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SEMISIMPLE COMPLETED GROUP RINGS

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Abstract. We give in this note a characterization of semisimple completed group rings. Our result extends theorem of Connell concerning the semisimplicity of group rings to the topological case.

1. Preliminaries

By a *semisimple ring* we mean a ring semisimple in the sense of Jacobson. By a *profinite ring* (*group*) we mean a compact totally disconnected ring (group).

Let R be a profinite ring with identity and G a profinite group. If V is a two sided ideal of R and N an invariant subgroup of G, consider the two sided ideal of the group ring $R[G]$,

$$(V, N) = V[G] + (1 - N),$$

where $(1 - N)$ is the ideal of $R[G]$ generated by the set $1 - N$ and

$$V[G] = \left\{ \sum_{i=1}^{n} v_i g_i : n \in \mathbb{N}^*,\ v_1, \ldots, v_k \in V,\ g_1, \ldots, g_k \in G \right\}.$$

We will remind for convenience some results from [5].

Lemma 1.1. *Let R, R' are two rings with identity and G, G' two groups. If $f\colon R \to R'$ is a ring homomorphism and $\varphi\colon G \to G'$ is a group homomorphism, then the kernel of the canonical homomorphism $\lambda\colon R[G] \to R'[G']$ extending f and φ is (V, N), where $V = \ker f$ and $N = \ker\varphi$.*

Lemma 1.2. *Let R be a profinite ring with identity and G a profinite group. Then $\cap (V, N) = \{0\}$, where V runs all open ideals of R and N all open invariant subgroups of G.*

A Hausdorff topological ring $(R, \mathfrak{T})$ is said to be *totally bounded* provided for every neighborhood V of zero there exists a finite subset F such that $R = F + V$. It is well known that a Hausdorff topological ring is totally bounded if and only if its completition $\left(\widehat{R}, \widehat{\mathfrak{T}}\right)$ is compact.

For a profinite ring R with identity and a profinite group G, the family $\{(V, N)\}$ where V runs all open ideals of R and N all open invariant subgroups of G gives a totally bounded ring topology $\mathfrak{T}$ on $R[G]$. The completion of the topological ring $(R[G], \mathfrak{T})$ is called the *completed group ring* (and is denoted by $R[[G]]$).

Lemma 1.3. *For each closed invariant subgroup N of G the ideal $(1 - N)$ of $R[G]$ is closed.*

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Lemma 1.4. *For each closed invariant subgroup N of G holds*

$$R\,[[G]]\Big/\overline{(1-N)}\cong R\,[[G/N]]$$

Theorem 1.5. *Let R be a profinite ring with identity and G a profinite group. If $\{N_\alpha\}_{\alpha\in\Omega}$ is a filter base consisting of closed invariant subgroups of G, then $R\,[[G]]$ is the inverse limit of rings $R\,[[G/N_\alpha]]$ and the canonical projections are onto.*

Remark 1.1. Theorem 1.5 and lemma 1.4 show that our definition of completed group ring is equivalent with the definition given in [3], through the inverse limit of the rings $R\,[G/N]$, where N runs all open invariant subgroups of G.

Question. For which profinite rings R and profinite groups G is the topological ring $(R\,[G]\,,\mathfrak{T})$ minimal?

Theorem 1.6 (The universal property of completed group rings)**.** *Let Λ be a profinite commutative ring with identity and G a profinite group. If A is a compact Λ-algebra with identity, then each continuous homomorphism α of G in $U\,(A)$ can be extended to a continuous homomorphism $\widehat{\alpha}$ of $\Lambda\,[[G]]$ in A.*

Remark 1.2. If $\alpha\colon G\to G'$ is a continuous homomorphism of a profinite group G in a profinite group G' and R is a profinite ring then there exists a continuous homomorphism $\widehat{\alpha}\colon R\,[[G]]\to R\,[[G']]$ extending α.

The following result of Connell [1] yields a characterization of semisimple group rings:

Theorem 1.7. *The group ring $R\,[G]$ is semisimple if and only if the following conditions are satisfied:*

1. *R is a semisimple ring;*
2. *G is a finite group;*
3. *the order of G is a unit in R.*

2. The main result

Theorem 2.1. *Let R be a profinite ring with identity and G a profinite group. The completed group ring $R\,[[G]]$ is semisimple if and only if are satisfied the following two conditions:*

1. *R is semisimple;*
2. *for every open ideal V of R and for every open invariant subgroup N of G the order of G/N is a unit in R/V.*

Proof. Suppose that $R\,[[G]]$ is semisimple. By theorem of Kaplansky [2],

$$R\,[[G]]\cong\prod_{i\in I}(F_i)_{n_i}\ ,$$

where each F_i is a finite field, n_i a natural number and $(F_i)_{n_i}$ is the ring of $n_i\times n_i$ matrices over F_i. In particular, $R\,[[G]]$ is a regular ring in the sense of von Neumann.

By remark 1.2 R is a continuous homomorphic image of $R\,[[G]]$, therefore R is regular, hence semisimple. If N is an open invariant subgroup of G and V an open ideal of R, then by lemma 1.1 there is a homomorphism α of $R\,[G]$ onto $(R/V)\,[G/N]$. Since $\ker\alpha=(V,N)$, α is continuous. We extend by the continuity the homomorphism α to a homomorphism $\widehat{\alpha}$ of $R\,[[G]]$ onto $(R/V)\,[G/N]$. Since $R\,[[G]]$ is regular, the ring $(R/V)\,[G/N]$ is regular too, therefore it is semisimple. According to theorem of Connell, the order of G/N is a unit in R/V.

Conversely, assume that R and G satisfy the conditions of theorem. We claim that the group ring $R\,[G/N]$ is semisimple for each open invariant subgroup N of G. For each open ideal V of R, f_V denotes the canonical homomorphism of $R\,[G/N]$ in $(R/V)\,[G/N]$. By the condition of theorem, $(R/V)\,[G/N]$ is semisimple. Since

$\cap I_V = \{0\}$, where $I_V = \ker f_V$, the ring $R[G/N]$ is semisimple as a subdirect product of semisimple rings. For each open invariant subgroup N of G, λ_N denotes the canonical homomorphism $R[[G]] \to R[G/N]$. The completed group ring $R[[G]]$ is according to lemma 1.4 a subdirect product of rings $R[G/N]$ where N runs all open invariant subgroups of G. We get that $R[[G]]$ is semisimple. □

Corollary 2.2. *If K is a finite field, then the completed group ring $K[[G]]$ is semisimple if and only if for every open invariant subgroup N of G, $\mathrm{char}(K) \nmid |G/N|$.*

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