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HENSTOCK-KURZWEIL TYPE INTEGRALS IN $\mathcal{P}$-ADIC HARMONIC ANALYSIS

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Dedicated to Professor W.R. Wade on the occasion of his 60th birthday

Abstract. Some recent results related to the $\mathcal{P}$-adic derivatives and integrals are surveyed. Applications of the Henstock-Kurzweil $\mathcal{P}$-integral and the Perron $\mathcal{P}$-integral to the problem of recovering the coefficients of series with respect to the Vilenkin system and the Haar system (both in one dimension and in higher dimensions) are discussed. The case of the continual analogue of the Vilenkin system is also considered.

1. Introduction

There are many areas in harmonic analysis which require integration processes more powerful than the Lebesgue integration and which involve some kinds of generalized derivatives. For example some important methods of summation of trigonometric series are based on symmetric derivatives of different orders. In particular the basic notion of the Riemann theory of trigonometric series is the Riemann-Schwarz second order symmetric derivative. Generalized integrals which solve the problem of recovering the coefficients of convegent trigonometric series from their sums, are also based on symmetric derivatives (see [58]).

In the case of the dyadic harmonic analysis and its $\mathcal{P}$-adic generalization, a role similar to the one of symmetric derivatives in the classical trigonometric case, is played by derivatives and integrals defined in terms of dyadic or more general $\mathcal{P}$-adic basis of differentiation.

In Section 3 of this paper we survey some new results related to the dyadic and $\mathcal{P}$-adic derivatives and integrals. Our primary concern here is Henstock-Kurzweil theory of integration in application to the dyadic and the $\mathcal{P}$-adic bases.

In Section 4 we consider some application of those derivatives and integrals to the theory of series with respect to Walsh, Haar and Vilenkin multiplicative systems and first of all, to the problem of recovering the coefficients of those series from their sums by generalized Fourier formulas. An extention of these results to the case of multiplicative transforms is given in Section 5.

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We should underline that talking about dyadic and $\mathcal{P}$-adic derivatives and integrals we do not refer to the popular notions introduced by Butzer and Wagner and widely used nowadays in dyadic analysis (see [37]), but we have in mind more classical concepts of differentiation and integration with respect to the dyadic or $\mathcal{P}$-adic interval bases. (In some literature, see for example [36], this kind of derivatives are called the derivatives with respect to a sequence of nets). This unfortunate but already established double meaning of the terms will not cause any confusion here because only the latter concept are discussed in this paper.

2. Preliminaries

The Vilenkin multiplicative systems are known to be the groups of characters of the respective compact abelian 0-dimensional groups. In particular the Walsh system is the group of characters of the Cantor dyadic group. Therefore those 0-dimensional groups represent natural domains on which the functions from this systems are defined, and the group structure plays an important role in the whole Walsh-Fourier and Vilenkin-Fourier analysis.

But in some cases it is more convenient having mapped those groups on the unit interval $[0, 1)$, to identify them with this interval and to use real line terminology also for elements of factor-groups of those groups. It is a point where the respective $\mathcal{P}$-adic intervals appear in this theory.

We recall some basic notations and definitions. For the sake of simplicity we restrict ourself here to a special case of Vilenkin 0-dimensional groups, namely $\mathcal{P}$-adic groups which are defined by sequenses $\mathcal{P}$ of natural numbers as follows. For a fixed sequence of natural numbers

$$\mathcal{P} = \{p_j\}_{j=0}^{\infty}, \qquad p_j \geq 2 \tag{1}$$

we define a group $G(\mathcal{P})$ of sequences $x = \{x_j\}_{j=0}^{\infty}$ where x_j are integers,

$$0 \leq x_j \leq p_j - 1$$

for all $j \geq 0$, and the group operation is defined as coordinatewise addition module p_j for the jth coordinate. So the group $G(\mathcal{P})$ is in fact the direct product of the discrete cyclic groups Z_{p_j}.

For the sequence $\mathcal{P}$ we put

$$m_0 = 1 \text{ and } m_k = \prod_{s=0}^{k-1} p_s \text{ for } k \geq 1. \tag{2}$$

Then each non-negative integer n has a unique representation of the form

$$n = \sum_{k=0}^{\infty} n_k m_k,$$

where each n_k is an integer satisfying $0 \leq n_k < p_k$. Using this representation, the characters χ_n of the group $G(\mathcal{P})$ can be described as

$$\chi_n(x) = \exp\left(2\pi i \sum_{k=0}^{\infty} \frac{x_k n_k}{p_k}\right). \tag{3}$$

We get in this way a family of Vilenkin multiplicative systems, each one corresponding to a fixed sequence $\mathcal{P}$ (see [1] for details). In a special case when the elements p_j of the sequence (1) are identically 2 we get the Walsh system in Paley enumeration.

Considering an element $x = \{x_0, x_1, x_2, \dots, x_j, \dots\}$ of the group $G(\mathcal{P})$ as a sequence of the coefficients of the $\mathcal{P}$-adic expansion

$$\sum_{j=0}^{\infty} \frac{x_j}{m_j}$$

of a number $\lambda(x)$, we can map this group on the interval $[0,1)$. This mapping is one-one if we agree to accept only finite expansions for $\mathcal{P}$-adic rational points of $[0,1)$, i.e. points of the form r/m_k, $r = 0, 1, \dots, m_k$. In this way we can consider the above system of characters χ_n as an orthogonal system of functions on $[0,1)$.

For a fixed sequence (1) we consider intervals

$$\left[\frac{r}{m_k}, \frac{r+1}{m_k}\right] = I_r^{(k)}, \qquad r = 0, 1, \dots, m_k - 1. \tag{4}$$

For a fixed $k = 0, 1, \dots,$ we call those intervals *$\mathcal{P}$-adic intervals* (or simply *$\mathcal{P}$-intervals*) *of rank k*. It is worth noting that under above mapping λ the interval $I_0^{(k)}$ is the closure of the image, under mapping λ, of a subgroup of the group $G(\mathcal{P})$ defined for $k \geq 1$ as $G_k(\mathcal{P}) = \{x : x_0 = x_1 = \dots = x_{k-1} = 0\}$, and the intervals $I_r^{(k)}$ are the closure of the images of the respective cosets of this subgroup in the group $G(\mathcal{P})$.

In what follows we shall identify a group element x with its image $\lambda(x)$. Note that for each $\mathcal{P}$-adic irrational point x, there exists only one $\mathcal{P}$-interval

$$I_x^{(k)} = [a_k(x), b_k(x)]$$

of rank k containing x so that $\{x\} = \bigcap_{k=0}^{\infty} I_x^{(k)}$ and we say that the sequence $\{I_x^{(k)}\}$ of nested $\mathcal{P}$-intervals is the *basic sequence of $\mathcal{P}$-intervals convergent to x*. If x is a $\mathcal{P}$-adic rational point different from 0 and 1, then there exist two decreasing sequences of $\mathcal{P}$-intervals for which x is a common end-point starting with some k, i.e. for such a point we have two basic sequences convergent to x: the left one and the right one.

Another orthogonal system which appears often in dyadic analysis is the Haar system. Although it is not a system of characters of any group it is very popular in harmonic analysis as being the simplest example of a wavelet system. It is closely related to the Walsh system and many problems in the Walsh-Fourier analysis can be reduced to the respective problems in terms of the Haar series (see [4], [18], [62]). We define the Haar system $\{h_n\}$ on the unit interval $[0,1]$, using open dyadic intervals in the following way. Let $h_0(x) = 1$ for all $x \in [0,1]$. If

$$n = 2^k + j,\ j = 0, \dots, 2^k - 1, \quad k \geq 0,$$

we put

$$h_n(x) = \begin{cases} 2^{k/2} & \text{if } x \in \left(\frac{2j}{2^{k+1}}, \frac{2j+1}{2^{k+1}}\right), \\ -2^{k/2} & \text{if } x \in \left(\frac{2j+1}{2^{k+1}}, \frac{2j+2}{2^{k+1}}\right), \\ 0 & \text{if } x \in [0,1] \setminus \left(\frac{j}{2^k}, \frac{j+1}{2^k}\right). \end{cases}$$

At 0 and 1 the functions h_n are defined by continuity from inside of $(0,1)$, and at the points where the Haar functions are left undefined up to now let them be equal to the average of their left and right limits.

We shall consider also a multidimensional setting. The multiple Vilenkin system is defined as m-dimensional sequence of products

$$\chi_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^{m} \chi_{n_i}(x^{(i)}), \qquad \mathbf{x} = (x^{(1)}, \ldots, x^{(m)}), \qquad \mathbf{n} = (n_1, \ldots, n_m).$$

In the same way the multiple Haar system $\{h_{\mathbf{n}}\}$ is defined.

An *interval* in $\mathbf{R}^m$ is always a compact set

$$(5) \qquad [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_m, b_m] \text{ with } a_i < b_i,\ i = 1, 2, \ldots, m.$$

A collection of intervals is called *nonoverlapping* whenever their interiors are disjoint. If each interval $[a_i, b_i]$ in (5) is $\mathcal{P}$-adic then m-dimensional interval (5) is also called $\mathcal{P}$-adic. More precisely, using notation (4) we call interval

$$I_{\mathbf{r}}^{(\mathbf{k})} = I_{r_1}^{(k_1)} \times \cdots \times I_{r_m}^{(k_m)},$$

where $\mathbf{k} = (k_1, \ldots, k_m)$, $\mathbf{r} = (r_1, \ldots, r_m)$, $\mathcal{P}$-interval of rank $\mathbf{k}$. Extending in a natural way, to the m-dimensional case, the introduced above notion of the basic sequence of $\mathcal{P}$-intervals convergent to a point $\mathbf{x}$, we denote this sequence by $\{I_{\mathbf{x}}^{(\mathbf{k})}\}$. If all the coordinates of $\mathbf{x}$ are $\mathcal{P}$-adic irrational then this m-dimensional sequence is unique. If exactly s coordinates, of $\mathbf{x}$, $0 \le s \le m$, are $\mathcal{P}$-adic rational then there are 2^s nested basic sequences convergent to $\mathbf{x}$.

We denote by $\mathcal{I}$ the family of all intervals in $\mathbf{R^m}$ and by $\mathcal{I}_{\mathcal{P}}$ the family of all m-dimensional $\mathcal{P}$-intervals.

If $E \subset \mathbf{R^m}$ then $|E|$ denotes the Lebesgue measure of E. The terms "almost everywhere" (br. a.e.) and "measurable" are always used in the sense of the Lebesgue measure. A δ-neighborhood of $\mathbf{x} \in \mathbf{R^m}$ is denoted by $U(\mathbf{x}, \delta)$.

3. Integrals associated with $\mathcal{P}$-adic derivate bases

$\mathcal{P}$-adic integrals we are going to consider here are generalizations of the original Henstock-Kurzweil integral and the classical Perron integral. A Riemann-type integral which turned out to be equivalent to the Denjoy-Perron integral was introduced independently by J. Kurzweil in [24] and R. Henstock in [21] and [22] to integrate real-valued functions defined on an interval of the real line. The idea of the definition is very simple and is based on replacing in the definition of the Riemann integral a positive constant δ which regulate the length of the intervals constituting a partition, by a positive function δ defined on the interval of integration.

Definition 1. A function f defined on an interval $[a, b]$ is said to be *Henstock-Kurzweil integrable* (or *HK-integrable*) on $[a, b]$, with HK-integral A, if for every $\varepsilon > 0$ there exists a positive function δ on $[a, b]$ such that

$$\left| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A \right| < \varepsilon \ ,$$

for any partition of $[a, b]$ which satisfies conditions

$$\xi_i \in [x_{i-1}, x_i] \subset (\xi - \delta(\xi_i), \xi + \delta(\xi_i))$$

for any $i = 1, 2, \ldots, n$, with $x_0 = a$ and $x_n = b$. We write $A = (HK) \int_a^b f$.

Those original papers written in the late fifties gave rise to a general theory of non-absolutely convergent integrals (see [20], [23], [27], [30], [32], [46]). A unifying notion in this theory is that of derivate basis (or basis of differentiation). It is usually defined in the classical abstract derivation theory as a family of sets contracting to a point (see [17]). In the theory we are discussing here, somewhat more subtle difinition is needed (see [23], [30], [56]).

A nonempty family $\mathcal{B}$ of subsets of the product $\mathcal{I} \times \mathbf{R}^\mathbf{m}$ is called a *derivate basis* (or simply a *basis*) on $\mathbf{R}^\mathbf{m}$ if certain conditions are fulfilled. We assume first of all that all bases $\mathcal{B}$ are filtering down, i.e., for every $\beta_1, \beta_2 \in \mathcal{B}$ there exists $\beta \in \mathcal{B}$ such that $\beta \subset \beta_1 \cap \beta_2$ (let us agree that $\mathcal{B}$ does not contain the empty set). We shall refer to the elements β of $\mathcal{B}$ as *basis sets*. In this paper we shall always suppose that $(I, \mathbf{x}) \in \beta$ implies $\mathbf{x} \in I$, although it is not the case in the general theory (see [32]). Given a basis $\mathcal{B}$, an interval I is called a $\mathcal{B}$*-interval* if $(I, \mathbf{x}) \in \beta$ for some $\mathbf{x}$ and some $\beta \in \mathcal{B}$. For a set $E \subset \mathbf{R}^\mathbf{m}$ and $\beta \in \mathcal{B}$ we write

$$\beta(E) = \{(I, \mathbf{x}) \in \beta : I \subset E\} \quad \text{and} \quad \beta[E] = \{(I, \mathbf{x}) \in \beta : \mathbf{x} \in E\}.$$

All the bases $\mathcal{B}$ we shall consider in this paper are so-called *Vitali bases*, i.e., such that for any $\mathbf{x}$ and for any $\delta > 0$ there is a basis set $\beta \in \mathcal{B}$ such that the set $\beta[\{\mathbf{x}\}]$ is nonempty and consists of only those pairs $(I, \mathbf{x})$ for which $I \subset U(\mathbf{x}, \delta)$. The simplest derivate basis on $\mathbf{R}^\mathbf{m}$ is the *full interval basis*. In this case, each basis set corresponds to a positive function δ defined on $\mathbf{R}^\mathbf{m}$ and called a *gage*. For a given gage δ, we denote

$$\beta_\delta = \{(I, \mathbf{x}) : I \in \mathcal{I},\ \mathbf{x} \in I \subset U(\mathbf{x}, \delta(\mathbf{x}))\}.$$

So the full interval basis is the family $\{\beta_\delta\}$ where δ runs over the set of all possible gages.

More general interval bases can be defined in a similar way by gages, if in the above definition of basis sets only those intervals are used which satisfy some additional properties. For example, if in the definition of the full interval basis we replace arbitrary intervals with intervals (5) subject to the regularity condition

$$\frac{\min_i(b_i - a_i)}{\max_i(b_i - a_i)} \geq r$$

with some fixed r, $0 < r \leq 1$, then we get an *r-regular interval basis.* If we denote the family of all the r-regular intervals by $\mathcal{I}_r$ then basis sets of the r-regular basis are defined by gages as

$$\beta_\delta^r = \{(I, \mathbf{x}) : I \in \mathcal{I}_r,\ \mathbf{x} \in I \subset U(\mathbf{x}, \delta(\mathbf{x}))\}.$$

Our primary concern here is $\mathcal{P}$*-adic bases* $\mathcal{B}_\mathcal{P}$ each of which corresponds to some fixed sequence (1). Basis sets of $\mathcal{B}_\mathcal{P}$ are defined by gages as

$$\beta_\delta^\mathcal{P} = \{(I, \mathbf{x}) : I \in \mathcal{I}_\mathcal{P},\ \mathbf{x} \in I \subset U(\mathbf{x}, \delta(\mathbf{x}))\}.$$

We get the $\mathcal{P}$*-adic r-regular* basis $\mathcal{B}_\mathcal{P}^r$ if we define basis sets as

$$\beta_\delta^{\mathcal{P}r} = \{(I, \mathbf{x}) : I \in \mathcal{I}_\mathcal{P} \cap \mathcal{I}_r,\ \mathbf{x} \in I \subset U(\mathbf{x}, \delta(\mathbf{x}))\}.$$

A finite collection $\pi \subset \beta$ is called a *β-partition* if for distinct elements (I', x') and (I'', x'') in π, the intervals I' and I'' are nonoverlapping. If a partition $\pi = \{(I_i, x_i)\} \subset \beta(I)$ for some $I \in \mathcal{I}$ is such that $\cup_i I_i = I$, then we say that π is a *β-partition of I*. We say that a basis $\mathcal{B}$ has the *partitioning property* if for any $\mathcal{B}$-interval I and for any $\beta \in \mathcal{B}$ there exists a β-partition of I. The partitioning property is not so trivial as it may seem to be at first glance. Whereas in the particular case of the full interval basis on $\mathbf{R}$ this property has long been known as the Cousin lemma, in the multidimensional case for some bases it was proved only recently (see [15]), and for some bases the property is not valid at all or holds true only in some weaker sense as it is in the case of the symmetric approximate basis (see [58]). As for the $\mathcal{P}$-adic bases, both regular and unregular, the partitioning property for these bases can be established without difficulty.

Given a basis $\mathcal{B}$, an interval function τ defined at least on all $\mathcal{B}$-intervals is said to be *$\mathcal{B}$-continuous* (or continuous with respect to the basis $\mathcal{B}$) *at a point* $\mathbf{x}$ if for any $\varepsilon > 0$ there exists $\beta \in \mathcal{B}$ such that $|\tau(I)| < \varepsilon$ whenever $(I, \mathbf{x}) \in \beta[\{\mathbf{x}\}]$. A $\mathcal{B}$-interval function τ is said to be *strongly $\mathcal{B}$-continuous at* $\mathbf{x}$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|\tau(I)| < \varepsilon$ for any $\mathcal{B}$-interval I whenever $\mathbf{x} \in I$ and $|I| < \delta$.

The *upper derivative* of a $\mathcal{B}$-interval function τ at a point $\mathbf{x}$ with respect to the basis $\mathcal{B}$ is defined as

$$\overline{D}_{\mathcal{B}}\tau(\mathbf{x}) = \inf_{\beta} \sup \left\{ \frac{\tau(I)}{|I|} : (I, \mathbf{x}) \in \beta[\{\mathbf{x}\}] \right\}.$$

Similarly, the *lower derivative* is defined as

$$\underline{D}_{\mathcal{B}}\tau(\mathbf{x}) = \sup_{\beta} \inf \left\{ \frac{\tau(I)}{|I|} : (I, \mathbf{x}) \in \beta[\{\mathbf{x}\}] \right\}.$$

If $\underline{D}_{\mathcal{B}}\tau(\mathbf{x}) = \overline{D}_{\mathcal{B}}\tau(\mathbf{x}) \neq \pm\infty$, we say that τ is *$\mathcal{B}$-differentiable* at $\mathbf{x}$ and denote the $\mathcal{B}$-derivative by $D_{\mathcal{B}}\tau(\mathbf{x})$.

In some cases below we shall talk about $\mathcal{B}$-derivative and $\mathcal{B}$-continuity of a point function F on $\mathbf{R}$ having in mind the respective notions for the associated with F an additive interval function defined by $\tau(I) = \Delta F(I) = F(d) - F(c)$, for each $\mathcal{B}$-interval $I = [c, d]$.

The above notions of $\mathcal{B}$-derivatives, if applied to the $\mathcal{P}$-adic bases $\mathcal{B}_{\mathcal{P}}$ or $\mathcal{B}^r_{\mathcal{P}}$ will be refered to as $\mathcal{P}$-derivatives (or $\mathcal{P}_r$-derivatives) with respective notations $\overline{D}_{\mathcal{P}}\tau$, $\underline{D}_{\mathcal{P}}\tau$, $D_{\mathcal{P}}\tau$ (or $\overline{D}_{\mathcal{P}_r}\tau$, $\underline{D}_{\mathcal{P}_r}\tau$, $D_{\mathcal{P}_r}\tau$). It is useful to note that in the case of the $\mathcal{P}$-adic basis the set $\beta_\delta[\{\mathbf{x}\}]$ is constituted by pairs $(I^{(\mathbf{k})}_{\mathbf{x}}, \mathbf{x})$ in which interval $I^{(\mathbf{k})}_{\mathbf{x}}$ represent all basic sequences convergent to $\mathbf{x}$.

For a basis $\mathcal{B}$ having the partitioning property, Definition 1 of integral can be extended in the following way (see [23], [30]).

Definition 2. A function f defined on a $\mathcal{B}$-interval J is said to be *$H_{\mathcal{B}}$-integrable* on J with integral A if for every $\varepsilon > 0$ there exists a basis set β such that

$$\left| \sum_{(I,x)\in\pi} f(x)|I| - A \right| < \varepsilon ,$$

for any β-partition π of J. We write $A = (H_{\mathcal{B}}) \int_J f$.

The $H_{\mathcal{B}}$-integral of a complex-valued function is defined in a natural way by the integrals of its real and imaginary parts.

The $H_{\mathcal{B}}$-integral with respect to any basis can be given an equivalent Perron-type definition (see [30]), where major and minor functions are defined by lower and upper derivatives with respect to this basis.

Definition 3. Let f be a function defined on a $\mathcal{B}$-interval J. An additive $\mathcal{B}$-interval function M (resp. m) is called a *$\mathcal{B}$-major* (resp. a *$\mathcal{B}$-minor*) function of f on J if

$$\underline{D}_{\mathcal{B}}M(\mathbf{x}) \geq f(\mathbf{x}) \quad (\text{resp. } \overline{D}_{\mathcal{B}}m(\mathbf{x}) \leq f(\mathbf{x})) \quad \text{for all} \quad \mathbf{x} \in J. \tag{6}$$

A function f is said to be *$P_{\mathcal{B}}$-integrable* on J if

$$-\infty < \inf_{M} \{M(J)\} = \sup_{m} \{m(J)\} < \infty,$$

where "inf" is taken over all major functions M and "sup" is taken over all minor functions m. The common value is denoted by $(P_{\mathcal{B}}) \int_J f$, and it is called the Perron integral with respect to $\mathcal{B}$ or $P_{\mathcal{B}}$-integral of f over J.

Remark 1. If the major and minor functions in the above definition are assumed to be $\mathcal{B}$-continuous, then it is possible to permit a countable exeptional set in the condition (6) getting an equvalent definition. In some cases (see Section 4) some special uncountable exceptional sets are allowed but in such cases we have to assume that the major and minor functions are strongly $\mathcal{B}$-continuous (see [41], [43]).

In the case of the ordinary Perron integral on the interval of the real line, the known Marcinkiewicz theorem (see [36]) states that a measurable function is Perron integrable (and therefore also HK-integrable) if it has at least a single pair of continuous major and minor functions. This theorem was extended to some generalized Perron integrals (see [11]). But at the same time it was shown that Perron integral with respect to symmetric basis and dyadic basis does not have the Marcinkiewicz property (see [45], [52]).

In the case of the $\mathcal{P}$-adic bases $\mathcal{B}_{\mathcal{P}}$ or $\mathcal{B}_{\mathcal{P}}^r$ we shall call the respective integrals in short $\mathcal{P}$-integral or $\mathcal{P}_r$-integral, for the Henstock-type integrals, and $P_{\mathcal{P}}$-integral or $P_{\mathcal{P}r}$-integral, for the Perron-type integrals. In the particular case of the sequence $\mathcal{P} = \{2, 2, \dots, 2, \dots\}$ we get the well known dyadic integral, different versions of which (including Henstock-type and Perron-type versions) were studied in numerous papers (see [8], [19], [25], [31], [38], [44], [46], [52]). For $\mathcal{P}$-integral see [9], [43] and [50].

It is easy to check that if a function f is $H_{\mathcal{B}}$-integrable on J, then it is also $H_{\mathcal{B}}$-integrable on each $\mathcal{B}$-interval $I \subset J$. Therefore the *indefinite $H_{\mathcal{B}}$-integral $F(I)$* is defined as an additive interval function at least on the family of all $\mathcal{B}$-intervals $I \subset J$. This function is $\mathcal{B}$-continuous. As for $\mathcal{B}$ differentiability of F it depends on whether the Vitali covering theorem holds true for this basis. For unregular bases this theorem fails to be true (see [17]). But if we assume regularity conditions then for a wide class of bases the indefinite integral is differentiable almost everywhere. In particular we have

Proposition 1. *The indefinite $\mathcal{P}_r$-integral F of a function f $\mathcal{P}_r$-integrable on a $\mathcal{P}$-interval J, is $\mathcal{P}_r$-continuous at each point of J and it is $\mathcal{P}_r$-differentiable a.e. with $D_{\mathcal{P}r}F(x) = f(x)$ a.e. on J.*

As for the m-dimensional $\mathcal{P}$-integral which is obviously included into $\mathcal{P}_r$-integral, we can state only its $\mathcal{P}_r$-differentiability a.e.

We mention also an integral which is intermediate between the $\mathcal{P}$-integral and any of the $\mathcal{P}_r$-integrals, $0 \leq r \leq 1$. It is a generalization, for the $\mathcal{P}$-adic case, of an integral introduced by Mawhin in [29] (see also [26]). We shall call it $\mathcal{P}_0$-integral.

Definition 4. A function f defined on $\mathcal{P}$-interval J is said to be $\mathcal{P}_0$-*integrable* on J with integral A if for each $r \in (0,1)$ f is $\mathcal{P}_r$-integrable on J with integral A. We write $A = (\mathcal{P}_0) \int_J f$.

This integral can be given an equivalent Perron-type definition.

Now we introduce a notion of variational measure generated by a function. This notion is due to B. Thomson (see [56], [57]) and is very useful in describing classes of indefinite $H_{\mathcal{B}}$-integrals and in obtaining in this way descriptive characterizations of the integrals.

Given a basis $\mathcal{B}$, a set $E \subset \mathbf{R}^m$ and an interval function τ, defined at least on all $\mathcal{B}$-intervals, we define

$$V_\tau(E) = \inf \sup \sum_{(I,x)\in\pi} |\tau(I)|\,,$$

where "sup" is taken over all $\pi \subset \beta[E]$ and "inf" is taken over all basis sets $\beta \in \mathcal{B}$. We call V_τ the *variational measure generated by* τ *with respect to the basis* $\mathcal{B}$. Note that V_τ is a metric outher measure in $\mathbf{R}^m$ (see [57]) and so its restriction to the Borel sets is a measure. A variational measure V_τ is called *absolutely continuous* (with respect to the Lebesgue measure) on a set E if $|N| = 0$ implies $V_\tau(N) = 0$ for any set $N \subset E$. It is easy to check (see [28] and [54]) that if a variational measure is absolutely continuous then it is a measure also on the family of Lebesgue measurable sets.

The following property is known for $H_{\mathcal{B}}$-integral associated with Vitali basis $\mathcal{B}$ (see [30], [56]):

Proposition 2. *The indefinite $H_{\mathcal{B}}$-integral F generates an absolutely continuous variational measure V_F with respect to $\mathcal{B}$.*

More delicate problem is a question whether the converse of this statement holds for a particular bases. It was established recently that for some bases the converse is true (see [5], [6], [7], [8], [10], [14], [28], [53], [59]) and it is not true in a general case. In particular for the $\mathcal{P}$-adic bases this result is true only in the case the sequence (1) is bounded. In the one-dimensional case it is proved in ([9]):

Theorem 1. *Let the sequence $\mathcal{P}$ be bounded. An additive $\mathcal{P}$-interval function F is the indefinite $\mathcal{P}$-integral of a function f on a $\mathcal{P}$-interval J if and only if F generates an absolutely continuous variational measure with respect to $\mathcal{B}_{\mathcal{P}}$ and $D_{\mathcal{P}}F(x) = f(x)$ a.e. on J.*

This theorem gives in fact a descriptive definition of the $\mathcal{P}$-integral associated with a bounded sequence $\mathcal{P}$.

In higher dimension analogous descriptive characterization is obtained for the dyadic integral in [53], and this result can be extended for the case of any bounded sequence $\mathcal{P}$.

As for any unbounded sequence $\mathcal{P}$, it is shown in [9], by constructing an example, that already in the one-dimensional case there exists an additive $\mathcal{P}$-interval function F which generates an absolutely continuous variational measure V_F and which is the indefinite $\mathcal{P}$-integral of no $\mathcal{P}$-integrable function.

The reason for this is the fact that the $\mathcal{P}$-adic basis defined by an unbounded sequence $\mathcal{P}$ fails to possess the so called Ward property. We say that a basis $\mathcal{B}$ *possesses Ward property* if each additive $\mathcal{B}$-interval function is $\mathcal{B}$-differentiable a.e.

on a set on which at least one of its extreme derivatives is finite. The following theorem is proved in [9] for the one-dimansional case.

Theorem 2. *A $\mathcal{P}$-adic basis possesses the Ward property if and only if the sequence $\mathcal{P}$ which defines this basis is bounded.*

The necessity part of this theorem is proved by constructing the following example.

Theorem 3. *For any unbounded sequence $\mathcal{P} = \{p_j\}_{j=0}^{\infty}$ there exist a closed set S of positive measure and*

(i) a continuous point function F on $[0,1]$ such that

$$\overline{D}_{\mathcal{P}}F(x) = +1 \quad \text{and} \quad \underline{D}_{\mathcal{P}}F(x) = -1 \tag{7}$$

for any $x \in S$;

(ii) a continuous point function G on $[0,1]$ such that

$$\overline{D}_{\mathcal{P}}G(x) = \underline{D}_{\mathcal{P}}G(x) = +\infty \tag{8}$$

for any $x \in S$.

The Ward property is used in the proof of Theorem 1 to establish the $\mathcal{P}$-differentiability a.e. of the function F, which generates an absolutely continuous variational measure. But if $\mathcal{B}$-differentiability a.e. of a function under consideration is assumed in advance then that kind of discriptive characterization of the $H_{\mathcal{B}}$-integral can be obtained for a wider class of bases. In particular we have the following result for the multidimensional $\mathcal{P}$-adic integrals.

Theorem 4. *An additive $\mathcal{P}$-interval function F is the indefinite $\mathcal{P}_r$-integral of a function f on a $\mathcal{P}$-interval J if and only if F generates an absolutely continuous variational measure with respect to $\mathcal{B}^r_{\mathcal{P}}$ and F is $\mathcal{P}_r$-differentiable a.e. with $D_{\mathcal{P}_r}F(x) = f(x)$ a.e. on J.*

As multidimensional $\mathcal{P}$-integral can fail to be $\mathcal{P}$-differentiable a.e., then the previous statement is true for this integral only in one direction:

Theorem 5. *If an additive $\mathcal{P}$-interval function F generates an absolutely continuous variational measure with respect to $\mathcal{B}_{\mathcal{P}}$ and F is $\mathcal{P}$-differentiable a.e. with $D_{\mathcal{P}}F(x) = f(x)$ a.e. on J then F is the indefinite $\mathcal{P}$-integral of f.*

The following theorem can be obtained from Theorems 4 and 5 as a corollary.

Theorem 6. *Let an additive $\mathcal{P}$-interval function F be $\mathcal{P}_r$-differentiable (or $\mathcal{P}$-differentiable) and $D_{\mathcal{P}_r}F(x) = f(x)$ (or $D_{\mathcal{P}}F(x) = f(x)$) everywhere on J outside a set E with $|E| = 0$. If the variational measure, with respect to $\mathcal{B}^r_{\mathcal{P}}$ $(\mathcal{B}_{\mathcal{P}})$, of E is equal zero then f is $\mathcal{P}_r$-integrable (respectively, $\mathcal{P}$-integrable) on J and F is its indefinite $\mathcal{P}_r$-integral ($\mathcal{P}$-integral).*

This theorem together with the Ward property implies the following proposision for the case of a bounded sequence $\mathcal{P}$

Proposition 3. *Let the sequence $\mathcal{P}$ be bounded and let the upper and the lower $\mathcal{P}_r$-derivatives of an additive $\mathcal{P}$-interval function F be finite everywhere on a $\mathcal{P}$-interval J except on a countable set where F is $\mathcal{P}$-continuous. Then its $\mathcal{P}_r$-derivative f (which exists a.e. by Ward property) is $\mathcal{P}_r$-integrable on J and F is the indefinite $\mathcal{P}$-integral of f on J.*

We conclude this section with some remarks concerning relationship between the class of $\mathcal{P}$-adic primitives (in the one-dimensional case) and the classical ACG- and VBG-classes which are known to play an important role in the theory of Denjoy integrals (see [36]).

We say that a function is *exact $\mathcal{P}$-primitive* if it has finite $\mathcal{P}$-derivative everywhere. Following [36], we call the integral defined by such primitives, the *Newton $\mathcal{P}$-integral*. So the above primitives constitute the class of the indefinite integrals in this sense.

It is shown in [44] that the indefinite $\mathcal{P}$-integral in the dyadic case can fail to be VBG and hence also to be ACG function. On the other hand it is proved in [38] that all exact dyadic primitives belong to the (ACG) class (recall that a function F is said to be (ACG) on E if $E = \bigcup_n E_n$ with F being AC on each E_n, so F is not supposed to be continuous as it is in the definition of ACG).

It turns out however that this difference in the behavior of the indefinite integrals in the Newton sense and in the Henstock-Kurzweil sense for the dyadic basis, is rather an exception. Already in the case of the triadic basis (i.e. when $p_j = 3$ for each $j \geq 0$ in (1)) *a continuous exact $\mathcal{P}$-primitive can fail to be a VBG function and to satisfy Lusin condition* (N) (see [9]).

On the other hand the function F, being exact $\mathcal{P}$-primitive, is by Proposition 3 the indefinite $\mathcal{P}$-integral of its $\mathcal{P}$-derivative and so by Theorem 1, generates an absolutely continuous $\mathcal{P}$-variational measure. Hence we have got the following result:

Theorem 7. *There exists, on an interval of the real line, a continuous function which generates absolutely continuous $\mathcal{P}$-variational measure but which is not VBG and does not satisfy Lusin condition (N).*

In some literature (see [27]) a function generating an absolutely continuous variational measure is said to satisfy the strong Lusin condition. So in this terminology we can say that strong Lusin condition with respect to $\mathcal{P}$-adic basis does not imply Lusin condition (N), in contrast with the case of usual interval basis and some other bases (see [16]).

Another consequence of the above example is the fact that *the Newton $\mathcal{P}$-integral, in the case of the triadic bases, and the Denjoy-Khintchine integral are noncompatible.* (As we have already mentioned, in the dyadic case the Newton $\mathcal{P}$-integral is (ACG) and this implies that it is compatible with the Denjoy-Khintchine integral.)

4. Application to the problem of recovering the coefficients

The problem of recovering the coefficients of orthogonal series from their sums is a generalization of the uniqueness problem for the coefficients of orthogonal series. It makes sense to consider this problem of recovering only for those orthogonal systems for which some kind of the usual uniqueness theorem is already established. The uniqueness can be related to pointwise convergent series or series which are summable in a certain sense, and the convergence or summability can be supposed everywhere or outside some exeptional set. It is natural sometimes to impose some kind of growth condition on the coefficients or the partial sums of the series. (For references to the literature on the rich theory of uniqueness of Walsh, Haar and Vilenkin series, including subtle theory of sets of uniqueness, see [1], [18], [37], [61], [63]. The classical trigonometric case is treated for example in [65])

If the uniqueness theorem is proved for a certain system and so the coefficients of an orthogonal series with respect to this system are uniquely determined by its sum, then it is natural to expect that they may be recovered from the sum by Fourier formulas, as it takes place in the simplest cases, for example in the case of the uniform convergence. Indeed for many known systems (trigonometric, Haar, Walsh, Vilenkin systems) it is true that every series with respect to those systems which converge everywhere to a summable function, is the Fourier series of this function. But the point is that the sum of everywhere convergent orthogonal series can fail to be Lebesgue integrable. For example, it is known (see [65]) that the series

$$\sum_{k=2}^{\infty} \frac{\sin kx}{\ln k}$$

converges everywhere but fails to be the Fourier-Lebesgue series. This kind of examples can be given for the other above mentioned systems as well. To integrate such series, one needs nonabsolutely convergent integrals. In the cases where the sum is integrable in one or another known general sense, the question is whether the coefficients can be determined by Fourier formulas in which the integral is understood in the same particular sense. The complete solution of the problem of recovering the coefficients of a convergent series with respect to some system is found if a general process of integration is developed so that any everywhere convergent series with respect to the considered system is the Fourier series of its sum in the sense of the defined integral.

It is no wonder that the theory started with the classical trigonometric case. Here the first solution of the problem of defining an integral so powerful that the sum of any everywhere convergent series is integrable and the coefficients can be computed by generalized Fourier formulas is due to Denjoy. He introduced in [13] a very complicated definition of a second order integral called the totalization T_{2s} , which recaptures a function from its second Riemann symmetric derivative. The difficulty of the T_{2s}-totalization process which involves a transfinite sequence of operations, led other authors to look for an easier solution of the coefficient problem. J. Marcinkiewicz and A. Zygmund, J.C. Burkill, R.D. James and some other authors produced a Perron type integrals reducing the problem to the one of recovering a function from its second order symmetric derivative (see [58] for details). The latest step in this direction was done by D. Preiss and B. Thomson in [34] who produced a first order Henstock-Kurzweil type integral which itegrates approximate symmetric derivatives. As for the multidimensional case, the uniqueness problem for rectangularly convergent multiple trigonometric series was solved only in 1991 by Tetunashvili [55] (see also [35]) and independently in 1993 by Ash, Freiling and Rinne [2]. This brought meaning to the problem of recovering the coefficients for these series. A multidimensional generalization of Preiss and Thomson symmetric integral or any other multidimensional integral which solves the coefficients problem in higher dimension is not obtained yet up to now. Nethertheless, the coefficient problem can be solved in a roundabout way on the bases of Tetunashvili method using iterated integration (see [48]).

The first step toward a resolution of the coefficient problem in the Walsh case was taken by Arutunyan and Talalyan [4] and by Crittenden and Shapiro [12] who solved the problem for the Walsh series convergent to a Lebesgue integrable function. In the former paper the case of Haar series was also covered. Next, different versions of the uniqeness theorem for Walsh and Haar series with sums integrable in the Denjoy-Perron and Denjoy-Khinchine sense were obtained (see [3], [39], [40], [60]).

The first integral solving the coefficient problem for Haar (and Walsh) series was introduced in [38] in a descriptive form. A costructive definition of Denjoy type, based on transfinite induction, was given in [42] and later, independently, in [25]. The coefficient problem for Vilenkin series was examined in [43]

It is an advantage of application of the Henstock-Kurzweil theory that it provides a unifying approach to the coefficient problem for many orthogonal systems, including the multidimensional case. The choice of the Henstock-Kurzweil type (or Perron-type) integral with respect to the particular basis is determined by the system and by the type of convergence under consideration. While the symmetric basis is a natural choice for the classical trigonometric case, the $\mathcal{P}$-adic basis and the respective derivatives and integrals appear in a natural way in the theory of the Vilenkin system.

A starting point for an application of the $\mathcal{P}$-derivative and the $\mathcal{P}$-integral to the theory of Vilenkin series is an observation that due to martingale properties of the partial sums S_{m_k} (where m_k are defined by (2)) of a series with respect to the system (3), the integral $\int_{I_r^k} S_{m_k}$ defines an additive interval-function $\tau(I)$ on the family of $\mathcal{P}$-intervals (Yoneda call it *quasi-measure*, see [37], [64] and [66]), and for this function

$$S_{m_k}(x) = \frac{\tau(I_x^{(k)})}{|I_x^{(k)}|}. \tag{9}$$

at each $\mathcal{P}$-adic irrational point. For $\mathcal{P}$-adic rational points the sutuation is complicated by the fact that there are two basic sequences $\{I_x^{(k)}\}$ convegent to x. So we prefer not to use the above equality at such points. (We would not have this problem if we considered the series on the group $G(\mathcal{P})$ instead of the interval $[0,1)$, with Haar measure on it and with additive function τ defined on the algebra generated by cosets instead of the one generated by $\mathcal{P}$-intervals. But we prefer the real line setting to be able to cover also the case of the Haar system.)

In higher dimension we consider rectangular partial sums

$$S_{\mathbf{j}}(\mathbf{x}) = \sum_{n_1=0}^{j_1} \cdots \sum_{n_m=0}^{j_m} a_{\mathbf{n}} \prod_{i=1}^{m} \chi_{n_i}(x_i) \tag{10}$$

where $\mathbf{j} = (j_1 \dots, j_m)$ and denoting $m_{\mathbf{k}} = (m_{k_1}, \dots, m_{k_m})$ where $\mathbf{k} = (k_1, \dots, k_m)$ we get for the additive function

$$\tau(I_{\mathbf{r}}^{(\mathbf{k})}) = \int_{I_{\mathbf{r}}^{(\mathbf{k})}} S_{m_{\mathbf{k}}} \tag{11}$$

a multidimensional analogue of (9):

$$S_{m_{\mathbf{k}}}(\mathbf{x}) = \frac{\tau(I_{\mathbf{x}}^{(\mathbf{k})})}{|I_{\mathbf{x}}^{(\mathbf{k})}|}. \tag{12}$$

for any point $\mathbf{x}$ with all the coordinates $\mathcal{P}$-adic irrational.

Another simple observation, which is essential for establishing that a given Vilenkin series is the Fourier series in the sense of some general integral, is the following

Proposition 4. *Let some integration process $\mathcal{A}$ be given which produces an integral additive on $\mathcal{I}_{\mathcal{P}}$. A series*

$$\sum a_{\mathbf{n}}\chi_{\mathbf{n}} \tag{13}$$

is the Fourier series of an $\mathcal{A}$-integrable function f if and only if $\tau(I) = (\mathcal{A})\int_I f$ for any $\mathcal{P}$-interval I

This means that in order to get a solution of the problem of recovering the coefficients from the sum f of a series (13) we are to prove that f is integrable in a certain sense $\mathcal{A}$ and $\mathcal{P}$-interval function (11), defined by this series, is the indefinite $\mathcal{A}$-integral of f.

Now the equalities (9) and (12) reveal a close connection between type of convergence of the series and differential properties of the function τ. In particular in one-dimentional case the equality (9) implies immediately that if a Vilenkin series converges at a $\mathcal{P}$-adic irrational point x to a sum $f(x)$ then τ is $\mathcal{P}$-differentiable at x and $f(x)$ is its $\mathcal{P}$-derivative.

To consider this relationship in higher dimension we need to recall definitions of different types of convergence of a multiple series. We say that series (13) is *rectangularly convergent* to $f(\mathbf{x})$ at $\mathbf{x}$ if its rectangular partial sums (10) are convergent to $f(\mathbf{x})$ when $\min_i j_i \to \infty$. We say that the series (13) is *r-regular rectangularly convergent*, $0 < r \leq 1$, to $f(\mathbf{x})$ at $\mathbf{x}$ if in the previous definition we consider only those rectangular sums (10) for which $\min_i j_i / \max_i j_i \geq r$. Therefore we get from (12) that if a series (13) converges rectangularly or r-regular rectangularly at a point $\mathbf{x}$ with $\mathcal{P}$-adic irrational coordinate to a sum $f(\mathbf{x})$ then τ is $\mathcal{P}$-differentiable or $\mathcal{P}_r$-differentiable at $\mathbf{x}$ and $D_{\mathcal{P}}\tau(\mathbf{x}) = f(\mathbf{x})$ or $D_{\mathcal{P}_r}\tau(\mathbf{x}) = f(\mathbf{x})$, respectively.

All the above observations imply that the problem of recovering the coefficients of a series (13) from its sum can be reduced to the problem of recovering the additive $\mathcal{P}$-interval function τ from its derivative in the respective sense, and this latter problem can be considered in terms of the integration theory discussed in Section 3.

As the equality (12) says nothing about the differentiability of τ at points with $\mathcal{P}$-adic rational coordinates and also at points of a possible exceptional set where we know nothing about convergence of the considered series, we need some information related to the behavior of τ on the set of nondifferentiability which would implies that the variational measure V_τ is equal zero on that exceptional set. Then we can hope to recover τ from its derivative, existing a.e., by using some of the results stated in Section 3. Such a nice behavior of τ on the exeptional set can be obtained either from the convergence condition or from some additional growth assumption imposed on the series. For example, it can be easily shown, in the one-dimensional case, that if the coefficints of a series (13) satisfy the condition $\lim_{n\to\infty} a_n = 0$ (which is a consequence of the convergence of the series at least at one point) then τ is $\mathcal{P}$-continuous everywhere. As $\mathcal{P}$-continuity of τ on a countable set E obviously implies $V_\tau(E) = 0$ we can apply Theorem 6 to get in the one-dimensional case (see [50])

Theorem 8. *Suppose that the series* (13) (*in one dimension*) *is convergent to a function f everywhere on* $[0, 1)$ *exept possibly on a countable set. Then f is $\mathcal{P}$-integrable on* $[0, 1]$ *and* (13) *is $\mathcal{P}$-Fourier series of f.*

In this theorem we can weaken the assumption of convergency supposing that the series is convergent only a.e. and its partial sums S_{m_k} are bounded at each point exept possibly on a countable set. In the case of bounded sequence $\mathcal{P}$ even the apriori assumption of convergency a.e. can be dropped. We can replace this assumption by boundeness of the sums $S_{m_k}(x)$ at each point and then the equality (9) and Proposition 3 can be applied to the real and to the imaginary parts of the series.

In the case of the multiple series (13) (and also multiple Haar series) convergent rectangularly, the coefficient can be recovered by a Perron-type integral $P^*_{\mathcal{P}}$ which is defined by strongly $\mathcal{P}$-continuous major and minor functions which satisfy (6) everywhere except, possibly, on a set of points with $\mathcal{P}$-irrational coordinates (see Remark 1 and [43]). So we have

Theorem 9. *Let the sequence* (1) *be bounded. If a series* (13) *is convergent rectangularly everywhere on the unit interval of* $\mathbf{R}^m$ *to a function* f *then* f *is* $P_{\mathcal{P}}$*-integrable and* (13) *is* $P_{\mathcal{P}}$*-Fourier series of* f*.*

Similar theorem holds for multiple Haar series.

In the case of r-regular rectangular convergence analogous theorems were obtained for Haar and Walsh series (see [33]), using Perron integral with respect to the regular dyadic basis defined by major and minor functions on which some more delicate continuity conditions are imposed.

We can also consider a case where series is r-regular convergent for any $r \in (0, 1]$ (note that it is not the same as just rectangular convergence!). In this case the Mawhin-type integral (see Definition 4) (or its Perron-type analogue) is appropriate.

In Section 3 we have mentioned that the Newton $\mathcal{P}$-integral and the Denjoy-Khinchine integral are noncompatible. The example by which this noncompatibility is obtained can be used to proof the following (see [51])

Theorem 10. *There exists a series* (13) (*in one dimension*) *with coefficients convergent to zero such that its sums* S_{m_k} *are convergent everywhere on* $[0, 1)$ *to a Denjoy-Khinchine-integrable function* f*, but this series is not the Fourier series of* f *in the sense of the Denjoy-Khinchine integral.*

However, if we replace the assumtion related to the convergence of the sums S_{m_k} in this theorem by the condition that the entire sequence of the partial sums converges, then the following result is true (see [51]).

Theorem 11. *If the series* (13) (*in one dimension*) *with respect to the Vilenkin system defined by a bounded sequence* $\mathcal{P}$ *converges everywhere on* $[0, 1)$ (*with the possible exeption of some countable set*) *to a Denjoy-Khinchine-integrable function* f*, then this series is the Fourier series of* f *in the sense of the Denjoy-Khinchine integral.*

On application of some other type of generalized integrals to the theory of Walsh and Haar series see [47] and [49]

5. $\mathcal{P}$-integral in inversion formula for multiplicative transform

The problem of uniqueness can be considered also for the continual analogues of the Vilenkin system, i.e. for the case where a $\mathcal{P}$-adic group not being compact, is locally compact and so the system of characters is not descrete.

We recall the appropriate definitions (see [1]). Consider a double sequence of natural numbers

$$\mathcal{R} = \{\ldots, p_{-j}, \ldots, p_{-2}, p_{-1}, p_1, p_2 \ldots, p_j, \ldots\} \tag{14}$$

where $p_j \geq 2$ for $j \in \mathbf{Z} \setminus \{0\}$ and two its subsequences: the right one $\mathcal{P} = \{p_j\}_{j=1}^{\infty}$ and the left one $\mathcal{P}' = \{p_{-j}\}_{j=1}^{\infty}$. We set $m_0 = 1, m_k = \prod_{s=1}^{k} p_s$ and $m_{-k} = \prod_{s=1}^{k} p_{-s}$.

For a given sequence (14) we define the group $G(\mathcal{R})$ of sequences

$$x = \{\ldots, x_{-j}, \ldots, x_{-2}, x_{-1}, x_1, x_2 \ldots, x_j, \ldots\}$$

where $0 \leq x_j \leq p_j - 1$, $j \in \mathbf{Z} \setminus \{0\}$ and $x_{-j} = 0$ for $j > k(x) \geq 1$ with group operation defined as coordinatewise addition module p_j for the jth coordinate. The dual group of the group $G(\mathcal{R})$ is isomorphic to the group $G(\mathcal{R}')$ which is defined in a similar way by a sequence symmetrical to the sequence (14):

$$\mathcal{R}' = \{\ldots, p'_{-j}, \ldots, p'_{-2}, p'_{-1}, p'_1, p'_2 \ldots, p'_j, \ldots\}$$

with $p'_j = p_{-j}$ for $j \in \mathbf{Z} \setminus \{0\}$ (see [1]). Then the character $\chi(\cdot, y)$ of $G(\mathcal{R})$ corresponding to $y \in G(\mathcal{R}')$ can be described as

$$\chi(x, y) = \exp\left(2\pi i \left(\sum_{j=1}^{k(y)} \frac{x_j\ y_{-j}}{p_j} + \sum_{j=1}^{k(x)} \frac{x_{-j}\ y_j}{p_{-j}}\right)\right).$$

As in the compact case, we can consider an element of the group $G(\mathcal{P})$ as the coefficients of the $\mathcal{R}$-adic expansion

$$\sum_{j=1}^{k(x)} x_{-j} m_{-j+1} + \sum_{j=1}^{\infty} \frac{x_j}{m_j}$$

of a non-negative number and in this way we can map this group on the set $\mathbf{R}_+$ of the non-negative reals. The same can be done with the group $G(\mathcal{R}')$. So the above characters $\chi(x, y)$ can be treated as functions on $\mathbf{R}_+ \times \mathbf{R}_+$.

The problem of recovering the coeficients can be generalized to the case of this system of characters coming to be in this case a problem of establishing an inversion formula for a multiplicative transform of the form $\int_0^{+\infty} a(x)\chi(x, y)dx$.

To formulate a result in this direction we define by the above sequences $\mathcal{P}$ and $\mathcal{P}'$ the respective integrals, as in Section 3, and extend those integrals to any interval $[0, n$, n=1,2,..., in a natural way. Then we get (see [50])

Theorem 12. *Suppose that a function $a : \mathbf{R}_+ \to \mathbf{C}$ is locally $\mathcal{P}$-integrable and the improper $\mathcal{P}$-integral*

$$\int_0^{+\infty} a(x)\chi(x, y)dx$$

is convergent everywhere on $\mathbf{R}_+$, except possibly on a countable set, to a finite function $f(y)$. Then f is locally $\mathcal{P}'$-integrable and

$$a(x) = \lim_{n\to\infty} (\mathcal{P}') \int_0^{m_n} f(y)\overline{\chi(x, y)}dy \qquad a.e.\ on\ \ \mathbf{R}_+.$$

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