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SOME PROPERTIES FOR INTEGRAL OPERATORS ON SOME ANALYTIC FUNCTIONS WITH COMPLEX ORDER

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Abstract. In this paper we obtain some properties for two general integral operators on analytic functions with complex order.

1. Introduction

Let $U = \{z \in C : |z| < 1\}$ be the unit disk of the complex plane and denote by $\mathcal{H}(U)$, the class of the holomorphic functions in U. Consider A the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk U. In [7] Wiatrowski introduced the class of convex functions of order $b \in C$ $(b \neq 0)$ defined as follows:

Definition 1. A function $f(z) \in A$ is said to be convex function of order $b, (b \in C - \{0\})$, that is, $f \in C(b)$, if and only if $f'(z) \neq 0$ in U and

$$\mathbf{Re}\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > 0, \tag{1}$$

for all $z \in U$

In [6] Nasr and Aouf introduced the class $S(1-b), b \neq 0$, complex, of starlike functions of order $1 - b$, defined as follows:

Definition 2. A function $f(z) \in A$ is said to be starlike function of order $1-b$ $(b \in C - \{0\})$, that is $f \in S(1 - b)$, if and only if $f(z)/z \neq 0$ in U and

$$\mathbf{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0, \tag{2}$$

for all $z \in U$.

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In [4] Frasin studied the classes $S^*_\alpha(b)$ and $C_\alpha(b)$, where the classes are defined as follows:

Definition 3. A function $f(z) \in A$ is said to be a starlike of complex order $b, (b \in C - \{0\})$ and type $\alpha, (0 \leq \alpha < 1)$, that is $f \in S^*_\alpha(b)$, if and only if

$$\mathbf{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > \alpha, \tag{3}$$

for all $z \in U$.

Definition 4. A function $f(z) \in \mathcal{A}$ is said to be convex of complex order $b, (b \in C - \{0\})$ and type $\alpha, (0 \leq \alpha < 1)$, that is $f \in C_\alpha(b)$, if and only if

$$\mathbf{Re}\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > \alpha, \tag{4}$$

for all $z \in U$.

Remark 1. We note that for $\alpha = 0$ we have $C_0(b) = C(b)$ and $S^*_0(b) = S(1-b)$. Also we note that $S^*_\alpha(\cos\lambda e^{-i\lambda}) = S^\lambda_\alpha(\|\lambda| < \frac{\pi}{2},\ 0 \leq \alpha < 1)$, the class of λ-spirallike functions of order α, was introduced by Libera [5] and $C_\alpha(\cos\lambda e^{-i\lambda}) = C^\lambda_\alpha(\|\lambda| < \frac{\pi}{2},\ 0 \leq \alpha < 1)$ the class of λ-Roberton function of order α was introduced by Chichra [3].

In this paper, we consider two integral operators, the first operator is defined as follows:

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt\ (\alpha_i > 0). \tag{5}$$

This operator was introduced by Breaz and Breaz [1] and this second operator is defined as follows:

$$F_{\alpha_1,\dots,\alpha_n}(z) = \int_0^z (f_1'(t))^{\alpha_1} \dots (f_n'(t))^{\alpha_n}\, dt\, (\alpha_i > 0). \tag{6}$$

This operator was introduced by Breaz, Owa and Breaz [2].

In this paper we shall study some properties for functions belonging to the classes $S^*_\alpha(b)$ and $C_\alpha(b)$.

2. Main Results

Theorem 1. *Let $\alpha_i, i \in \{1,\dots,n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1,\dots,n\}$, and*

$$0 \leq 1 - \sum_{i=1}^{n} \alpha_i < 1. \tag{7}$$

We suppose that the functions $f_i \in S(1-b)$ for $i = \{1,\dots,n\}$ and $b \in C - \{0\}$. Then we have the integral operator $F_n \in C_\gamma(b)$, where $\gamma = 1 - \sum_{i=1}^{n} \alpha_i$.

Proof. We calculate for F_n the derivatives of the first and second order. From (5) we obtain:

$$F_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdots \left(\frac{f_n(z)}{z}\right)^{\alpha_n}$$

and

$$F_n''(z) = \sum_{i=1}^{n} \alpha_i \left(\frac{z f_i'(z) - f_i(z)}{z f_i(z)}\right) F_n'(z).$$

Then we have

$$(8)\qquad \begin{aligned} \frac{F_n''(z)}{F_n'(z)} &= \alpha_1 \left(\frac{z f_1'(z) - f_1(z)}{z f_1(z)}\right) + \cdots + \alpha_n \left(\frac{z f_n'(z) - f_n(z)}{z f_n(z)}\right), \\ \frac{F_n''(z)}{F_n'(z)} &= \alpha_1 \left(\frac{f_1'(z)}{f_1(z)} - \frac{1}{z}\right) + \cdots + \alpha_n \left(\frac{f_n'(z)}{f_n(z)} - \frac{1}{z}\right). \end{aligned}$$

Multiply the relation (8) with z we obtain:

$$(9)\qquad \frac{z F_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{z f_i'(z)}{f_i(z)} - 1\right).$$

Multiply the relation (9) with $\frac{1}{b}$ we obtain:
(10)

$$\frac{1}{b}\frac{z F_n''(z)}{F_n'(z)} = \frac{1}{b}\sum_{i=1}^{n} \alpha_i \left(\frac{z f_i'(z)}{f_i(z)} - 1\right) = \sum_{i=1}^{n} \alpha_i \left[1 + \frac{1}{b}\left(\frac{z f_i'(z)}{f_i(z)} - 1\right)\right] - \sum_{i=1}^{n} \alpha_i.$$

The relation (10) is equivalent to:

$$(11)\qquad 1 + \frac{1}{b}\frac{z F_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left[1 + \frac{1}{b}\left(\frac{z f_i'(z)}{f_i(z)} - 1\right)\right] - \sum_{i=1}^{n} \alpha_i + 1.$$

By equalising the real parts of the above inequality we obtain:

$$(12)\qquad \mathbf{Re}\left\{1 + \frac{1}{b}\frac{z F_n''(z)}{F_n'(z)}\right\} = \sum_{i=1}^{n} \alpha_i \mathbf{Re}\left[1 + \frac{1}{b}\left(\frac{z f_i'(z)}{f_i(z)} - 1\right)\right] - \sum_{i=1}^{n} \alpha_i + 1.$$

Since $f_i \in S(1-b)$ for $i = \{1, \ldots, n\}$, we apply in the above relation the inequality (2) and obtain:

$$(13)\qquad \mathbf{Re}\left\{1 + \frac{1}{b}\frac{z F_n''(z)}{F_n'(z)}\right\} > 1 - \sum_{i=1}^{n} \alpha_i.$$

Because $0 \le 1 - \sum_{i=1}^{n} \alpha_i < 1$, we have that $F_n \in C_\gamma(b)$, where $\gamma = 1 - \sum_{i=1}^{n} \alpha_i$. □

Corollary 1. *Let $\alpha_1 > 0$. If $0 \le 1 - \alpha_1 < 1$ and the function $f_1 \in S(1-b)$, then the integral operator $F_1 \in \mathcal{C}_\rho(b)$, where $\rho = 1 - \alpha_1$.*

Proof. Putting $n = 1$ in Theorem 1 we obtain the result. □

Putting $b = \cos\lambda e^{-i\lambda}(|\lambda| < \frac{\pi}{2}|)$ in Theorem 1, we obtain the following corollary:

Corollary 2. *Let the functions* $f_i \in S^\lambda(|\lambda| < \frac{\pi}{2}|)$, *for all* $i \in \{1, 2, \ldots, n\}$. *Then the integral operator* $F_n \in C^\lambda(\gamma)$, *where* $\gamma = 1 - \sum_{i=1}^{n} \alpha_i$, $\alpha_i > 0$ *and* $0 \leq 1 - \sum_{i=1}^{n} \alpha_i < 1$.

Theorem 2. *Let the functions* $f_i \in C(b)$, *and* $b \in C - \{0\}$, *for all* $i \in \{1, \ldots, n\}$. *Then the integral operator* $F_{\alpha_1,\ldots,\alpha_n} \in C_\eta(b)$, *where* $\eta = 1 - \sum_{i=1}^{n} \alpha_i$ *and* $0 \leq 1 - \sum_{i=1}^{n} \alpha_i < 1$.

Proof. From (6), we have

$$\frac{F''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)} = \alpha_1 \frac{f''_1(z)}{f'_1(z)} + \cdots + \alpha_n \frac{f''_n(z)}{f'_n(z)}. \tag{14}$$

We multiply both sides of (14) with $\frac{z}{b}$, we obtain that

$$\frac{1}{b}\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)} = \alpha_1 \frac{1}{b}\frac{zf''_1(z)}{f'_1(z)} + \cdots + \alpha_n \frac{1}{b}\frac{zf''_n(z)}{f'_n(z)}. \tag{15}$$

From the relation (15) we obtain that:

$$\mathbf{Re}\left(\frac{1}{b}\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)} + 1\right) = \sum_{i=1}^{n} \alpha_i \mathbf{Re}\left(1 + \frac{1}{b}\frac{zf''_i(z)}{f'_i(z)}\right) - \sum_{i=1}^{n} \alpha_i + 1. \tag{16}$$

Since $f_i \in C(b)$, we have

$$\mathbf{Re}\left(\frac{1}{b}\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)} + 1\right) > 1 - \sum_{i=1}^{n} \alpha_i. \tag{17}$$

Since $0 \leq 1 - \sum_{i=1}^{n} \alpha_i < 1$, the relation (17) implies that the integral operator $F_{\alpha_1,\ldots,\alpha_n} \in C_\eta(b)$, where $\eta = 1 - \sum_{i=1}^{n} \alpha_i$. □

Corollary 3. *Let the function* $f_1 \in C(b)$. *Then the integral operator* $F_{\alpha_1} \in \mathcal{C}_\sigma(b)$, *where* $\sigma = 1 - \alpha_1$ *and* $0 \leq 1 - \alpha_1 < 1$.

Proof. Putting $n = 1$ in Theorem 2, we obtain the result. □

Remark 2. Putting $b = \cos\lambda e^{-i\lambda}(|\lambda| < \frac{\pi}{2}|)$ in Theorem 2, we have the following corollary:

Corollary 4. *Let the functions* $f_i \in C^{\lambda}(|\lambda| < \frac{\pi}{2}|)$*, for all* $i \in \{1, 2, \ldots, n\}$*. Then the integral operator* $F_{\alpha_1,\ldots,\alpha_n} \in C^{\lambda}(\eta)$*, where* $\eta = 1 - \sum_{i=1}^{n} \alpha_i$ *and* $0 \leq 1 - \sum_{i=1}^{n} \alpha_i < 1$.

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