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ON WEAKLY SYMMETRIC AND WEAKLY RICCI-SYMMETRIC K-CONTACT MANIFOLDS

By U. C. De, T. Q. Binh and A. A. Shaikh

1. Introduction

The notion of locally symmetric Riemannian manifolds has been weakened in several ways and to a different extent by M. C. Chaki [2], [3], M. C. Chaki and S. K. Saha [4], L. Tamássy and T. Q. Binh [9], introducing the notions of pseudosymmetric, pseudo Ricci-symmetric, weakly symmetric and weakly projective symmetric Riemannian manifolds. Manifolds of such kinds were investigated by the above authors in the above mentioned papers and also in [5], [6]. On the other hand, Debasis Tarafdar and U. C. De [8] revealed the incompatibility of K-contact structure with pseudosymmetry and pseudo Ricci-symmetry, provieded these notions do not reduce to simple symmetry.

In this paper we shall give necessary conditions for the compatibility of several K-contact structures with weak symmetry and weak Ricci-symmetry and weak Ricci-symmetry, provided they do not reduce to the common local symmetry, that is, they are proper. Thus weak symmetry and weak Ricci-symmetry are weaker than pseudosymmetry and pseudo Ricci-symmetry respectively.

In a recent paper [10] L. Tamássy and T. Q. Binh studied weakly symmetric and weakly Ricci-symmetric Sasakian manifolds. It is known that every Sasakian manifold is K-contact, but the converse is not true in general. However, a 3-dimensional K-contact manifold is Sasakian. This enables us to get back Tamśsy and Binh's result [10] from our theorems.

2. Weakly symmetric and weakly Ricci-symmetric manifolds

The notions of weakly symmetric and weakly Ricci-symmetric manifolds were introduced by L. Tamássy and T. Q. Binh [9], [10].

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called *weakly symmetric* if there exist 1-forms α, β, γ, δ and σ such that

$$(2.1)\qquad \begin{aligned}(\nabla_X R)(Y,Z,U,V) = {} & \alpha(X)R(Y,Z,U,V) + \beta(Y)R((X,Z,U,V) \\ & + \gamma(Z)R(Y,X,U,V) + \delta(U)R(Y,X,V) \\ & + \sigma(V)R(Y,Z,U,X)\end{aligned}$$

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holds for all vector fields $X, Y, \dots, V \in \mathfrak{X}(M)$, where R is the Riemannian curvature tensor of (M^n, g) of type $(0,4)$ and ∇ is the covariant differentiation with respect to the Riemannian metric g A weakly symmetric manifold is said to be *proper* if $\alpha = \beta = \gamma = \delta = \sigma = 0$ is not the case.

A Rimennian manifold (M^n, g) $(n > 2)$ is called *weakly Ricci-symmetric* if there exist 1-forms ρ, μ, ν such that the relation

$$(V_X S)(Y,Z) = \rho(X)S(Y,Z) + \mu(Y)S(X,Z) + \nu(Z)S(X,Y) \tag{2.2}$$

holds for any vector fields X, Y, Z where S is the Ricci tensor of type $(0,2)$ of the manifold M^n. A weakly Ricci-symmetric manifold is said to be proper if $\rho = \nu = \mu = 0$ is not the case.

Recently U. C. De and S. Bandyopadhyay [6] gave an example of a weakly symmetric manifold and found its reduced form as follows:

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) = {}& \alpha(X)R(Y,Z,U,V) + \beta(Y)R(X,Z,U,V) \\ & + \beta(Z)R(Y,X,U,V) + \delta(U)R(Y,Z,X,V) \\ & + \sigma(V)R(Y,Z,U,X).\end{aligned} \tag{2.3}$$

Let $\{e_i\}$, $(i = 1, 2, \dots, n)$ be an orthonormal basis of the tangent space at point of the manifold. Then, putting $Y = V = e_i$ in (2.3) and taking summation for $1 \le i \le n$, we obtain

$$\begin{aligned}(V_X S)(Z,U) = {}& \alpha(X)S(Z,U) + \beta(Z)S(X,U) + \delta(U)S(Z,X) \\ & + \beta(R(X,Z)U) + \delta(R(X,U)Z).\end{aligned} \tag{2.4}$$

3. K-contact Riemannian manifolds

Let (M^n, g) be a contact Riemannian manifold with contact form η, associated vector field ξ, $(1,1)$-tensor field φ and associated Riemannian metric g. If ξ is a Killing vector field, then (M^n, g) is called a K-contact Riemannian manifold [1], [7]. A K-contact manifold is Sasakin of and only if the relation

$$(\nabla_X \varphi)(Y) = g(X,Y)\xi - \eta(Y)X \quad \text{holds for all } X, Y. \tag{3.1}$$

In a contact Riemannian manifold (M^n, g) the following relations hold [1], [7]:

$$\varphi\xi = 0, \quad \eta(\xi) = 1, \quad \eta o \varphi = 0, \tag{3.2}$$

$$\varphi^2 X = -X + \eta(X)\xi, \tag{3.3}$$

$$g(X,\xi) = \eta(X), \tag{3.4}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{3.5}$$

for any vector fields X, Y.

If (M^n, g) is a K-contact manifold, then besides (3.2)–(3.5), the following relations hold [1], [7]:

$$\nabla_X \xi = -\varphi X, \tag{3.6}$$

$$g(R(\xi, X)Y, \xi) = \eta(R(\xi, X)Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3.7}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{3.8}$$

$$R(\xi, X)\xi = -X + \eta(X)\xi, \tag{3.9}$$

$$(\nabla_X \varphi)(Y) = R(\xi, X)Y \quad \text{for any vector fields } X, Y. \tag{3.10}$$

Further, since ξ is a Killing vector, the Lie derivative of S and the scalar curvature r vanish, i.e.

$$L_\xi S = 0 \tag{3.11}$$

and

$$L_\xi r = 0. \tag{3.12}$$

4. Weakly symmetric K-contact manifolds

We suppose that the weakly symmetric manifold (M^n, g) $(n > 3)$ is K-contact. Since the manifold is weakly symmetric, we have (2.4) which, by putting $X = \xi$ and using (3.8), yields

$$\begin{aligned}(\nabla_\xi S)(Z, U) = \alpha(\xi)S(Z, U) + (n-1)\,[\beta(Z)\eta(U) + \delta(U)\eta(Z)] \\ + \beta(R(\xi, Z)U) + \delta(R(\xi, U)Z).\end{aligned} \tag{4.1}$$

¿From (3.11), it follows that

$$(\nabla_\xi S)(Z, U) = -S(\nabla_Z \xi, U) - S(Z, \nabla_U \xi). \tag{4.2}$$

By virtue of (3.6), we get from (4.2)

$$(\nabla_\xi S)(Z, U) = S(\varphi Z, U) + S(Z, \varphi U). \tag{4.3}$$

Now, since φ is skew-symmetric, the Ricci operator Q is symmetric and $Q\varphi = \varphi Q$ in a K-contact manifold, we obtain from (4.3)

$$(\nabla_\xi S)(Z, U) = 0, \tag{4.4}$$

where the Ricci operator Q is associated with S by $g(QX, Y) = S(X, Y)$.
¿From (4.1) and (4.4), we have

$$\begin{aligned}\alpha(\xi)S(Z, U) + (n-1)\,[\beta(Z)\eta(U) + \delta(U)\eta(Z)] \\ +\beta(R(\xi, Z)U) + \delta(R(\xi, U)Z) = 0.\end{aligned} \tag{4.5}$$

Putting $Z = U = \xi$ in (4.5) and then using (3.8) and (3.2), we get

$$(n-1)\,[\alpha(\xi) + \beta(\xi) + \delta(\xi)] = 0,$$

which gives us (since $n > 3$),

$$\alpha(\xi) + \beta(\xi) + \delta(\xi) = 0. \tag{4.6}$$

This means that the vanishing of the 1-form $\alpha + \beta + \delta$ over the Killing vector field ξ of (M^n, g) $(n > 3)$ is necessary in order that the manifold (M^n, g) $(n > 3)$ be a K-contact manifold. We show that $\alpha + \beta + \delta = 0$ is also necessary for this.

Now, putting $U = \xi$ in (2.4) and then using (3.8), we get

$$\begin{aligned}(\nabla_X S)(Z, \xi) = (n-1)\,[\alpha(X)\eta(Z) + \beta(Z)\eta(X)] + \delta(\xi)S(Z, X) \\ + \beta(R(X, Z)\xi) + \delta(R(X, \xi)Z).\end{aligned} \tag{4.7}$$

We also have

$$(\nabla_X S)(Z, \xi) = \nabla_X S(Z, \xi) - S(\nabla_X Z, \xi) - S(Z, \nabla_X \xi),$$

which by virtue of (3.8) and (3.6) yields,

$$(\nabla_X S)(Z, \xi) = (n-1)\,[\nabla_X \eta(Z) - \eta(\nabla_X Z)] + S(Z, \varphi X).$$

The above relation can also be written by means of (3.6) in the from

$$(\nabla_X S)(Z, \xi) = -(n-1)g(Z, \varphi X) + S(Z, \varphi X). \tag{4.8}$$

Hence from (4.7) and (4.8) we get

$$\begin{aligned}(n-1)\,[\alpha(X)\eta(Z) + \beta(Z)\eta(X)] + \delta(\xi)S(Z, X) + \beta(R(X, Z)\xi) \\ + \delta(R(X, \xi)Z) = -(n-1)g(Z, \varphi X) + S(Z, \varphi X).\end{aligned} \tag{4.9}$$

Putting $X = \xi$ in (4.9) and then using (3.2), (3.8) and (3.9), we obtain (since $R(\xi, \xi)Z = 0$),

$$(n-1)\,[\alpha(\xi) + \delta(\xi)]\,\eta(Z) + (n-2)\beta(Z) + \beta(\xi)\eta(Z) = 0. \tag{4.10}$$

Replacing Z by X in (4.10) we have

$$(n-1)\,[\alpha(\xi) + \delta(\xi)]\,\eta(X) + (n-2)\beta(X) + \beta(\xi)\eta(X) = 0. \tag{4.11}$$

Again, substituting Z by ξ in (4.9), by virtue of (3.2), (3.8) and (3.9) we get,

$$(n-1)\,[\alpha(X) + \beta(X)] + \delta(X) + (n-2)\,[\beta(\xi) + \delta(\xi)]\,\eta(X) = 0. \tag{4.12}$$

Adding (4.11) and (4.12), we obtain

$$\begin{aligned}(n-1)\,[\alpha(X) + \beta(X)] + \delta(X) + (n-1)\,[\alpha(\xi) + \beta(\xi) + \delta(\xi)]\,\eta(X) \\ + (n-2)\delta(\xi)\eta(X) = 0.\end{aligned} \tag{4.13}$$

Using (4.6) in (4.13), we have

$$(n-1)\,[\alpha(X) + \beta(X)] + \delta(X) + (n-2)\delta(\xi)\eta(X) = 0. \tag{4.14}$$

Now, putting $Z = \xi$ in (4.5) and using (3.2), (3.8) and (3.9), we get

$$(n-1)\,[\alpha(\xi) + \beta(\xi)]\,\eta(U) + (n-2)\delta(U) + \delta(\xi)\eta(U) = 0. \tag{4.15}$$

Replacing U by X in (4.15) we have

$$(n-1)\,[\alpha(\xi) + \beta(\xi)]\,\eta(X) + (n-2)\delta(X) + \delta(\xi)\eta(X) = 0. \tag{4.16}$$

Addition of (4.14) and (4.16) gives by virtue of (4.6)

$$\alpha(X) + \beta(X) + \delta(X) = 0 \quad \text{for all } X. \tag{4.17}$$

Hence from (4.17) we can state the following

Theorem 1. *There exist no weakly symmetric K-contact manifolds $M^n(\varphi,\eta,\xi,g)$ $(n>3)$ if $\alpha+\beta+\delta$ is not everywhere zero.*

Since every Sasakian manifold is K-contact we also can state the

Corollary 4.1. *There exist no weakly symmetric Sasakian manifolds $M^n(\varphi,\eta,\xi,g)$ $(n<2)$ if $\alpha+\beta+\delta$ is not everywhere zero.*

The above Corollary 4.1 has been proved by Tamássy and Binh [10].

5. Weakly Ricci symmetric K-contact manifolds

Let us consider a weakly Ricci-symmetric K-contact manifold (M^n,g) $(n>3)$. By virtue of (3.8), the relation (2.2) gives us

$$(\nabla_\xi S)(Y,Z)=\rho(\xi)S(Y,Z)+(n-1)\,[\mu(Y)\eta(Z)+\nu(Z)\eta(Y)] \tag{5.1}$$

By virtue of (4.4) and (5.1), we have

$$\rho(\xi)S(Y,Z)+(n-1)\,[\mu(Y)\eta(Z)+\nu(Z)\eta(Y)]=0. \tag{5.2}$$

Putting $Y=Z=\xi$ in (5.2) and then using (3.2) and (3.8), we obtain (since $n>3$)

$$\rho(\xi)+\mu(\xi)+\nu(\xi)=0. \tag{5.3}$$

Again, replacing Y by ξ in (2.2), by virue of (4.8) and (3.8) we get,

$$\begin{aligned}(n-1)\,[\rho(X)\eta(Z)+\nu(Z)\eta(X)]+\mu(\xi)S(X,Z)\\ =-(n-1)g(Z,\varphi X)+S(Z,\varphi X).\end{aligned} \tag{5.4}$$

Putting $Z=\xi$ in (5.4) and then using (3.2) and (3.8), we get

$$(n-1)\,[\rho(X)+\mu(\xi)\eta(X)+\nu(\xi)\eta(X)]=0. \tag{5.5}$$

Also, substituting X by ξ in (5.4), by virtue of (3.2) and (3.8) we obtain,

$$(n-1)\,[\rho(\xi)\eta(Z)+\nu(Z)+\mu(\xi)\eta(Z)]=0 \quad \text{for all } Z.$$

Replacing Z by X, the above relation reduces to

$$(n-1)\,[\rho(\xi)\eta(X)+\nu(X)+\mu(\xi)\eta(X)]=0. \tag{5.6}$$

Adding (5.5) and (5.6), we have by virtue of (5.3),

$$\rho(X)+\nu(X)+\mu(\xi)\eta(X)=0. \tag{5.7}$$

Now, putting $Z=\xi$ in (5.2) and then using (3.2) and (3.8), we get (since $n>3$),

$$\mu(Y)+[\rho(\xi)+\nu(\xi)]\,\eta(Y)=0, \qquad \text{for all } Y,$$

from which follows that

$$\mu(X)+[\rho(\xi)+\nu(\xi)]\,\eta(X)=0. \tag{5.8}$$

Adding (5.7) and (5.8) and then using (5.3), we obtain

$$\rho(X)+\mu(X)+\nu(X)=0 \quad \text{for all } X. \tag{5.9}$$

Hence we can state the following

Theorem 2. *There exists no weakly Ricci symmetric K-contact manifold $M^n(\varphi,\eta,\xi,g)$ $(n>3)$ if $\rho+\mu+\nu$ is not everywhere zero.*

Corollary 5.1. *There exists no weakly Ricci-symmetric Sasakian manifold $M^n(\varphi,\eta,\xi,g)$ $(n>2)$ if $\rho+\mu+\nu$ is not everywhere zero.*

This Corollary has been proved in [10].

6. Weakly symmetric almost Einstein manifolds

Definition 6.1. A Riemannian manifold (M^n, g) is said to be *almost Einstein* if the Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\omega(X)\omega(Y), \quad \text{for all } X, Y, \tag{6.1}$$

where a and b are costants, and ω is a non-zero 1-form defined by $\omega(X) = g(X, \widetilde{\rho})$.

Wow consider a weakly symmetric manifold (M^n, g) which is almost Einsten. Then we have (2.4). Now (4.2) can be written as

$$\begin{aligned}(\nabla_X S)(Y,Z) &= \alpha(X)S(Y,Z) + \beta(Y)S(X,Z) + \delta(Z)S(X,Y)\\ &= \beta(R(X,Y)Z) + \delta(R(X,Z)Y).\end{aligned} \tag{6.2}$$

Let $g(X, L) = \beta(X)$ and $g(X, M) = \delta(X)$. From (6.1) we get

$$(\nabla_X S)(Y,Z) = b\left[(\nabla_X\omega)(Y)\omega(Z) + \omega(Y)(\nabla_X\omega)(Z)\right]. \tag{6.3}$$

¿From (6.2) and (6.3) we obtain

$$\begin{aligned}&\alpha(X)S(Y,Z) + \beta(Y)S(X,Z) + \delta(Z)S(X,Y) + \beta(R(X,Y)Z)\\ &\quad + \delta(R(X,Z)Y) = b\left[(\nabla_X\omega)(Y)\omega(Z) + \omega(Y)(\nabla_X\omega)(Z)\right].\end{aligned} \tag{6.4}$$

Putting $Y = Z = e_i$ in (6.4) and then taking the sum for $1 \leq i \leq n$, we get

$$\alpha(X)r + 2S(X,L) + 2S(X,M) = 2b\sum_{i=1}^{n}(\nabla_X\omega)(e_i)\omega(e_i).$$

Using (6.1) in the above equation, we obtain

$$\begin{aligned}&\alpha(X)r + 2\left[ag(X,L) + b\omega(X)\omega(L)\right] + 2\left[ag(X,M) + b\omega(X)\omega(M)\right]\\ &\qquad = 2b\sum_{i=1}^{n}(\nabla_X\omega)(e_i)\omega(e_i).\end{aligned} \tag{6.5}$$

The right hand side of (6.5) can be written as

$$2b\sum_{i=1}^{n}\left[X\omega(e_i) - \omega(\nabla_X e_i)\right]\omega(e_i).$$

Let $\{e_i\}$ be an orhonormal basis at $T_\rho M$, $\rho \in M$. Let us translate these e_i parallel from ρ in any direction X_ρ. Then $(\nabla_X e_i)_\rho = 0$. Then the right hand side of (6.5) reduces to

$$\begin{aligned}2b\sum_{i=1}^{n}(X\omega(e_i))\omega(e_i) &= 2b\sum_{i=1}^{n}(Xg(e_i,\widetilde{\rho}))g(e_i,\widetilde{\rho})\\ &= 2b\sum_{i=1}^{n}g(\nabla_X\widetilde{\rho}, e_i)g(e_i,\widetilde{\rho})\\ &= 2bg(\nabla_X\widetilde{\rho},\widetilde{\rho}) = bXg(\widetilde{\rho},\widetilde{\rho}) = bX\|\omega\|^2.\end{aligned}$$

Hence (6.5) takes the form

$$r\alpha + 2a\beta + 2a\delta + 2b\omega(L)\widetilde{\rho} + 2b\omega(M)\widetilde{\rho} = b\|\omega\|^2.$$

Thus we have the following

Theorem 3. *There exists no weakly symmetric almost Einstein manifold if* $r\alpha + 2a(\beta+\delta) + 2b(\omega(L)\widetilde{\rho} + \omega(M)\widetilde{\rho}) - b\|\omega\|^2$ *is not everywhere zero.*

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T. Q. Binh
Lajos Kossuth University
Institute of Mathematics and Informatics
H-4010 Debrecen, P.O. Box 12
Hungary
E-mail address: binh@math.klte.hu

U. C. De and A. A. Shaikh
Department of Mathematics
University of Kalyani
Kalyani-741235
W.B., India
E-mail address: ucde@klyuniv.ernet.in