_1(p) = A(p)$. If f and g are analytic in U, we say that f is subordinate to g, written symbolically as follows: $f \prec g$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$ $(z \in U)$. In particular, if the function g is univalent in U, then we have the following equivalence (cf. [2, 14], see also [15, p. 4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f \in A_n(p)$, given by (1.1), and $g \in A_n(p)$ given by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_{k+p} z^{k+p} \quad (p, n \in \mathbb{N}), \tag{1.2}$$

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then the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z) \quad (p, n \in \mathbb{N};\ z \in U). \tag{1.3}$$

Let $\alpha_1, A_1, \ldots, \alpha_q, A_q$ and $\beta_1,\ B_1, \ldots, \beta_s, B_s \quad (q, s \in \mathbb{N})$ be positive real parameters such that

$$1 + \sum_{i=1}^{s} B_i - \sum_{i=1}^{q} A_i \geq 0.$$

The Wright generalized hypergeometric function [31] (see also [28])

$$ {}_q\Psi_s\left[(\alpha_1, A_1), \ldots, (\alpha_q, A_q); (\beta_1, B_1), \ldots, (\beta_s, B_s); z\right] = {}_q\Psi_s\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right]$$

is defined by

$$ {}_q\Psi_s\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{q} \Gamma(\alpha_i + nA_i)}{\prod_{i=1}^{s} \Gamma(\beta_i + nB_i)} \cdot \frac{z^n}{n!} \quad (z \in U).$$

If $A_i = 1 (i = 1, \ldots, q)$ and $B_i = 1 (i = 1, \ldots, s)$, we have the relationship:

$$\Omega_q\Psi_s\left[(\alpha_i, 1)_{1,q}; (\beta_i, 1)_{1,s}; z\right] = {}_q F_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)\ ,$$

where ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ is the generalized hypergeometric function (see [28]) and

$$\Omega = \frac{\prod_{i=1}^{s} \Gamma(\beta_i)}{\prod_{i=1}^{q} \Gamma(\alpha_i)}. \tag{1.4}$$

The Wright generalized hypergeometric functions were invoked in the geometric function theory (see [23, 24]).

By using the generalized hypergeometric function Dziok and Srivastava [7] introduced a linear operator. In [8] Dziok and Raina and in [1] Aouf and Dziok extended the linear operator by using the Wright generalized hypergeometric function.

First we define a function ${}_q\Phi_s^p\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right]$ by

$$ {}_q\Phi_s^p\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right] = \Omega z^p_q\Psi_s\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right]$$

and consider the following linear operator

$$\theta_p\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}\right] : A_n(p) \to A_n(p),$$

defined by the convolution

$$\theta_p\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}\right] f(z) = {}_q\Phi_s^p\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right] * f(z)\ .$$

We observe that, for a function f of the form (1.1), we have

$$\theta_p\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}\right] f(z) = z^p + \sum_{k=n}^{\infty} \Omega \sigma_n(\alpha_1) a_{k+p} z^{k+p}, \tag{1.5}$$

where Ω is given by (1.4) and $\sigma_n(\alpha_1)$ is defined by

$$\sigma_n(\alpha_1) = \frac{[\Gamma(\alpha_1 + A_1 n) \dots \Gamma(\alpha_q + A_q n)]}{\Gamma(\beta_1 + B_1 n) \dots \Gamma(\beta_s + B_s n) n!}. \tag{1.6}$$

If, for convenience, we write

$$\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] = \theta_p\left[(\alpha_1, A_1), \dots, (\alpha_q, A_q); (\beta_1, B_1), \dots, (\beta_s, B_s)\right] f(z),$$

then one can easily verify from the definition (1.5) that

$$zA_1(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z))' = \alpha_1 \theta_{p,q,s}\left[\alpha_1 + 1, A_1, B_1\right] f(z)$$

$$-(\alpha_1 - pA_1)\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z) \ (A_1 > 0). \tag{1.7}$$

The linear operator $\theta_{1,q,s}\left[\alpha_1, A_1, B_1\right] = \theta\left[\alpha_1\right]$ was introduced by Dziok and Raina [8] and studied by Aouf and Dziok [1].

We note that, for $f \in A_n(p), A_i = 1 (i = 1, \dots, q), B_i = 1 (i = 1, \dots, s)$ and by specializing the parameters $\alpha_i \ (i = 1, \dots, q), \ \beta_i \ (i = 1, \dots, s), \ q$ and s we obtain the following operators studied by various authors:

(i) $\theta_{p,q,s}\left[\alpha_1\right] f(z) = H_{p,q,s}(\alpha_1) f(z)$ (see Patel et al.[22]);

(ii) $\theta_{p,2,1}\left[a, 1; c\right] f(z) = L_p(a, c) f(z) (a > 0, c > 0)$ (see Carlson and Shaffer [3] and Saitoh [25]);

(iii) $\theta_{p,2,1}\left[\mu + p, 1; 1\right] f(z) = D^{\mu+p-1} f(z) (\mu > -p)$, where $D^{\mu+p-1} f(z)$ is the $(\mu + p - 1) - th$ order Ruscheweyh derivative of a function $f \in A_n(p)$ (see Kumar and Shukla [10]);

(iv) $\theta_{p,2,1}\left[1 + p, 1; 1 + p - \mu\right] f(z) = \Omega_z^{(\mu,p)} f(z)$, where the operator $\Omega_z^{(\mu,p)} f(z)$ is defined by (see Srivastava and Aouf [27])

$$\Omega_z^{(\mu,p)} f(z) = \frac{\Gamma(1 + p - \mu)}{\Gamma(1 + p)} z^\mu D_z^\mu f(z) (0 \le \mu < 1; p \in \mathbb{N}),$$

where $\Omega_z^\mu f(z)$ is the fractional derivative operator (see, for details, [6] and [18] and [19]);

(v) $\theta_{p,2,1}\left[\delta + p, 1; \delta + p + 1\right] f(z) = F_{\delta,p}(f)(z)$, where $F_{\delta,p}(f)$ is the generalized Bernardi-Libera-Livingston operator (see [5]), defined by

$$F_{\delta,p}(f)(z) = \frac{\delta + p}{z^\delta} \int_0^z t^{\delta-1} f(t) dt (\delta > -p; p \in \mathbb{N});$$

(vi) $\theta_{p,2,1}\left[p + 1, 1; m + p\right] f(z) = I_{m,p} f(z) (m \in \mathbb{Z}; m > -p)$, where the operator $I_{m,p}$ is the $(m + p - 1) - th$ Noor operator, considered by Liu and Noor [12];

(vii) $\theta_{p,2,1}\left[\lambda + p, c; a\right] f(z) = I_p^\lambda(a, c) f(z) (a, c \in \mathbb{R} \backslash \mathbb{N}_0^-; \lambda > -p)$, where $I_p^\lambda(a, c) f(z)$ is the Cho-Kwon-Srivastava operator (see [4]).

For fixed parameters A and $B(-1 \leq B < A \leq 1)$, we say that a function $f \in A_n(p)$ is in the class $Q^n_{p,q,s}(\alpha_1, A_1, B_1; A, B)$, if it satisfies the following subordination condition:

$$\frac{\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}(p \in \mathbb{N}). \tag{1.8}$$

In view of the definition of subordination, (1.8) is equivalent to the following condition:

$$\left| \frac{\frac{\left(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)\right)'}{z^{p-1}} - p}{B\frac{\left(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)\right)'}{z^{p-1}} - pA} \right| < 1 \quad (z \in U). \tag{1.9}$$

For convenience, we write $Q^n_{p,q,s}(\alpha_1, A_1, B; 1 - \frac{2\theta}{p}, -1) = Q^n_{p,q,s}(\alpha_1, A_1, B_1; \theta)$, where $Q^n_{p,q,s}(\alpha_1, A_1, B_1; \theta)$ denote the class of functions in $A_n(p)$ satisfying the following inequality:

$$\operatorname{Re} \frac{\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z)\right)'}{z^{p-1}} > \theta \quad (0 \leq \theta < p; p \in \mathbb{N};\ z \in U). \tag{1.10}$$

2. Preliminaries

To establish our main results, we shall need the following lemmas.

Lemma 1 ([9]). *Let the function h be analytic and convex (univalent) in U with $h(0) = 1$ Suppose also the function φ given by*

$$\varphi(z) = 1 + a_n z^n + a_{n+1} z^{n+1} + \ldots, \tag{2.1}$$

is analytic in U. If

$$\varphi(z) + \frac{1}{\gamma} z\varphi'(z) \prec h(z), \tag{2.2}$$

where $\gamma \neq 0$ and $\operatorname{Re}\gamma \geq 0$. Then

$$\varphi(z) \prec \Psi(z) = \frac{\gamma}{n} z^{-\frac{\gamma}{n}} \int_0^z t^{\frac{\gamma}{n}-1} h(t)dt \prec h(z), \tag{2.3}$$

and Ψ is the best dominant of (2.2).

With a view to stating a well-known result (Lemma 2 below), we denote by $P(\delta)$ the class of functions Φ given by

$$\Phi(z) = 1 + c_1 z + c_2 z^2 + \ldots\ , \tag{2.4}$$

which are analytic in U and satisfy the following inequality:

$$\operatorname{Re}\Phi(z) > \delta(0 \leq \delta < 1; z \in U).$$

Lemma 2 ([20]). *Let the function* Φ*, given by (2.4), be in the class* $P(\delta)$*. Then*

$$\operatorname{Re}\Phi(z) \geq 2\delta - 1 + \frac{2(1-\delta)}{1+|z|}\ (0 \leq \delta < 1; z \in U).$$

Lemma 3 ([29]). *For* $0 \leq \gamma_1, \gamma_2 < 1$,

$$P(\gamma_1) * P(\gamma_2) \subset P(\gamma_3)\ (\gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2)).$$

The result is the best possible.

Lemma 4 ([26]). *Let* Φ *be analytic in* U *with*

$$\Phi(0) = 1 \text{ and } \operatorname{Re}\Phi(z) > \frac{1}{2} \quad (z \in U).$$

Then, for any function F *analytic in* U, $(\Phi * F)(U)$ *is contained in the convex hull of* $F(U)$.

Lemma 5 ([17]). *Let* φ *be analytic in* U *with* $\varphi(0) = 1$ *and* $\varphi(z) \neq 0$ *for* $0 < |z| < 1$*, and let* $A, B \in \mathbb{C}$ *with* $A \neq B$ *and* $|B| \leq 1$.

(i) Let $B \neq 0$ *and* $\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ *satisfy either*

$$\left|\frac{\gamma(A-B)}{B} - 1\right| \leq 1 or \left|\frac{\gamma(A-B)}{B} + 1\right| \leq 1.$$

If φ *satisfies*

$$1 + \frac{z\varphi'(z)}{\gamma\varphi(z)} \prec \frac{1+Az}{1+Bz},$$

then

$$\varphi(z) \prec (1+Bz)^{\gamma(\frac{A-B}{B})}$$

and this is the best dominant.

(ii) Let $B = 0$ *and* $\gamma \in \mathbb{C}^*$ *be such that* $|\gamma A| < \pi$*. If* φ *satisfies*

$$1 + \frac{z\varphi'(z)}{\gamma\varphi(z)} \prec 1 + Az \quad ,$$

then

$$\varphi(z) \prec e^{\gamma A z}$$

and this is the best dominant.

For real or complex numbers a, b and $c (c \notin \mathbb{Z}_0^-)$, the Gaussian hypergeometric function is defined by

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c} \cdot \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \cdots .$$

We note that the above series converges absolutely for $z \in U$ and hence represents an analytic function in U (see, for details, [30, Chapter 14]).

Each of the identities (asserted by Lemma 6 below) is well-known (cf., e.g., [30, Chapter 14].

Lemma 6 ([30]). *For real or complex parameters a, b and $c (c \notin \mathbb{Z}_0^-)$,*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a,b;c;z) \quad (\mathrm{Re}(c) > \mathrm{Re}(b) > 0); \tag{2.5}$$

$${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z); \tag{2.6}$$

and

$${}_2F_1(a,b;c;z) = (1-z)^{-a}_2F_1(a,c-b;c;\frac{z}{z-1}). \tag{2.7}$$

3. Main Results

Unless otherwise mentioned we shall assume through this paper that $-1 \leq B < A \leq 1$, λ, $A_1 > 0$ and $p, n \in \mathbb{N}$.

Theorem 1. *Let the function f defined by (1.1) satisfy the following subordination condition*

$$(1-\lambda)\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} + \lambda\frac{(\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z))'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}. \tag{3.1}$$

Then

$$\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} \prec Q(z) \prec \frac{1+Az}{1+Bz}, \tag{3.2}$$

where the function Q given by

$$Q(z) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}_2F_1(1,1;\frac{\alpha_1}{A_1\lambda n}+1;\frac{Bz}{1+Bz}) & (B \neq 0), \\ 1 + \frac{A\alpha_1}{A_1\lambda n + \alpha_1}z & (B = 0), \end{cases} \tag{3.3}$$

is the best dominant of (3.2). Furthermore

$$\mathrm{Re}\,\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} > \rho \ \ (z \in U), \tag{3.4}$$

where

$$\rho(z) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}_2F_1(1,1;\frac{\alpha_1}{A_1\lambda n}+1;\frac{B}{B-1}) & (B \neq 0), \\ 1 - \frac{A\alpha_1}{A_1\lambda n + \alpha_1} & (B = 0), \end{cases} \tag{3.5}$$

the estimate in (3.4) is the best possible.

Proof. Consider the function φ defined by

$$\varphi(z)=\frac{\left(\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f(z)\right)'}{pz^{p-1}}\quad(z\in U).\tag{3.6}$$

Then φ is of the form (2.1) and is analytic in U. Applying the identity (1.7) in (3.6) and differentiating the resulting equation with respect to z, we get

$$(1-\lambda)\frac{\left(\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f(z)\right)'}{pz^{p-1}}+\lambda\frac{\left(\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]f(z)\right)'}{pz^{p-1}}$$
$$=\varphi(z)+\frac{A_1\lambda}{\alpha_1}z\varphi'(z)\prec\frac{1+Az}{1+Bz}\ .$$

Now, by using Lemma 1 for $\gamma=\frac{\alpha_1}{A_1\lambda}$, we obtain

$$\frac{\left(\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f(z)\right)'}{pz^{p-1}}\prec Q(z)\prec\frac{\alpha_1}{A_1\lambda n}z^{\frac{-\alpha_1}{A_1\lambda n}}\int_0^z t^{\frac{\alpha_1}{A_1\lambda n}-1}\left(\frac{1+At}{1+Bt}\right)dt$$

$$=\begin{cases}\dfrac{A}{B}+\left(1-\dfrac{A}{B}\right)(1-Bz)^{-1}{}_2F_1\left(1,1;\dfrac{\alpha_1}{A_1\lambda n}+1;\dfrac{Bz}{1+Bz}\right) & (B\neq 0),\\ 1+\dfrac{\alpha_1 A}{A_1\lambda n+\alpha_1}z & (B=0),\end{cases}$$

by change of variables followed by use of the identities (2.5), (2.6) and (2.7) (with $a=1,c=b+1,b=\dfrac{\alpha_1}{A_1\lambda n}$). This proves the assertion (3.2) of Theorem 1.

Next, in order to prove the assertion (3.4) of Theorem 1, it suffices to show that

$$\inf_{|z|<1}\{\operatorname{Re}Q(z)\}=Q(-1).\tag{3.7}$$

Indeed we have, for $|z|\leq r<1$,

$$\operatorname{Re}\frac{1+Az}{1+Bz}\geq\frac{1-Ar}{1-Br}.$$

Upon setting

$$g(\zeta,z)=\frac{1+A\zeta z}{1+B\zeta z}\text{and } dv(\zeta)=\frac{\alpha_1}{A_1\lambda n}\zeta^{\frac{\alpha_1}{A_1\lambda n}-1}d\zeta\quad(0\leq\zeta\leq 1),$$

which is a positive measure on the closed interval $[0,1]$, we get

$$Q(z)=\int_0^1 g(\zeta,z)dv(\zeta),$$

so that

$$\operatorname{Re} Q(z) \geq \int_0^1 \left(\frac{1 - A\zeta r}{1 - B\zeta r}\right) dv(\zeta) = Q(-r) \quad (|z| \leq r < 1).$$

Letting $r \to 1^-$ in the above inequalities, we obtain the assertion (3.4) of Theorem 1. Finally, the estimate in (3.4) is the best possible as the function Q is the best dominant of (3.2). □

Taking $\lambda = 1, A = 1 - \frac{2\sigma}{p}(0 \leq \sigma < p)$ and $B = -1$ in Theorem 1, we obtain the following corollary.

Corollary 1. *The following inclusion property holds true for the class* $Q^n_{p,q,s}(\alpha_1, A_1, B_1; \theta)$:

$$Q^n_{p,q,s}(\alpha_1 + 1, A_1, B_1; \theta) \subset Q^n_{p,q,s}(\alpha_1, A_1, B_1; \beta(p, n, \alpha_1, A_1, \theta)) \subset Q^n_{p,q,s}(\alpha_1, A_1, B_1; \theta),$$

where

$$\beta(p, n, \alpha_1, A_1, \theta) = \theta + (p - \theta)\left\{ {}_2F_1(1, 1; \frac{\alpha_1}{A_1 n} + 1; \frac{1}{2}) - 1 \right\}.$$

The result is the best possible.

Taking $\lambda = 1$ in Theorem 1, we obtain the following corollary.

Corollary 2. *The following inclusion property holds true for the function class* $Q^n_{p,q,s}(\alpha_1, A_1, B_1; A, B)$ *:*

$$Q^n_{p,q,s}(\alpha_1 + 1, A_1, B_1; A, B) \subset Q^n_{p,q,s}(\alpha_1, A_1, B_1; 1 - \frac{2\theta}{p}, -1) \subset Q^n_{p,q,s}(\alpha_1, A_1, B_1; A, B), 0 \leq \theta < p,$$

where

$$\theta = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 - B)^{-1}_2 F_1(1, 1; \frac{\alpha_1}{A_1 n} + 1; \frac{B}{B-1}) & (B \neq 0), \\ 1 - \frac{\alpha_1 A}{A_1 n + \alpha_1} & (B = 0). \end{cases}$$

The result is the best possible.

Theorem 2. *If* $f \in Q^n_{p,q,s}(\alpha_1, A_1, B_1; \theta)(0 \leq \theta < 1)$*, then*

$$\text{(3.8)} \quad \operatorname{Re} \frac{(1 - \lambda)(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))' + \lambda(\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] f(z))'}{pz^{p-1}} > \theta(|z| < R),$$

where

$$R = \left\{ \frac{\sqrt{\alpha_1^2 + \lambda^2 A_1^2 n^2} - \lambda A_1 n}{\alpha_1} \right\}^{\frac{1}{n}}.$$

The result is the best possible.

Proof. Since $f \in Q^n_{p,q,s}(\alpha_1, A_1, B_1; \theta)$, we write

$$\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1; \theta] f(z))'}{pz^{p-1}} = \theta + (1-\theta)u(z) \ (z \in U). \tag{3.9}$$

Then, clearly, u is of the form (2.1), is analytic in U, and has a positive real part in U. Making use of the identity (1.7) in (3.9) and differentiating the resulting equation with respect to z, we obtain

(3.10)
$$\frac{1}{(1-\theta)}\left\{\frac{(1-\lambda)(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))' + \lambda(\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z))'}{pz^{p-1}} - \theta\right\} = u(z) + \frac{A_1\lambda}{\alpha_1} zu'(z).$$

Now, by applying the well-known estimate [13]

$$\frac{|zu'(z)|}{\operatorname{Re} u(z)} \leq \frac{2nr^n}{1-r^{2n}} \quad (|z| = r < 1)$$

in (3.10), we get

(3.11)
$$\frac{1}{(1-\theta)}\operatorname{Re}\left\{\frac{(1-\lambda)(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))' + \lambda(\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z))'}{pz^{p-1}} - \theta\right\} \geq \operatorname{Re} u(z)\left(1 - \frac{2A_1\lambda nr^n}{\alpha_1(1-r^{2n})}\right).$$

It is easily seen that the right-hand side of (3.11) is positive provided that $r < R$, where R is given as in Theorem 2. This proves the assertion (3.8) of Theorem 2.

In order to show that the bound R is the best possible, we consider the function $f \in A_n(p)$ defined by

$$\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} = \theta + (1-\theta)\frac{1+z^n}{1-z^n} \quad (0 \leq \theta < 1; z \in U).$$

Noting that

$$\frac{1}{(1-\theta)}\left\{\frac{(1-\lambda)(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))' + \lambda(\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z))'}{pz^{p-1}} - \theta\right\} = \frac{\alpha_1 - \alpha_1 z^{2n} - 2A_1\lambda nz^n}{\alpha_1(1-z^n)^2} = 0,$$

for $z = R\exp(\frac{i\pi}{n})$. This completes the proof of Theorem 2. □

Putting $\lambda = 1$ in Theorem 2 , we obtain the following result.

Corollary 3. *If* $f \in Q^n_{p,q,s}(\alpha_1, A_1, B_1; \theta)(0 \leq \theta < 1)$*, then* $f \in Q^n_{p,q,s}(\alpha_1 + 1, A_1, B_1; \theta)$ *for* $|z| < \widetilde{R}$*, where*

$$\widetilde{R} = \left\{ \frac{\sqrt{\alpha_1^2 + A_1^2 n^2} - A_1 n}{\alpha_1} \right\}^{\frac{1}{n}}.$$

The result is the best possible.

For a function $f \in A_n(p)$*, the generalized Bernardi-Libera-Livingston integral operator* $F_{\delta,p}$ *is defined by*

$$\begin{aligned} F_{\delta,p}(f)(z) &= \frac{\delta + p}{z^p} \int_0^z t^{\delta-1} f(t)dt \\ &= \left(z^p + \sum_{k=n}^{\infty} \frac{\delta + p}{\delta + p + k} z^{p+k} \right) * f(z) \ (\delta > -p) \\ &= z^p {}_2F_1(1, \delta + p, \delta + p + 1; z) * f(z). \end{aligned} \tag{3.12}$$

Theorem 3. *Let* $f \in Q^n_{p,q,s}(\alpha_1, A_1, B_1; A, B)$ *and let the operator* $F_{\delta,p}(f)$ *defined by (3.12). Then*

$$\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))'}{pz^{p-1}} \prec \theta(z) \prec \frac{1 + Az}{1 + Bz}, \tag{3.13}$$

where the function θ *given by*

$$\theta(z) = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 + Bz)^{-1}_2 F_1(1, 1; \frac{p+\delta}{n} + 1; \frac{Bz}{Bz+1}) & (B \neq 0), \\ 1 + \frac{(p+\delta)A}{p+\delta+n} z & (B = 0). \end{cases} \tag{3.14}$$

is the best dominant of (3.13). Furthermore,

$$\operatorname{Re}\left\{ \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))'}{pz^{p-1}} \right\} > \xi^* \quad (z \in U), \tag{3.15}$$

where

$$\xi^* = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 - B)^{-1}_2 F_1(1, 1; \frac{p+\delta}{n} + 1; \frac{B}{B+1}) & (B \neq 0) \\ 1 - \frac{(p+\delta)}{p+\delta+n} A & (B = 0). \end{cases} \tag{3.16}$$

The result is the best possible.

Proof. From (1.7) and (3.12) it follows that

$$\begin{aligned} z(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))' &= (p + \delta)(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)) - \\ &\quad \delta(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))'. \end{aligned} \tag{3.17}$$

By setting

$$\varphi(z) = \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))'}{pz^{p-1}} \quad (z \in U), \tag{3.18}$$

we note that $\varphi(z)$ is of the form (2.1) and is analytic in U. Using the identity (3.17) in (3.18), and then differentiating the resulting equation with respect to z, we obtain

$$\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} = \varphi(z) + \frac{z\varphi'(z)}{p+\delta} \prec \frac{1+Az}{1+Bz}.$$

Now the remaining part of Theorem 3 follows by employing the techniques that we used in proving Theorem 1 above. □

Remark 1. We observe that.

$$\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))}{pz^{p-1}} = \frac{\delta+p}{pz^{\delta+p}} \int_0^z t^\delta (\theta_{p,q,s}[\alpha_1, A_1, B_1] f(t))' dt \tag{3.19}$$

$$(f \in A_n(p); z \in U).$$

In view of (3.19), Theorem 3 for $A = 1 - 2\mu (0 \le \mu < 1)$ and $B = -1$ yields the following corollary.

Corollary 4. *If $\delta > 0$ and if $f \in A_n(p)$satisfies the following inequality*

$$\operatorname{Re} \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} > \mu \ (0 \le \mu < 1; z \in U),$$

then

$$\operatorname{Re} \frac{\delta+p}{pz^{p+\delta}} \int_0^z t^\delta (\theta_{p,q,s}[\alpha_1, A_1, B_1] f(t))' dt >$$

$$\mu + (1-\mu)\left[{}_2F_1\left(1, 1; \frac{p+\delta}{n} + 1; \frac{1}{2}\right) - 1\right] (z \in U).$$

The result is the best possible.

Theorem 4. *Let $f \in A_n(p)$. Suppose also that $g \in A_n(p)$ satisfies the following inequality:*

$$\operatorname{Re} \frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)}{z^p} > 0 \ (z \in U).$$

If

$$\left| \frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)}{\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)} - 1 \right| < 1 \ (z \in U),$$

then

$$\operatorname{Re} \frac{z(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)} > 0 \quad (|z| < R_0),$$

where

$$R_0 = \frac{\sqrt{9n^2 + 4p(p+n)} - 3n}{2(p+n)}.$$

Proof. Letting

$$w(z) = \frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)}{\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)} - 1 = k_n z^n + k_{n+1} z^{n+1} + \dots, \tag{3.20}$$

we note that w is analytic in U, with

$$w(0) = 0 \text{and } |w(z)| \leq |z|^n \quad (z \in U).$$

Then, by applying the familiar Schwarz lemma [16], we obtain

$$w(z) = z^n \Psi(z),$$

where the function Ψ is analytic in U and $|\Psi(z)| \leq 1 (z \in U)$. Therefore, (3.20) leads us to

$$\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z) = \theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)(1 + z^n \Psi(z)) \ (z \in U). \tag{3.21}$$

Differentiating (3.21) logarithmically with respect to z, we obtain

$$\frac{z(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)} = \frac{z(\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z))'}{\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)} + \frac{z^n\{n\Psi(z) + z\Psi'(z)\}}{1 + z^n \Psi(z)}. \tag{3.22}$$

Putting $\varphi(z) = \frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)}{z^p}$,we see that the function $\varphi(z)$ is of the form (2.1), is analytic in U, $\operatorname{Re}\varphi(z) > 0 (z \in U)$ and

$$\frac{z(\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z))'}{\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)} = \frac{z\varphi'(z)}{\varphi(z)} + p,$$

so that we find from (3.22) that

$$\operatorname{Re} \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)} \geq p - \left|\frac{z\varphi'(z)}{\varphi(z)}\right| - \left|\frac{z^n\left\{n\Psi(z) + z\Psi'(z)\right\}}{1 + z^n \Psi(z)}\right| \quad (z \in U). \tag{3.23}$$

Now, by using the following known estimates [13] (see also [21]) :

$$\left|\frac{\varphi'(z)}{\varphi(z)}\right| \leq \frac{2nr^{n-1}}{1 - r^{2n}} \text{ and } \left|\frac{n\Psi(z) + z\Psi'(z)}{1 + z^n \Psi(z)}\right| \leq \frac{n}{1 - r^n} \quad (|z| = r < 1),$$

in (3.21), we obtain

$$\operatorname{Re} \frac{z(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)} \geq \frac{p - 3nr^n - (p+n)r^{2n}}{1 - r^{2n}} \ (|z| = r < 1),$$

which is certainly positive, provided that $r < R_0, R_0$ being given as in Theorem 4. □

Theorem 5. *Let $-1 \leq D_j < C_j \leq 1 (j = 1, 2)$. If each of the functions $f_j \in A_n(p)$ satisfies the following subordination condition:*

$$(1-\lambda)\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f_j(z)}{z^p} + \lambda\frac{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] f_j(z)}{z^p} \prec \frac{1 + C_j z}{1 + D_j z}, \tag{3.24}$$

then

$$(1-\lambda)\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] G(z)}{z^p} + \lambda\frac{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] G(z)}{z^p} \prec \frac{1 + (1-2\eta)z}{1 - z}, \tag{3.25}$$

where

$$G(z) = \theta_{p,q,s}[\alpha_1, A_1, B_1](f_1 * f_2)(z)$$

and

$$\eta = 1 - \frac{4(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)}\left[1 - \frac{1}{2}{}_2F_1(1, 1; \frac{\alpha_1}{A_1\lambda} + 1; \frac{1}{2})\right].$$

The result is the best possible when $D_1 = D_2 = -1$.

Proof. Suppose that each of the functions $f_j \in A_n(p) (j = 1, 2)$ satisfies the condition (3.24). Then, by letting

$$\varphi_j(z) = (1-\lambda)\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f_j(z)}{z^p} + \lambda\frac{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] f_j(z)}{z^p} \quad (j = 1, 2), \tag{3.26}$$

we have

$$\varphi_j(z) \in P(\gamma_j) \quad \left(\gamma_j = \frac{1 - C_j}{1 - D_j}; j = 1, 2\right).$$

By making use of identity (1.7) in (3.26), we observe that

$$\theta_{p,q,s}[\alpha_1, A_1, B_1] f_j(z) = \frac{\alpha_1}{A_1\lambda} z^{p - \frac{\alpha_1}{A_1\lambda}} \int_0^z t^{\frac{\alpha_1}{A_1\lambda} - 1} \varphi_j(t)dt \quad (j = 1, 2),$$

which in view of the definition of G given already with (3.25) yields

$$\theta_{p,q,s}[\alpha_1, A_1, B_1] G(z) = \frac{\alpha_1}{A_1\lambda} z^{p - \frac{\alpha_1}{A_1\lambda}} \int_0^z t^{\frac{\alpha_1}{A_1\lambda} - 1} \varphi_0(t)dt, \tag{3.27}$$

where, for convenience,

$$\begin{aligned} \varphi_0(z) &= (1-\lambda)\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] G(z)}{z^p} + \frac{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] G(z)}{z^p} \\ &= \frac{\alpha_1}{A_1\lambda} z^{-\frac{\alpha_1}{A_1\lambda}} \int_0^z t^{\frac{\alpha_1}{A_1\lambda} - 1} (\varphi_1 * \varphi_2)(t)dt. \end{aligned} \tag{3.28}$$

Since $\varphi_1 \in P(\gamma_1)$ and $\varphi_2 \in P(\gamma_2)$, it follows from Lemma 3 that

$$(\varphi_1 * \varphi_2)(z) \in P(\gamma_3)\ (\gamma_3 = 1 - 2(1-\gamma_1)((1-\gamma_2)). \tag{3.29}$$

Now, by using (3.29) in (3.28) and then appealing to Lemma 2 and Lemma 4, we get

$$\begin{aligned}
\operatorname{Re}\{\varphi_0(z)\} &= \frac{\alpha_1}{A_1\lambda}\int_0^1 u^{\frac{\alpha_1}{A_1\lambda}-1}\operatorname{Re}\{(\varphi_1 * \varphi_2)\}(uz)du \\
&\geq \frac{\alpha_1}{A_1\lambda}\int_0^1 u^{\frac{\alpha_1}{A_1\lambda}-1}(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u|z|})du \\
&> \frac{\alpha_1}{A_1\lambda}\int_0^1 u^{\frac{\alpha_1}{A_1\lambda}-1}(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u})du \\
&= 1 - \frac{4(C_1-D_1)(C_2-D_2)}{(1-D_1)(1-D_2)}\left(1 - \frac{\alpha_1}{A_1\lambda}\int_0^1 u^{\frac{\alpha_1}{A_1\lambda}-1}(1+u)^{-1}du\right) \\
&= 1 - \frac{4(C_1-D_1)(C_2-D_2)}{(1-D_1)(1-D_2)}\left[1 - \frac{1}{2}{}_2F_1(1,1;\frac{\alpha_1}{A_1\lambda}+1;\frac{1}{2})\right] \\
&= \eta (z \in U).
\end{aligned}$$

When $D_1 = D_2 = -1$, we consider the functions $f_j(z) \in A_n(p) (j = 1,2)$, which satisfy the hypothesis (3.24) of Theorem 5 and are defined by

$$\theta_{p,q,s}[\alpha_1, A_1, B_1] f_j(z) = \frac{\alpha_1}{A_1\lambda} z^{-\frac{\alpha_1}{A_1\lambda}} \int_0^z t^{\frac{\alpha_1}{A_1\lambda}-1}(\frac{1+C_j t}{1-t})dt \ \ (j = 1,2).$$

Thus it follows from (3.28) and Lemma 2 that

$$\begin{aligned}
\varphi_0(z) &= \frac{\alpha_1}{A_1\lambda}\int_0^1 u^{\frac{\alpha_1}{A_1\lambda}-1}\left\{1 - (1+C_1)(1+C_2) + \frac{(1+C_1)(1+C_2)}{1-uz}\right\}du \\
&= 1 - (1+C_1)(1+C_2) + (1+C_1)(1+C_2)(1-z)^{-1}_2F_1\left(1,1;\frac{\alpha_1}{A_1\lambda}+1;\frac{z}{z-1}\right) \\
&\to 1 - (1+C_1)(1+C_2) + \frac{1}{2}(1+C_1)(1+C_2){}_2F_1\left(1,1;\frac{\alpha_1}{A_1\lambda}+1;\frac{1}{2}\right)
\end{aligned}$$

as $z \to 1^-$, which completes the proof of Theorem 5. □

Remark 2. Taking $A_i = 1 (i = 1, \ldots, q), B_i = 1 (i = 1, \ldots, s)$ and $j = 1$ in Theorem 5, we obtain the result obtained by Liu [11, Theorem 2.4].

Putting $A_i = 1 (i = 1, \ldots, q), B_i = 1 (i = 1, \ldots, s), C_j = 1 - 2\theta_j (0 \leq \theta_j < 1), D_j = 1 (j = 1, 2), q = s + 1, \alpha_1 = \beta_1 = p, \alpha_j = 1 (j = 2, 3, \ldots, s + 1)$ and $\beta_j = 1 (j = 2, 3, \ldots, s)$ in Theorem 5, we obtain the following result.

Corollary 5. *If the functions $f_j \in A_n(p) (j = 1, 2)$ satisfy the following inequality:*

$$\operatorname{Re}\left\{(1-\lambda)\frac{f_j(z)}{z^p} + \lambda\frac{f_j'(z)}{pz^{p-1}}\right\} > \theta_j \ \ (0 \leq \theta_j < 1; j = 1, 2; z \in U), \tag{3.30}$$

then

$$\operatorname{Re}\left\{(1-\lambda)\frac{(f_1 * f_2)(z)}{z^p} + \lambda\frac{z(f_1 * f_2)'(z)}{pz^{p-1}}\right\} > \eta_0 \ \ (z \in U),$$

where

$$\eta_0 = 1 - 4(1-\theta_1)(1-\theta_2)\left[1 - \frac{1}{2}{}_2F_1(1, 1; \frac{p}{\lambda} + 1; \frac{1}{2})\right].$$

The result is the best possible.

Theorem 6. *Let the function f be defined by (1.1) be in the class $Q^n_{p,q,s}[\alpha_1, A_1, B_1; A, B]$ and let $g \in A_n(p)$ satisfy the following inequality:*

$$\operatorname{Re}\frac{g(z)}{z^p} > \frac{1}{2} \quad (z \in U).$$

Then

$$(f * g)(z) \in Q^n_{p,q,s}[\alpha_1, A_1, B_1; A, B].$$

Proof. We have

$$\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1](f * g)(z))'}{pz^{p-1}} = \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1]f(z))'}{pz^{p-1}} * \frac{g(z)}{z^p} \ (z \in U).$$

Since

$$\operatorname{Re}\frac{g(z)}{z^p} > \frac{1}{2} \quad (z \in U)$$

and the function

$$\frac{1 + Az}{1 + Bz}$$

is convex (univalent) in U, it follows from (1.8) and Lemma 4 that $(f * g) \in Q^n_{p,q,s}[\alpha_1, A_1, B_1; A, B]$. □

Theorem 7. *Let $\alpha_1 > 0, \nu \in \mathbb{C}^*$ and let $A, B \in \mathbb{C}$ with $A \neq B$ and $|B| \leq 1$. Suppose that*

$$\left|\frac{\frac{\nu\alpha_1}{A_1}(A-B)}{B} - 1\right| \leq 1$$

or

$$\left|\frac{\frac{\nu\alpha_1}{A_1}(A-B)}{B} + 1\right| \leq 1$$

if $B \neq 0$, *and*

$$|\nu\pi| \leq \frac{A_1\pi}{\alpha_1},$$

if $B = 0$.

If $f \in A_n(p)$ *with* $\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z) \neq 0$ *for all* $z \in U^* = U\backslash\{0\}$, *then*

$$\frac{\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z)}{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)} \prec \frac{1+Az}{1+Bz}$$

implies

$$\left(\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)}{z^p}\right)^{\nu} \prec q_1(z),$$

where

$$q_1(z) = \begin{cases} (1+Bz)^{\frac{v\alpha_1}{A_1}(\frac{A-B}{B})}, & \text{if } B \neq 0, \\ e^{\frac{\nu\alpha_1}{A_1}Az}, & \text{if } B = 0, \end{cases}$$

is the best dominant. (All the powers are the principal ones).

Proof. Let us put

$$\varphi(z) = \left(\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)}{z^p}\right)^{\nu} \quad (z \in U), \tag{3.31}$$

where the power is the principal one, then $\varphi(z)$ is analytic in U, $\varphi(0) = 1$ and $\varphi(z) \neq 0$ for $z \in U$. Taking the logarithmic derivatives in both sides of (3.31), multiplying by z and using the identity (1.7), we have

$$1 + \frac{z\varphi'(z)}{\frac{\nu\alpha_1}{A_1}\varphi(z)} = \frac{\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z)}{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)} \prec \frac{1+Az}{1+Bz}.$$

Now the assertions of Theorem 7 follows by using Lemma 5 with $\gamma = \frac{\nu\alpha_1}{A_1}$. This completes the proof of Theorem 7. □

Remark 3. Putting $A_i = 1 (i = 1, \dots, q), B_i = 1 (i = 1, \dots, s), A = 1 - 2\rho, 0 \leq \rho < 1$ and $B = -1$in Theorem 7, we obtain the result obtained by Liu [11, Theorem 5].

Putting $A = 1 - \frac{2\eta}{p}(0 \leq \eta < p), B = -1, A_i = 1(i = 1, \dots, q), B_i = 1(i = 1, \dots, s), n = 1, q = s+1, \alpha_1 = \beta_1 = p, \alpha_j = 1(j = 2, \dots, s+1)$, and $\beta_j = 1(j = 2, \dots s)$ in Theorem 7, we obtain the following corollary.

Corollary 6. *Assume that* $\nu \in \mathbb{C}^*$ *satisfies either*

$$|2\nu(\eta - p) - 1| \leq 1 \ \text{or} \ |2\nu(\eta - p) + 1| \leq 1.$$

If the function $f \in A(p)$ *with* $f(z) \neq 0$ *for* $z \in U^*$ *satisfy the following inequality:*

$$\text{Re}\,\frac{zf'(z)}{f(z)} > \eta (0 \leq \eta < p),$$

then

$$\left(\frac{f(z)}{z^p}\right)^{\nu} \prec q_2(z) \qquad (z \in U) \ ,$$

where

$$q_2(z) = (1-z)^{-2\nu(p-\eta)}(z \in U),$$

is the best dominant.

References

[1] M. K. Aouf and J. Dziok, Certain class of analytic functions associated with the Wright generalized hypergeometric function, J. Math. Appl. 30(2008), 23-32.

[2] T. Bulboaca, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.

[3] B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, SIAM J. Math. Anal. 15(1984), 737-745.

[4] N. E. Cho, O. S. Kwon and H. M. Srivastava, Inclusion properties and arguments properties for certain subclasses of multivalent functions associated with a family of linear operators, J. Math. Anal. Appl. 292(2004), 470-483.

[5] J. H. Choi, M. Saigo and H. M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl. 276(2002), 432-445.

[6] J. Dziok, Classes of analytic functions involving some integral operator, Folia Sci. Univ. Tech.Resoviensis 20(1995), 21-39.

[7] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput. 103 (1999), 1-13.

[8] J. Dziok and R. K. Raina, Families of analytic functions associated with the Wright generalized hypergeometric function, Demonstratio Math. 37(2004), no.3, 533-542.

[9] D. J. Hallenbeck and St. Ruscheweyh, Subordinations by convex functions, Proc. Amer. Math. Soc. 52(1975), 191-195.

[10] Vinod Kumar and S. L. Shukla, Multivalent functions defined by Ruscheweyh derivatives. I and II, Indian J. Pure Appl. Math. 15(1984), no. 11, 1216-1227, 15(1984), no. 1, 1228-1238.

[11] J. -L. Liu, On subordinations for certain multivalent analytic functions associated with the generalized hypergeometric function, J. Inequal. Pure Appl. Math. 7(2006), no. 4, Art. 131, 1-6.

[12] J. -L. Liu and K. I. Noor, Some properties of Noor integral operator, J. Natur. Geom. 21(2002), 81-90.

[13] T. H. MacGregor, Radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 14(1963), 514-520.

[14] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J. 28(1981), 157-171.

[15] S. S. Miller and P. T. Mocanu, Differential Subordinations : Theory and Applications, Series on Monographs and Texbooks in Pure and Applied Mathematics, Vol.225, Marcel Dekker, New York and Basel, 2000.

[16] Z. Nehari, Conformal Mapping, MaGraw-Hill, New York, 1952.

[17] M. Obradovic and S. Owa, On certain properties for some classes of starlike functions, J. Math. Anal. Appl. 145(1990), 357-364.

[18] S.Owa, On the distortion theorems. I, Kyungpook Math. J. 18(1978) 53-59.

[19] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, Canad. J. Math. 39(1987), 1057-1077.

[20] D. Z. Pashkouleva, The starlikeness and spiral-convexity of certain subclasses of analytic functions, in : H. M. Srivastava and S. Owa (Editors), Current Topics in Analytic Function Theory, 266-273, World Scientific Publishing and Hong Kong, 1992.
[21] J. Patel, Radii of $\gamma-$ spiralikeness of certain analytic functions, J. Math. Phys. Sci. 27(1993), 321-334.
[22] J. Patel, A. K. Mishra and H. M. Srivastava, Classes of multivalent analytic functions involving the Dziok-Srivastava operator, Comput. Math. Appl. 54(2007), 599-616.
[23] R. K. Raina, On certain classes of analytic functions and application to fractional calculas operator, Integral Transform. Spec. Funct. 5(1997), 247-260.
[24] R. K. Raina and T. S. Nahar, On univalent and starlike Wright generalized hypergeometric functions, Rend. Sen. Math. Univ. Padova 95(1996),11-22.
[25] H. Saitoh, A linear operator and its applications of first order differential subordinations, Math. Japon. 44(1996), 31-38.
[26] R. Singh and S. Singh, Convolution properties of a class of starlike functions, Proc. Amer. Math. Soc. 106(1989), 145-152.
[27] H. M. Srivastava and M. K. Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. I and II, J. Math. Anal. Appl. 171(1992), 1-13, 192(1995), 673-688.
[28] H. M. Srivastava and P. W. Karlsson, Multiple Gausian Hypergeometric Series, Halsted Press, Ellis Horwood Limited, Chichester, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1985.
[29] J. Stankiewicz and Z. Stankiewicz, Some applications of the Hadamard convolution in the theory of functions Ann. Univ. Mariae Curie-Sklodowska, Sect. A 40(1986), 251-165.
[30] E. T. Whittaker and G. N. Wastson, A Course on Modern Analysis : An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Accout of the Principal Transcenclental Functions, Fourth Edition (Reprinted), Cambridg Univ. Press, Camridge, 1927.
[31] E. M. Wright, The asymptotic expansion of the generalized hypergeometric functions, Proc. London Math. Soc. 46(1946), 389-408.

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