SELFINJECTIVE KOSZUL ALGEBRAS

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The study of Koszul algebras and their representations has accelerated significantly in the last few years. They have been used in Topology, Algebraic Geometry and Commutative Algebra and they are used more and more frequently in Representation Theory, see for instance [BGS],[GTM],[M],[MZ] and [R]. The aim of this paper is to present some of the results presented by the second author at the Luminy conference that will appear in [MZ], as well as some new facts about the shapes of the components of the Auslander-Reiten quivers of selfinjective Koszul algebras. Namely, we show that if we have a selfinjective Koszul algebra of Loewy length greater than three having a noetherian Koszul dual, then each component of its graded A-R quiver is of the form $\mathbf{Z}A_\infty$ and we show that this need not be the case for the usual A-R quiver if we ignore the grading. By a Koszul algebra we mean a graded associative algebra over a field K, $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \dots$ satisfying the following conditions: (1) for each i, the K- dimension of Λ_i is finite, and for each $i, j \geq 0$ we have $\Lambda_i\Lambda_j = \Lambda_{i+j}$; (2) $\Lambda_0 \simeq K \times \cdots \times K$, and (3) the Yoneda ext-algebra of Λ, $E(\Lambda) = \oplus_{n\geq 0} \operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)$ is generated as a graded K-algebra by the degree zero and one parts. For more general facts about Koszul algebras and modules with linear resolutions we refer to the wonderful survey article of Fröberg, as well as to [GM]. The ext algebra of a Koszul algebra Λ is also denoted $\Lambda^!$ and is also called the Koszul dual, or the shriek algebra of Λ. Recall that a graded module has a linear (graded) projective resolution, if the module is generated in degree zero, and, there exists a graded resolution of the module such that the first projective module in the resolution is generated in degree zero, the second one in degree one, the third one in degree two, and so on. So in a way, these modules have projective resolutions whose terms have the generators distributed in the best possible places. Note that linear resolutions are always minimal. The modules having these types of resolutions, and their graded shifts are also called Koszul modules, and let $\mathcal{K}_\Lambda$ be the subcategory of $\operatorname{gr}\Lambda$ consisting of all the modules having linear resolutions. If Λ is a Koszul algebra, then so is $E(\Lambda)$ and we have a contravariant functor $E : \operatorname{mod}\Lambda \to \operatorname{gr} E(\Lambda)$ that induces a duality between the subcategories $\mathcal{K}_\Lambda$ and $\mathcal{K}_{E(\Lambda)}$. The notion of Koszul modules was generalized in [GM] to what we call now weakly a Koszul module. Observe that the definition makes sense even in the nongraded case. If Λ is a Koszul algebra with graded radical J, then a module M is weakly Koszul, if there exists a projective resolution of M:$\cdots \to \mathcal{P}_n \xrightarrow{f_n} \mathcal{P}_{n-1} \to \cdots \to \mathcal{P}_0 \xrightarrow{f_0} M \to 0$ such that for each $i, k \geq 0$ we have $J^{k+1}\mathcal{P}_i \cap \operatorname{Ker} f_i = J^k \operatorname{Ker} f_i$. Note again, that such a resolution, if it exists, must be minimal. Throughout this paper we will always assume that we deal with Koszul algebras. The modules will always be finitely generated and graded, and unless specified, the homomorphisms will be degree zero homomorphisms.

1. Weakly Koszul modules

1.1. We start first by giving equivalent characterizations to the notion of weakly Koszul module. Note that the class of weakly Koszul modules contains the Koszul modules and

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is also closed under syzygies. It is also closed under certain types of extensions. In fact, we have the following result, see [GM1]:

Proposition. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of graded Λ-modules and assume that $J^kB \cap A = J^kA$ for each $k \geq 0$.(i) If A, B are weakly Koszul, then so is C.(ii) If A and C are weakly Koszul then B is also weakly Koszul.(iii) If A, B and C are weakly Koszul, then we have a short exact sequence of weakly Koszul modules $0 \to \Omega A \to \Omega B \to \Omega C \to 0$, where Ω denotes the first syzygy, and we also have $J^k\Omega B \cap \Omega A = J^k\Omega A$ for each $k \geq 0$.*

We can obtain very nice filtrations for our weakly Koszul modules and then we can develop inductive ways of proving nice things about them. Using the above lemma we can prove the following:

Theorem. *Let $M = M_i \oplus M_{i+1} \oplus \ldots$ be a graded weakly Koszul module with $M_i \neq 0$, and let $K_M =< M_i >$ be the submodule of M generated by the degree i part. Then: (i)] K_M is a Koszul module; (ii)] M/K_M is a weakly Koszul module.*

Proof. We first see that for each $k \geq 0$ we have that $J^kM \cap K_M = J^kK_M$; we don't even need the fact that M is weakly Koszul for this to hold. Next, one can show by induction that K_M is a Koszul module, and then we apply part (i) of the previous proposition and obtain our result. □

1.2. Let Λ be a Koszul algebra and let J denote its radical. Let M be a graded Λ-module. Its associated graded module is $G(M) = M/JM \oplus JM/J^2M \oplus \ldots$. The associated graded module of M is a graded module over the associated graded algebra $G(\Lambda) = \Lambda/J \oplus J/J^2 \oplus \ldots$. In our case, since Λ is Koszul, it is isomorphic as a graded algebra to its associated graded algebra so $G(M)$ can be viewed as a graded Λ module. We state now the main result of this section:

Theorem. *The following statements are equivalent about a graded Λ-module M: (i) M is a weakly Koszul module; (ii) $G(M)$ is a Koszul Λ-module generated in degree zero; (iii) $E(M)$ is a Koszul $E(\Lambda)$-module generated in degree zero.*

Proof. The equivalence of the first two statements follows from 1.1 by induction on p where M is generated in degrees $i_0 < i_1 < \ldots < i_p$. Let us sketch now one direction of the equivalence of (i) and (iii). Assume that M is a weakly Koszul module. Then it is easy to show that JM is again weakly Koszul ([GM1]), and by applying the functor E to the short exact sequence $0 \to JM \to M \to M/JM \to 0$, we obtain a short exact sequence of graded $E(\Lambda)$-modules

$$0 \to E(JM)(-1) \to E(M/JM) \to E(M) \to 0$$

Since JM is again weakly Koszul, we obtain in a similar fashion a short exact sequence of graded $E(\Lambda)$-modules

$$0 \to E(J^2M)(-2) \to E(JM/J^2M)(-1) \to E(JM)(-1) \to 0$$

By induction $E(M)$ has a linear resolution over $E(\Lambda)$ so $E(M)$ is a Koszul module over $E(\Lambda)$ generated in degree zero. □

It would be interesting to describe the graded Λ-modules M having the property that $E(M)$ is weakly Koszul.

1.3. We now consider the case where Λ is a finite dimensional Koszul algebra whose ext-algebra is noetherian. It turns out then that $\Lambda^! = E(\Lambda)$ has in fact finite global dimension. To see this we start by showing that each graded simple $\Lambda^!$-module has finite graded projective dimension, therefore it has finite projective dimension. The next step is showing that every finitely generated graded module over the Koszul dual has finite projective dimension. To obtain the passage from graded to the nongraded case, we use the well known inequality $\operatorname{pd} M_\Lambda \leq \operatorname{pd} G(M)_{G(\Lambda)}$ where $G(M)$ denotes the associated graded module of M over $\Lambda \cong G(\Lambda)$. Using this observation we have the following theorem:

Theorem. *Let Λ be a finite dimensional Koszul algebra whose ext-algebra is noetherian, and let M be a finitely generated graded Λ-module. Then there exists a positive integer k such that $\Omega^k M$ is weakly Koszul. If in addition Λ is a selfinjective algebra, then there exists a k such that $\tau^k M$ is weakly Koszul, where τ denotes the Auslander-Reiten translate.*

Proof. For the selfinjective case, one has first to show that if $\nu = DHom_\Lambda(-,\Lambda)$ is the Nakayama equivalence, then ν takes weakly Koszul modules into weakly Koszul modules, and then we use the fact that $\tau = \nu\Omega^2$ and the first part of the theorem. □

1.4. We have a very nice application of the previous theorem to the rationality of Poincaré series. Recall that if M is a finitely generated graded Λ module, then its Poincaré series $P_\Lambda^M(t)$ is defined as $P_\Lambda^M(t) = \sum_{n\geq 0} \dim_K \operatorname{Ext}^n_\Lambda(M,\Lambda_0)t^n$. It is well known that each finitely generated Koszul module has a rational Poincaré series if the algebra is a finite dimensional Koszul algebra, and using 1.1 and induction, one can prove the same thing about weakly Koszul modules. If the algebra is a commutative noetherian local algebra such that its Koszul dual is noetherian, it is well known that our algebra is a complete intersection, and in that case it is known that every module has a rational Poincaré series. Using the previous remarks and 1.3., we have that, if Λ is a finite dimensional Koszul algebra whose ext-algebra is noetherian, and if M is a finitely generated graded Λ-module, then M has a rational Poincaré series.

2. Selfinjective Koszul algebras

2.1. The primary motivation for this part came from a paper by Ringel. He proved that if Λ is a selfinjective Koszul algebra of Loewy length greater than three, then if a component of the Auslander-Reiten quiver contains a Koszul module, then the stable part of this component is of the type $\mathbf{Z}A_\infty$ and moreover, the Koszul module is at the mouth of the component ([R]). We can generalize this result as follows:

Theorem. *Let Λ be a selfinjective Koszul algebra of Loewy length greater than three. Let $\mathcal{C}$ be a connected component of the (graded) Auslander-Reiten quiver of Λ containing a weakly Koszul module. Then the stable part of $\mathcal{C}$ is of type $\mathbf{Z}A_\infty$. If, in addition the ext-algebra $E(\Lambda)$ is noetherian, then every stable component of the Auslander-Reiten quiver is of the type $\mathbf{Z}A_\infty$.*

Proof. To show that the first part holds, we use ideas from [ABPRS]. Namely, we show first that the almost split sequence ending at a weakly Koszul module has at most two indecomposable middle terms and that if there are two indecomposable summands in the middle, then the map from the middle term to the right hand term is the sum of an irreducible epimorphism and an irreducible monomorphism. Then we look at the weakly Koszul of smallest length lying in the component, and conclude that it must lie on the mouth so that the stable part is of the form $\mathbf{Z}A_\infty$. For the last part, if the ext-algebra of Λ is noetherian, then each graded component contains a weakly Koszul module, and then we apply the first part. Note that in order to show that the stable part of a component is of the form $\mathbf{Z}A_\infty$ we could have also used Shiping Liu's methods by proving there there is an infinite sectional path in the component □

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For a while we thought that every component containing a weakly Koszul module must also contain a Koszul module (lying at the mouth of course, by [R]). It turns out that this not need be the case as Jin Guo has kindly pointed out to us. Observe also that, each component of the Auslander-Reiten quiver containing a weakly Koszul module of Loewy length less that $n-1$ where n denotes the Loewy length of Λ is still of type $\mathbf{Z}A_\infty$ even if we forget about the grading. This raises the question whether we can have tubes in the ordinary Auslander-Reiten quiver. It turns out that this is indeed the case, even in the nice case of the exterior algebra, as we will see in the next section.

3. Skew tensor products

3.1. Since our construction is done over the exterior algebra, we need to recall some facts about skew tensor products. Recall that, if Λ and Γ are two graded K-algebras, then their skew tensor product $\Lambda \boxtimes \Gamma$ is defined as the usual tensor product of vector spaces $\Lambda \otimes_K \Gamma$, and where the product is defined as follows: if λ_1, λ_2 are homogeneous elements of Λ, and if γ_1, γ_2 are homogeneous elements of Γ, then we let $(\lambda_1 \otimes \gamma_1)(\lambda_2 \otimes \gamma_2) = (-1)^{(\deg \gamma_1)(\deg \lambda_2)}\lambda_1\lambda_2 \otimes \gamma_1\gamma_2$.. In particular, $(\lambda \otimes 1)(1 \otimes \gamma) = \lambda \otimes \gamma$, and if both λ and γ are homogeneous elements of degree 1, then $(1 \otimes \gamma)(\lambda \otimes 1) = -(\lambda \otimes 1)(1 \otimes \gamma)$, so the multiplication is anti-commutative. If M and N are two graded left Λ and Γ modules respectively, then we can define their skew tensor product $M \boxtimes N$ as the usual tensor product of K-vectors spaces $M \otimes_K N$, and with the left action of $\Lambda \boxtimes \Gamma$ defined as $(\lambda \otimes \gamma)(m \otimes n) = (-1)^{(\deg \gamma)(\deg m)}\lambda m \otimes \gamma n$. It is easy to see that if Λ and Γ are two graded algebras, and if $f : M \to X$, and $g : N \to Y$ are two graded maps of Λ-modules (Γ-modules respectively), then we have an induced $\Lambda \boxtimes \Gamma$-graded map between the correponding skew tensor products $f \boxtimes g : M \boxtimes N \to X \boxtimes Y$. It also turns out that skew tensoring is an exact functor, so we also obtain that the skew tensor product commutes with finite direct sums and we have natural isomorphisms of graded $\Lambda \boxtimes \Gamma$-modules $M \boxtimes (X \oplus Y) \cong (M \boxtimes X) \oplus (M \boxtimes Y)$ and also $(M \oplus N) \boxtimes X \cong (M \boxtimes X) \oplus (N \boxtimes X)$.

Let $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \dots$, and $\Gamma = \Gamma_0 \oplus \Gamma_1 \oplus \dots$ be two graded K-algebras where the initial subalgebras Λ_0 and Γ_0 are both products of finitely many copies of K, and let $E(\Lambda)$ and $E(\Gamma)$ be their ext-algebras. Let $M \in \operatorname{gr}\Lambda$ and $N \in \operatorname{gr}\Gamma$ and set $E(M) = \oplus_{n\geq 0} \operatorname{Ext}^n_\Lambda(M, \Lambda_0)$ in $\operatorname{gr} E(\Lambda)$, and define $E(N)$ in $\operatorname{gr} E(\Gamma)$ similarly. Finally, let $\Lambda \otimes_K \Gamma$ denote the usual tensor product of graded algebras. Using the usual standard arguments one can prove the following

Theorem. *There exist a natural isomorphism of graded algebras $\Psi : E(\Lambda) \boxtimes E(\Gamma) \to E(\Lambda \otimes_K \Gamma)$, and a natural isomorphism of graded $E(\Lambda)\boxtimes E(\Gamma)$-modules $\Phi : E(M)\boxtimes E(N) \to E(M \otimes_K N)$*

In particular, if we let $\Lambda = \bigwedge V$, and $\Gamma = \bigwedge W$ be the exterior algebras of V and W where V and W are two K-vector spaces, then the skew tensor product $\Lambda \boxtimes \Gamma$ is the exterior algebra $\bigwedge(V \oplus W)$. We will need the following result:

Proposition. *Assume that K is an algebraically closed field and that Λ and Γ are two graded K-algebras. Let M be finitely generated indecomposable graded Λ module, and let N be a finitely generated indecomposable Γ-module. Then $M \otimes_K N$ is an indecomposable $\Lambda \otimes_K \Gamma$-module.*

Proof. To prove the indecomposability of $M \otimes_K N$, is enough to show that $M \otimes_K N$ has a local endomorphism ring. Observe first that since both M and N are indecomposable, their endomorphism rings are local. Next, we have $\operatorname{Hom}_\Lambda(M, M) \otimes_K \operatorname{Hom}_\Gamma(N, N) \cong \operatorname{Hom}_{\Lambda\otimes_K\Gamma}(M \otimes_K N, M \otimes_K N)$ and therefore, since the radical of $\operatorname{End}_{\Lambda\otimes_K\Gamma}(M \otimes_K N)$ is $\mathbf{r} = J_{\operatorname{End}(M)} \otimes_K \operatorname{End}_\Gamma(N, N) + \operatorname{End}_\Lambda(M, M) \otimes_K J_{\operatorname{End}(N)}$, we obtain

$$\frac{\mathrm{End}_{\Lambda\otimes_K\Gamma}(\mathrm{M} \otimes_K \mathrm{N})}{\mathbf{r}} \cong \frac{\mathrm{End}_\Lambda(\mathrm{M}, \mathrm{M})}{\mathrm{J}_{\mathrm{End}(M)}} \otimes_K \frac{\mathrm{End}_\Gamma(\mathrm{N}, \mathrm{N})}{\mathrm{J}_{\mathrm{End}(N)}} \cong K \otimes_K K \cong K$$

thus $\operatorname{End}_{\Lambda\otimes_K\Gamma}(M\otimes_K N)$ is local, hence $M\otimes_K N$ is an indecomposable $\Lambda\otimes_K\Gamma$-module. □

Let Λ and Γ be two Koszul algebras. Then $\Lambda \boxtimes \Gamma \cong E(E(\Lambda) \otimes_K E(\Gamma))$, so that $\Lambda \boxtimes \Gamma$ is also a Koszul algebra. We have the following easy consequence:

3.2. The following shows one way of constructing indecomposable Koszul modules over the skew tensor product of two Koszul algebras.

Proposition. *Let Λ and Γ be two Koszul algebras, and let M be an indecomposable Koszul Λ-module, and let and N be an indecomposable Koszul Γ-module. Then $M \boxtimes N$ is an indecomposable Koszul $\Lambda \boxtimes \Gamma$-module.*

Proof. We only need to observe that we have an isomorphism of graded $\Lambda \otimes_K \Gamma$-modules $E(M \boxtimes N) \cong E(M) \otimes_K E(N)$ and apply the previous proposition. □

3.3. We want to compute the Auslander-Reiten translate of a $\Lambda \boxtimes \Gamma$-Koszul module of the form $\Lambda \boxtimes M$ where M is an indecomposable nonprojective Koszul Γ-module. If Λ is a connected Koszul algebra, then the previous result implies that $\Lambda \boxtimes M$ is an indecomposable nonprojective Koszul $\Lambda \boxtimes \Gamma$-module. We observe first that since skew tensoring is an exact functor, we have an isomorphism of graded $\Lambda \boxtimes \Gamma$-modules $\Omega_{\Lambda \boxtimes \Gamma}(\Lambda \boxtimes M) \cong \Lambda \boxtimes \Omega_\Gamma M$. Then, using the previous observations, a rather standard argument shows that if M is an indecomposable Koszul Γ-module, then $\tau_{\Lambda \boxtimes \Gamma}(\Lambda \boxtimes M)$ is isomorphic to $\Lambda \boxtimes \tau_\Gamma(M)$. Let now Γ be the exterior algebra in two variables, so Γ is a tame algebra. We know then that there exist Koszul Γ-modules M of period two, that is, we have $M \cong \tau_\Gamma(M)$. Then, if we let Λ denote the exterior algebra in, say $n-2$ variables where $n > 2$, then $\Delta = \Lambda \boxtimes \Gamma$ is isomorphic to the exterior algebra in n variables. Moreover the Koszul Δ module $\Lambda \boxtimes M$ lies in a tube of period two in the ungraded Auslander-Reiten quiver of Δ, but in a $\mathbf{Z}A_\infty$ component of the graded Auslander-Reiten quiver of Δ.

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