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ON THE GENERALIZED CESÀRO SUMMABILITY FACTORS

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Abstract. In this paper a general theorem concerning the $\psi - \mid C,\alpha;\delta \mid_k$ summability factors of infinite series has been proved.

1. Introduction. A sequence (w_n) of positive numbers is said to be δ-quasi monotone, if $w_n \to 0$, $w_n > 0$ ultimately and $\Delta w_n \geq -\delta_n$, where (δ_n) is a sequence of positive numbers (see[1]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). We define A_n^α by identity

(1) $$\sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-\alpha-1}.$$

The sequence-to-sequence transformations given by

(2) $$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v$$

(3) $$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,$$

define the (C,α) means of the sequences (s_n) and (na_n), respectively.

The series $\sum a_n$ is said to be summable $\mid C,\alpha \mid_k$, $k \geq 1$ and $\alpha > -1$, if (see [3])

(4) $$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^\alpha - u_{n-1}^\alpha \mid^k < \infty.$$

If we take $\alpha = 1$, then $\mid C,\alpha \mid_k$ summability is the same as $\mid C,1 \mid_k$ summability. Let (ψ_n) be a sequence of positive real numbers. We say that the series $\sum a_n$ is said to be summable $\psi - \mid C,\alpha;\delta \mid_k$, $k \geq 1$, $\alpha > -1$ and $\delta \geq 0$, if

(5) $$\sum_{n=1}^{\infty} \psi_n^{\delta k+k-1} \mid u_n^\alpha - u_{n-1}^\alpha \mid^k < \infty.$$

But since $t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha)$ (see [4]) condition (5) can also be written as

(6) $$\sum_{n=1}^{\infty} \psi_n^{\delta k+k-1} n^{-k} \mid t_n^\alpha \mid^k < \infty.$$

If we take $\delta = 0$ and $\psi_n = n$ (resp. $\delta = 0$, $\alpha = 1$ and $\psi_n = n$), then $\psi - \mid C,\alpha;\delta \mid_k$ summability is the same as $\mid C,\alpha \mid_k$ (resp. $\mid C,1 \mid_k$) summability.

Remark. Since (ψ_n) is a sequence of positive real numbers the summability

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method $\psi- \mid C,\alpha;\delta \mid_k$ is a new method and general than the $\mid C,\alpha;\delta \mid_k$ summability method. On the other hand $\mid C,\alpha;\delta \mid_k$ and $\psi- \mid C,\alpha;\delta \mid_k$ summability methods are different from each other. That is they have got different summability fields. Therefore, we take the sequence (ψ_n) instead of n.

2. The following theorem is known.

Theorem A([2]). Let t_n^α be the n-th Cesàro mean of order α, with $\alpha \geq 1$, of the sequence (na_n) such that $a_n \geq 0$ for all $n \geq 1$ whenever $\alpha > 1$ and let $\lambda_n \to 0 \quad as \quad n \to \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi monotone with $\sum n^\alpha \delta_n \log n < \infty$, $\sum B_n \log n$ is convergent and $\mid \Delta\lambda_n \mid \leq \mid B_n \mid$ for all n.

$$\sum_{n=1}^{m} \mid \Delta(n^\alpha) \mid\mid B_{n+1} \mid \log n = O(1), \tag{7}$$

$$\sum_{n=1}^{m} \frac{1}{n} \mid t_n^\alpha \mid^k = O(\log m) \text{ as } m \to \infty, \tag{8}$$

then the series $\sum a_n\lambda_n$ is summable $\mid C,\alpha \mid_k$, $k \geq 1$.

3. The aim of this paper is to generalize Theorem A in the following form.

Theorem. Let $k \geq 1$ and $\delta \geq 0$. Let t_n^α be the n-th Cesàro mean of order α, with $\alpha \geq 1$, of the sequence (na_n) such that $a_n \geq 0$ for all $n \geq 1$ whenever $\alpha > 1$ and let $\lambda_n \to 0 \quad as \quad n \to \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi monotone with $\sum n^\alpha \delta_n \log n < \infty$, $\sum B_n \log n$ is convergent, $\mid \Delta\lambda_n \mid \leq \mid B_n \mid$ for all n and that condition (7) of Theorem A is satisfied. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k}\psi_n^{\delta k+k-1})$ is non-increasing and

$$\sum_{n=1}^{m} \psi_n^{\delta k+k-1} n^{-k} \mid t_n^\alpha \mid^k = O(\log m) \text{ as } m \to \infty, \tag{9}$$

then the series $\sum a_n\lambda_n$ is summable $\psi- \mid C,\alpha;\delta \mid_k$.

If we take $\delta = 0$, $\epsilon = 1$ and $\psi_n = n$ in this theorem, then we get Theorem A.

4. We need the following lemmas for the proof of our theorem.

Lemma 1 ([5]). If $\sigma > \delta > 0$, then

$$\sum_{n=v+1}^{m} \frac{A_{n-v}^{\delta-1}}{A_n^\sigma} = \sum_{n=v+1}^{m} \frac{(n-v)^{\delta-1}}{n^\sigma} = O(v^{\delta-\sigma}) \text{ as } m \to \infty. \tag{10}$$

Lemma 2 ([2]). Let $\lambda_n \to 0 \quad as \quad n \to \infty$. Suppose that there exists a sequence of numbers (B_n) which is δ-quasi monotone with $\sum B_n \log n$ is convergent and $\mid \Delta\lambda_n \mid \leq \mid B_n \mid$ for all n, then

$$\mid \lambda_n \mid \log n = O(1) \text{ as } n \to \infty. \tag{11}$$

Lemma 3 ([2]). Let $\alpha \geq 1$. If (B_n) is δ-quasi monotone with $\sum n^\alpha \delta_n \log n < \infty$ and $\sum B_n \log n$ is convergent, then

$$m^\alpha B_m \log m = O(1) \text{ as } m \to \infty, \tag{12}$$

$$\sum_{n=1}^{\infty} n^\alpha \mid \Delta B_n \mid \log n < \infty. \tag{13}$$

Lemma 4 ([2]). Let t_n^α be the n-th Cesàro mean of order α, with $\alpha \geq 1$, of the sequence (na_n) such that $a_n \geq 0$ for all $n \geq 1$ whenever $\alpha > 1$. If $n \geq v$, then

$$\mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p \mid \leq A_{n-v}^{\alpha-1} A_v^\alpha \mid t_v^\alpha \mid . \tag{14}$$

5. Proof of the Theorem. Let (T_n^α) be the n-th (C,α), with $\alpha \geq 1$, means of the sequence $(na_n\lambda_n)$. Then, by (3), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v. \tag{15}$$

Using Abel's transformation, we get

$$\begin{aligned} T_n^\alpha &= \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \\ &= \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \lambda_n t_n^\alpha \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha, \text{ say .} \end{aligned}$$

Since

$$\mid T_{n,1}^\alpha + T_{n,2}^\alpha \mid^k \leq 2^k(\mid T_{n,1}^\alpha \mid^k + \mid T_{n,2}^\alpha \mid^k),$$

to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} \psi_n^{\delta k+k-1} n^{-k} \mid T_{n,r}^\alpha \mid^k < \infty \text{ for } r = 1, 2, \text{ by (6).} \tag{16}$$

Firstly, when $k > 1$, using Lemma 4 and after applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned} \sum_{n=2}^{m+1} \psi_n^{\delta k+k-1} n^{-k} \mid T_{n,1}^\alpha \mid^k &= \sum_{n=2}^{m+1} \psi_n^{\delta k+k-1} n^{-k} \mid \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p \mid^k \\ &\leq \sum_{n=2}^{m+1} \psi_n^{\delta k+k-1} n^{-k} (A_n^\alpha)^{-k} \{\sum_{v=1}^{n-1} \mid B_v \mid A_v^\alpha A_{n-v}^{\alpha-1} \mid t_v^\alpha \mid\}^k \\ &= O(1)\sum_{n=2}^{m+1} \psi_n^{\delta k+k-1} n^{-k} (A_n^\alpha)^{-k} \{\sum_{v=1}^{n-1} v^\alpha \mid B_v \mid A_{n-v}^{\alpha-1} \mid t_v^\alpha \mid\}^k \\ &= O(1)\sum_{n=2}^{m+1} \psi_n^{\delta k+k-1} n^{-k} A_n^\alpha \sum_{v=1}^{n-1} (v^\alpha \mid B_v \mid)^k A_{n-v}^{\alpha-1} \mid t_v^\alpha \mid^k \\ &\times \{\sum_{v=1}^{n-1} \frac{A_{n-v}^{\alpha-1}}{A_n^\alpha}\}^{k-1} \\ &= O(1)\sum_{v=1}^{m} (v^\alpha \mid B_v \mid)^{k-1} (v^\alpha \mid B_v \mid) \mid t_v^\alpha \mid^k \sum_{n=v}^{m+1} \frac{\psi_n^{\delta k+k-1} A_{n-v}^{\alpha-1}}{n^k A_n^\alpha} \\ &= O(1)\sum_{v=1}^{m} v^\alpha \mid B_v \mid\mid t_v^\alpha \mid^k \sum_{n=v}^{m+1} \frac{\psi_n^{\delta k+k-1} n^{\epsilon-k} (n-v)^{\alpha-1}}{n^{\alpha+\epsilon}} \\ &= O(1)\sum_{v=1}^{m} v^\alpha \mid B_v \mid\mid t_v^\alpha \mid^k \psi_v^{\delta k+k-1} v^{\epsilon-k} \sum_{n=v}^{m+1} \frac{(n-v)^{\alpha-1}}{n^{\alpha+\epsilon}} \\ &= O(1)\sum_{v=1}^{m} v^\alpha \mid B_v \mid\mid t_v^\alpha \mid^k \psi_v^{\delta k+k-1} v^{-k}, \end{aligned}$$

by Lemma 1. Thus

$$\sum_{n=2}^{m+1} \psi_n^{\delta k+k-1} n^{-k} \mid T_{n,1}^\alpha \mid^k = O(1)\sum_{v=1}^{m-1} \Delta(v^\alpha \mid B_v \mid) \sum_{p=1}^{v} \psi_p^{\delta k+k-1} p^{-k} \mid t_p^\alpha \mid^k$$

$$+ O(1)m^{\alpha} \mid B_m \mid \sum_{v=1}^{m} \psi_v^{\delta k+k-1} v^{-k} \mid t_v^{\alpha} \mid^k$$

$$= O(1) \sum_{v=1}^{m-1} \Delta(v^{\alpha} \mid B_v \mid) \log v + O(1)m^{\alpha} \mid B_m \mid \log m$$

$$= O(1) \sum_{v=1}^{m-1} v^{\alpha} \mid \Delta B_v \mid \log v + O(1) \sum_{v=1}^{m-1} \mid \Delta(v^{\alpha}) \mid\mid B_{v+1} \mid \log v$$

$$+ O(1)m^{\alpha} \mid B_m \mid \log m = O(1) \text{ as } m \to \infty,$$

by virtue of the hypotheses of the Theorem and Lemma 3.
Again, since $\mid \lambda_n \mid= O(1)$, we have that

$$\sum_{n=1}^{m} \psi_n^{\delta k+k-1} n^{-k} \mid T_{n,2}^{\alpha} \mid^k = \sum_{n=1}^{m} \psi_n^{\delta k+k-1} n^{-k} \mid \lambda_n t_n^{\alpha} \mid^k$$

$$= \sum_{n=1}^{m} \psi_n^{\delta k+k-1} n^{-k} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid\mid t_n^{\alpha} \mid^k$$

$$= O(1) \sum_{n=1}^{m} \psi_n^{\delta k+k-1} n^{-k} \mid \lambda_n \mid\mid t_n^{\alpha} \mid^k$$

$$= O(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{p=1}^{n} \psi_p^{\delta k+k-1} p^{-k} \mid t_p^{\alpha} \mid^k$$

$$+ O(1) \mid \lambda_m \mid \sum_{n=1}^{m} \psi_n^{\delta k+k-1} n^{-k} \mid t_n^{\alpha} \mid^k$$

$$= O(1) \sum_{n=1}^{m-1} \mid \Delta \lambda_n \mid \log n + O(1) \mid \lambda_m \mid \log m$$

$$= O(1) \sum_{n=1}^{m-1} \mid B_n \mid \log n + O(1) \mid \lambda_m \mid \log m$$

$$= O(1) \text{ as } m \to \infty,$$

by virtue of the hypotheses of the Theorem and Lemma 2.
Therefore, we get (16). This completes the proof of the Theorem.

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