

SOME APPLICATIONS OF NEVANLINNA THEORY
TO ENTIRE FUNCTIONS THAT SHARE A SMALL FUNCTION
WITH TWO DIFFERENCE OPERATORS

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ABSTRACT. In this work, we are implementing some applications of Nevanlinna theory to entire functions that share a small function with two difference operators and we also generalize one of the results in the paper [3].

1. INTRODUCTION

Nevanlinna theory is considered one of the most important theories in complex analysis, especially in the study of entire functions. We say that an entire function $a(z)$ is a small function of $f(z)$ if $T(r, a) = S(r, f)$, where $S(r, f) = o(T(r, f))$, as $r \rightarrow \infty$ outside of a possible exceptional set of finite logarithmic measure. We use $S(f)$ to denote the family of all small functions with respect to $f(z)$. For an entire function $f(z)$ we define its shift by

$$f_c(z) = f(z + c),$$

and its difference operators by

$$L_c^n f(z) = \alpha_n f(z + nc) + \cdots + \alpha_1 f(z + c) + \alpha_0 f(z), \quad n \in \mathbb{N}, n \geq 1$$

where $\alpha_n (\neq 0), \dots, \alpha_1, \alpha_0$ are complex numbers. In particular for the case

$$\alpha_j = \binom{n}{j} (-1)^{n-j}, \quad j \in \mathbb{N}, \quad 0 \leq j \leq n$$

we define its difference operators by

$$\Delta_c f(z) = f(z + c) - f(z), \quad L_c^n f(z) = \Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)), \quad n \in \mathbb{N}, n \geq 2.$$

We say that $f(z)$ and $g(z)$ share $a(z)$ *CM* (counting multiplicities), provided that $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros counting multiplicities. The uniqueness of meromorphic functions sharing values with their difference operators has been studied in many papers see e.g. [1, 2, 3, 4, 8, 11], and sharing values with their shifts has been investigated by many authors see e.g. [3, 6, 7, 9].

In 2015, A. El Farissi, Z. Latreuch and A. Asiri [3] proved:

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Theorem A. *Let $f(z)$ be a transcendental entire function of finite order such that $f(z) \not\equiv f(z + c)$. Then $f(z)$, $f(z + c)$ and $\Delta_c f(z)$ can not share any finite value $a \neq 0$ CM. Furthermore; if $a = 0$, $f(z)$ must be of the following form $f(z) = h(z)e^{\frac{\beta}{c}z}$, where $\alpha \neq 0$ and $h(z)$ is a periodic entire function of period c .*

It is interesting now to see what is happening when $f(z)$, $f(z + c)$ and $L_c^n f(z)$ ($n \geq 1$) share $a(z)$ CM. The main result of this paper is to prove that the conclusion of Theorem A, remains valid when we replace $\Delta_c f(z)$ by $L_c^n f(z)$, and we obtain the following results.

Theorem 1.1. *Let $f(z)$ be an entire function of finite order such that $f(z) \not\equiv f_c(z)$, and let $a(z) \in S(f)$ be a periodic entire function with period c . If $f(z)$, $f(z + c)$ and $L_c^n f(z)$ ($n \geq 1$) share $a(z)$ CM, then*

$$f(z) = h(z)e^{\frac{\beta}{c}z} + a(z) \quad \text{and} \quad a(z) = 0 \quad \text{or} \quad \sum_{i=0}^n \alpha_i - 1 = 0,$$

where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c .

In the following examples, we take several cases to illustrate Theorem 1.1:

Example 1.1. In this example we illustrate the case $\sum_{i=0}^n \alpha_i - 1 = 0$ and $a(z) \neq 0$.

The entire function $f(z) = \cos(z) e^{\frac{1}{2\pi}z} + e$ satisfies

$$f(z + 2\pi n) = e^n \cos(z) e^{\frac{1}{2\pi}z} + e, \quad n \in \mathbb{N}.$$

We put $\alpha_i = -1, 1 \leq i \leq n, \alpha_0 = n + 1$ and $a(z) = e$, then

$$L_{2\pi}^n f(z) = -f(z + 2\pi n) - \dots - f(z + 2\pi) + (n + 1)f(z), \quad n \in \mathbb{N}^*.$$

We can get

$$\frac{f(z + 2\pi) - a(z)}{f(z) - a(z)} = \frac{e \cos(z) e^{\frac{1}{2\pi}z}}{\cos(z) e^{\frac{1}{2\pi}z}} = e,$$

and

$$\begin{aligned} \frac{L_{2\pi}^n f(z) - a(z)}{f(z) - a(z)} &= \frac{-\cos(z) e^{\frac{1}{2\pi}z} (e^n + \dots + e) - ne}{\cos(z) e^{\frac{1}{2\pi}z}} \\ &+ \frac{(n + 1) \cos(z) e^{\frac{1}{2\pi}z} + (n + 1)e - e}{\cos(z) e^{\frac{1}{2\pi}z}} = \frac{e^{n+1} - e}{1 - e} + n + 1, \end{aligned}$$

and hence $f(z)$, $f(z + 2\pi)$ and $L_{2\pi}^n f(z)$ share $a(z)$ CM.

Example 1.2. In this example we illustrate the case $a(z) = 0$ and $\sum_{i=0}^n \alpha_i - 1 \neq 0$.

The entire function $f(z) = e^z$ satisfies

$$f(z + n) = e^n e^z, \quad n \in \mathbb{N}.$$

We put $\alpha_i = 1, 0 \leq i \leq n$, then

$$L_1^n f(z) = f(z + n) + \dots + f(z + 1) + f(z), \quad n \in \mathbb{N}^*.$$

We can get

$$\frac{f(z+1)}{f(z)} = \frac{ee^z}{e^z} = e,$$

and

$$\begin{aligned} \frac{L_1^n f(z)}{f(z)} &= \frac{e^z(e^n + \dots + 1)}{e^z} \\ &= \frac{1 - e^{n+1}}{1 - e}, \end{aligned}$$

and hence $f(z)$, $f(z+1)$ and $L_1^n f(z)$ share 0 CM.

Example 1.3. In this example we illustrate the case $\sum_{i=0}^n \alpha_i - 1 = 0$ and $a(z) = 0$.

The entire function $f(z) = \sin(z)e^z$ satisfies

$$f(z + 2\pi n) = e^{2\pi n} \sin(z)e^z, \quad n \in \mathbb{N}.$$

We put $\alpha_i = 1, 1 \leq i \leq n, \alpha_0 = -(n-1)$ and $a(z) = 0$, then

$$L_{2\pi}^n f(z) = f(z + 2\pi n) + \dots + f(z + 2\pi) - (n-1)f(z), \quad n \in \mathbb{N}^*.$$

We can get

$$\frac{f(z + 2\pi)}{f(z)} = \frac{e^{2\pi} \sin(z)e^z}{\sin(z)e^z} = e^{2\pi},$$

and

$$\begin{aligned} \frac{L_{2\pi}^n f(z)}{f(z)} &= \frac{\sin(z)e^z(e^{2\pi n} + \dots + e^{2\pi} - (n-1))}{\sin(z)e^z} \\ &= \frac{e^{2\pi(n+1)} - e^{2\pi}}{e^{2\pi} - 1} - n + 1, \end{aligned}$$

and hence $f(z)$, $f(z + 2\pi)$ and $L_{2\pi}^n f(z)$ share 0 CM.

In this corollary, we have replaced $f(z+c)$ with $L_c^n f(z+c)$ in Theorem 1.1 and we obtained the same result.

Corollary 1.1. *Let $f(z)$ be an entire function of finite order such that $f(z) \not\equiv f_c(z)$, and let $a(z) \in S(f)$ be a periodic entire function with period c . If $f(z)$, $L_c^n f(z)$ and $L_c^n f(z+c)$ ($n \geq 1$) share $a(z)$ CM, then*

$$f(z) = h(z)e^{\frac{\beta}{c}z} + a(z) \quad \text{and} \quad a(z) = 0 \quad \text{or} \quad \sum_{i=0}^n \alpha_i - 1 = 0,$$

where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c .

Example 1.4. The entire function $f(z) = e^{\frac{1}{b}z}$, where $b \neq 0$ satisfies

$$f(z + nb) = e^n e^{\frac{1}{b}z}, \quad n \in \mathbb{N}.$$

We put $\alpha_i = 1$, $0 \leq i \leq n$ and $a(z) = 0$, then

$$L_b^n f(z) = f(z + nb) + \cdots + f(z + b) + f(z), \quad n \in \mathbb{N}^*$$

and

$$L_b^n f(z + b) = f(z + (n + 1)b) + \cdots + f(z + 2b) + f(z + b), \quad n \in \mathbb{N}^*,$$

we can get

$$\frac{L_b^n f(z) - a(z)}{f(z) - a(z)} = \frac{f(z)(e^n + \cdots + e + 1)}{f(z)} = \frac{1 - e^{n+1}}{1 - e},$$

and

$$\begin{aligned} \frac{L_b^n f(z + b) - a(z)}{f(z) - a(z)} &= \frac{f(z)(e^{n+1} + \cdots + e)}{f(z)} \\ &= \frac{e - e^{n+2}}{1 - e}, \end{aligned}$$

and hence $f(z)$, $L_b^n f(z)$ and $L_b^n f(z + b)$ share 0 CM.

It is natural to ask what happens if $L_c^n f(z)$ is replaced by $\Delta_c^n f(z)$ in Theorem 1.1. Corresponding to this question, we obtain the following result.

Theorem 1.2. *Let $f(z)$ be an entire function of finite order such that $f(z) \not\equiv f_c(z)$, and let $a(z) \in S(f)$ be a periodic entire function with period c . If $f(z)$, $f(z + c)$ and $\Delta_c^n f(z)$ ($n \geq 1$) share $a(z)$ CM, then*

$$f(z) = h(z)e^{\frac{\beta}{c}z}, \quad \text{and } a(z) = 0,$$

where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c .

Example 1.5. The entire function $f(z) = \sin(z)e^{\frac{1}{2\pi}z}$ satisfies

$$f(z + 2\pi n) = e^n f(z), \quad n \in \mathbb{N}.$$

We can get

$$\frac{f(z + 2\pi)}{f(z)} = \frac{ef(z)}{f(z)} = e,$$

and

$$\frac{\Delta_{2\pi}^n f(z)}{f(z)} = \frac{f(z) \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} e^i}{f(z)} = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} e^i,$$

and hence $f(z)$, $f(z + 2\pi)$ and $\Delta_{2\pi}^n f(z)$ share 0 CM.

In the following corollary we have replaced $f(z + c)$ with $\Delta_c^n f(z + c)$ in Theorem 1.2, and we obtained the same result.

Corollary 1.2. *Let $f(z)$ be an entire function of finite order such that $f(z) \not\equiv f_c(z)$, and let $a(z) \in S(f)$ be a periodic entire function with period c . If $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^n f(z+c)$ ($n \geq 1$) share a (z) CM, then*

$$f(z) = h(z)e^{\frac{\beta}{c}z}, \quad \text{and} \quad a(z) = 0,$$

where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c .

Example 1.6. The entire function $f(z) = \sin(z) e^{\frac{1}{2\pi}z}$ satisfies

$$f(z + 2\pi n) = e^n f(z), \quad n \in \mathbb{N}.$$

We can get

$$\frac{\Delta_{2\pi}^n f(z)}{f(z)} = \frac{f(z) \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} e^i}{f(z)} = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} e^i,$$

and

$$\frac{\Delta_{2\pi}^n f(z + 2\pi)}{f(z)} = \frac{f(z) \sum_{i=0}^n C_n^i (-1)^{n-i} e^{i+1}}{f(z)} = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} e^{i+1},$$

and hence $f(z)$, $f(z + 2\pi)$ and $\Delta_{2\pi}^n f(z)$ share 0 CM.

2. LEMMAS

For the proof of our results, we need the following lemmas.

Lemma 2.1 ([10]). *Let $c \in \mathbb{C}$, $n \in \mathbb{N}$, and let $f(z)$ be a meromorphic function of finite order. Then for any small periodic function $a(z)$ with period c , with respect to $f(z)$,*

$$m\left(r, \frac{\Delta_c^n f}{f - a}\right) = S(r, f)$$

where the exceptional set associated with $S(r, f)$ is of at most finite logarithmic measure.

Lemma 2.2 ([5]). *Suppose that $f_1(z), f_2(z), \dots, f_n(z)$ ($n \geq 2$) are some meromorphic functions and $g_1(z), g_2(z), \dots, g_n(z)$ ($n \geq 1$) entire functions satisfying the following conditions:*

- (i) $\sum_{j=1}^n f_j(z) e^{g_j(z)} \equiv 0$;
- (ii) $g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$;
- (iii) for $1 \leq j \leq n$, $1 \leq j < k \leq n$. $T(r, f_j) = o\{T(r, e^{g_j(z) - g_k(z)})\}$ ($r \rightarrow \infty$, $r \notin E$).

Then $f_j(z) \equiv 0$, ($j = 1, 2, \dots, n$).

Lemma 2.3 ([5]). *Let $f(z)$ be a non-constant meromorphic function in the complex plane and*

$$R(f) = \frac{P(f)}{Q(f)},$$

where $P(f) = \sum_{k=0}^p a_k f^k$ and $Q(f) = \sum_{j=0}^q b_j f^j$ are two mutually prime polynomials in $f(z)$. If the coefficients a_k, b_j are small functions of $f(z)$ and $a_p(z) \not\equiv 0, b_q(z) \not\equiv 0$, then

$$T(r, R(f)) = \max \{p, q\} T(r, f) + S(r, f) .$$

3. PROOFS OF THE THEOREMS AND COROLLARIES

Proof of Theorem 1.1. Suppose that $f(z), f(z + c)$ and $L_c^n f(z)$ share $a(z)$ CM. Then

$$(3.1) \quad \frac{f(z + c) - a(z)}{f(z) - a(z)} = e^{p(z)},$$

and

$$(3.2) \quad \frac{L_c^n f(z) - a(z)}{f(z) - a(z)} = e^{q(z)},$$

where p and q are polynomials. From (3.1) it's easy to prove the following

$$(3.3) \quad f(z + nc) - a(z) = [f(z) - a(z)] e^{\sum_{i=0}^{n-1} p(z+ic)},$$

by using equations (3.2), and (3.3), we obtain

$$\frac{\alpha_n [f(z) - a(z)] e^{\sum_{i=0}^{n-1} p(z+ic)} + a(z) \alpha_n + \dots + \alpha_0 [f(z) - a(z)] + a(z) \alpha_0 - a(z)}{f(z) - a(z)} = e^{q(z)},$$

then

$$(3.4) \quad \alpha_n e^{\sum_{i=0}^{n-1} p(z+ic)} + \dots + \alpha_1 e^{p(z)} + \alpha_0 + \frac{a(z) \left(\sum_{i=0}^n \alpha_i - 1 \right)}{f(z) - a(z)} = e^{q(z)} .$$

From (3.1) and (3.2), we get

$$(3.5) \quad \frac{L_c^n f(z + c) - L_c^n f(z)}{f(z) - a(z)} = \frac{\alpha_n \Delta_c f(z + nc) + \dots + \alpha_0 \Delta_c f(z)}{f(z) - a(z)} = e^{p(z)+q_c(z)} - e^{q(z)} .$$

Set

$$\varphi(z) = e^{p(z)+q_c(z)} - e^{q(z)} .$$

We show that $\varphi(z) \not\equiv 0$. If $\varphi(z) \equiv 0$, then

$$(3.6) \quad e^{p(z)} = e^{q(z)-q_c(z)},$$

thus, by equations (3.1) and (3.6), we have

$$(3.7) \quad \frac{f(z+c) - a(z)}{f(z) - a(z)} = e^{q(z)-q_c(z)}.$$

If $e^{p(z)}$ is a constant or $\deg q(z) = 1$, then by equations (3.1) and (3.7) respectively, we get

$$\frac{f(z+c) - a(z)}{f(z) - a(z)} = e^\beta,$$

where $\beta \neq 0$, we leave this situation to come back to it in the end.

If $q(z)$ is a constant, then by (3.7) we get the following contradiction

$$f(z+c) = f(z).$$

By equation (3.7) it is easy to see

$$(3.8) \quad \frac{f(z+nc) - a(z)}{f(z) - a(z)} = e^{q(z)-q_{nc}(z)}.$$

By using equations (3.3), (3.8) and (3.4), we have

$$(3.9) \quad -\frac{a(z) \left(\sum_{i=0}^n \alpha_i - 1 \right)}{f(z) - a(z)} = \alpha_n e^{q(z)-q_{nc}(z)} + \dots + \alpha_1 e^{q(z)-q_c(z)} + \alpha_0 - e^{q(z)}.$$

If $a(z) \left(\sum_{i=0}^n \alpha_i - 1 \right) = 0$, then

$$e^{q(z)} = \alpha_n e^{q(z)-q_{nc}(z)} + \dots + \alpha_1 e^{q(z)-q_c(z)} + \alpha_0,$$

we get the following contradiction

$$T(r, e^q) = S(r, e^q).$$

If $a(z) \left(\sum_{i=0}^n \alpha_i - 1 \right) \neq 0$, then by using equations (3.7) and (3.9), we have

$$(3.10) \quad -\frac{a(z) \left(\sum_{i=0}^n \alpha_i - 1 \right)}{f(z) - a(z)} = \alpha_n e^{q(z)-q_{(n+1)c}(z)} + \dots + \alpha_0 e^{q(z)-q_c(z)} - e^{q(z)},$$

thus, by equations (3.9) and (3.10), we have

$$\alpha_n e^{q(z)-q_{(n+1)c}(z)} + (\alpha_{n-1} - \alpha_n) e^{q(z)-q_{nc}(z)} + \dots + (\alpha_0 - \alpha_1) e^{q(z)-q_c(z)} - \alpha_0 = 0,$$

as we know from the above that $\deg q(z) \geq 2$, then by Lemma 2.2 we get the following contradiction

$$\alpha_n = \alpha_{j-1} - \alpha_j = \alpha_0 = 0, \quad 0 \leq j \leq n,$$

thus, we deduce $\varphi(z) \not\equiv 0$.

Since $\varphi(z) \not\equiv 0$, by Lemma 2.1 and equation (3.5), we deduce that

$$\begin{aligned}
 T(r, \varphi) &= m(r, \varphi) \\
 (3.11) \quad &\leq m\left(r, \frac{\Delta_c f(z + nc)}{f - a(z)}\right) + \dots + m\left(r, \frac{\Delta_c f(z)}{f - a(z)}\right) + S(r, f) = S(r, f).
 \end{aligned}$$

Note that $\frac{e^{p(z)+q_c(z)}}{\varphi(z)} - \frac{e^{q(z)}}{\varphi(z)} = 1$. By using the second main theorem and equation (3.11), we have

$$\begin{aligned}
 T\left(r, \frac{e^q}{\varphi}\right) &\leq \overline{N}\left(r, \frac{e^q}{\varphi}\right) + \overline{N}\left(r, \frac{\varphi}{e^q}\right) + \overline{N}\left(r, \frac{1}{\frac{e^q}{\varphi} + 1}\right) + S\left(r, \frac{e^q}{\varphi}\right) \\
 &= \overline{N}\left(r, \frac{e^q}{\varphi}\right) + \overline{N}\left(r, \frac{\varphi}{e^q}\right) + \overline{N}\left(r, \frac{\varphi}{e^{p+q_c}}\right) + S\left(r, \frac{e^q}{\varphi}\right) \\
 (3.12) \quad &= S(r, f) + S\left(r, \frac{e^q}{\varphi}\right).
 \end{aligned}$$

Thus, by equations (3.11) and (3.12), we have

$$(3.13) \quad T(r, e^q) = S(r, f).$$

Similarly, we get

$$(3.14) \quad T(r, e^p) = S(r, f).$$

By using the first main theorem, we have

$$(3.15) \quad T\left(r, \frac{a(z)\left(\sum_{i=0}^n \alpha_i - 1\right)}{f - a(z)}\right) = T(r, f) + S(r, f).$$

From equations (3.4) and (3.15), we deduce that

$$(3.16) \quad T(r, f) \leq T(r, e^{i=0}^{n-1}) + \dots + T(r, e^p) + T(r, e^q) + S(r, f).$$

If $a(z)\left(\sum_{i=0}^n \alpha_i - 1\right) \neq 0$, by equations (3.13), (3.14) and (3.16), we deduce the contradiction

$$T(r, f) \leq S(r, f)$$

and from this, we deduce that either $a(z) = 0$ or $\sum_{i=0}^n \alpha_i - 1 = 0$, and by equation (3.4), we have

$$(3.17) \quad \alpha_n e^{i=0}^{n-1} + \dots + \alpha_1 e^{p(z)} + \alpha_0 = e^{q(z)}.$$

Next, we prove that $p(z)$ and $q(z)$ are constants. We need to treat the following cases:

First of all, we set

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0 = a_n z^n + \alpha(z)$$

and

$$q(z) = b_m z^m + b_{m-1} z^{m-1} + \dots + b_0 = b_m z^m + \gamma(z),$$

where $a_n \neq 0, a_{n-1}, \dots, a_0, b_m \neq 0, b_{m-1}, \dots, b_0$ are constants, α and γ are polynomials where $\deg \alpha \leq n - 1$, and $\deg \gamma \leq m - 1$.

On the other hand we have

$$\begin{aligned} \sum_{i=0}^{j-1} p(z + ic) &= p(z) + p(z + c) + \dots + p(z + (j - 1)c) \\ &= ja_n z^n + \lambda_j(z), \end{aligned}$$

where λ_j are polynomials with degree at most $n - 1$ for $j = 1, 2, \dots, n$. By equation (3.17), we have

$$(3.18) \quad \alpha_n \left(e^{a_n z^n} \right)^n e^{\lambda_n(z)} + \dots + \alpha_1 e^{a_1 z^n} e^{\lambda_1(z)} + \alpha_0 = e^{b_m z^m} e^{\gamma(z)}.$$

Define functions $H(z) = e^{a_n z^n}$, and $G(z) = e^{b_m z^m}$. Then, equation (3.18) becomes

$$(3.19) \quad \alpha_n [H(z)]^n e^{\lambda_n(z)} + \dots + \alpha_1 H(z) e^{\lambda_1(z)} + \alpha_0 = G(z) e^{\gamma(z)}.$$

(i) If $m \neq n$, then we have two subcases:

Case (A): If $m < n$, then by using equation (3.19) and applying Lemma 2.3, we see that

$$nT(r, H) = S(r, H),$$

which is impossible.

Case (B): If $n < m$, then, by using equation (3.19) and applying Lemma 2.3, we see that

$$T(r, G) = S(r, G),$$

which is impossible.

(ii) If $n = m \neq 0$, then we have two subcases:

Case (A): If $b_m = ja_n, 1 \leq j \leq n$, then by using equation (3.18), we have $\alpha_n [H(z)]^n e^{\lambda_n(z)} + \dots + \alpha_j [H(z)]^j \left(e^{\lambda_j(z)} - e^{\gamma(z)} \right) + \dots + H(z) e^{\lambda_1(z)} + \alpha_0 = 0$, then by Lemma 2.3 we deduce the contradiction

$$nT(r, H) = S(r, H).$$

Case (B): If $b_m \neq ja_n, 1 \leq j \leq n$, then by using equation (3.18), we have

$$\alpha_n e^{na_n z^n + \lambda_n(z)} + \dots + \alpha_1 e^{a_1 z^n + \lambda_1(z)} + \alpha_0 = e^{b_m z^n + \gamma(z)},$$

then by Lemma 2.2 we deduce the contradiction

$$1 = \alpha_n = \alpha_j = \alpha_0 = 0, \quad 0 \leq j \leq n.$$

Finally, we conclude that $p(z)$ and $q(z)$ are constants, suppose that $e^{p(z)} = e^\beta$ (the same situation we left earlier) where $\beta \neq 0$, from equation (3.1), we have

$$(3.20) \quad f(z + c) - a(z) = e^\beta [f(z) - a(z)].$$

If $f(z)$ and $g(z)$ are two solutions of the equation (3.20), then $h(z) = \frac{f(z)-a(z)}{g(z)-a(z)}$ is a periodic function of period c . Obviously $g(z) = e^{\frac{\beta}{c}z} + a(z)$ is solution of (3.20). Hence the entire solution of (3.20) must be of the form $f(z) = h(z)e^{\frac{\beta}{c}z} + a(z)$, where $h(z)$ is a periodic entire function of period c . \square

Proof of Corollary 1.1. Suppose that $f(z)$, $L_c^n f(z)$ and $L_c^n f(z+c)$ share $a(z)$ CM. Then

$$(3.21) \quad \frac{L_c^n f(z) - a(z)}{f(z) - a(z)} = e^{p(z)},$$

and

$$(3.22) \quad \frac{L_c^n f(z+c) - a(z)}{f(z) - a(z)} = e^{q(z)},$$

where p and q are polynomials. By using equation (3.21), we deduce that

$$(3.23) \quad \frac{L_c^n f(z+c) - a(z)}{f(z+c) - a(z)} = e^{p(z+c)}.$$

By equations (3.22) and (3.23), we get the following result

$$(3.24) \quad \frac{f(z+c) - a(z)}{f(z) - a(z)} = e^{q(z)-p(z+c)},$$

and finally, using equations (3.21) and (3.24), we can deduce $f(z)$, $f(z+c)$ and $L_c^n f(z)$ ($n \geq 1$) share $a(z)$ CM, then by Theorem 1.1 we conclude that

$$f(z) = h(z)e^{\frac{\beta}{c}z} + a(z) \quad \text{and} \quad a(z) = 0 \quad \text{or} \quad \sum_{i=0}^n \alpha_i - 1 = 0,$$

Where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c . \square

Proof of Theorem 1.2. Suppose that $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$ share $a(z)$ CM, we follow the same steps as in Proof of Theorem 1.1. Equation (3.4) becomes

$$\alpha_n e^{\sum_{i=0}^{n-1} p(z+ic)} + \dots + \alpha_1 e^{p(z)} + \alpha_0 - \frac{a(z)}{f(z) - a(z)} = e^{q(z)},$$

because we know that, if

$$L_c^n f(z) = \Delta_c^n f(z),$$

then

$$\sum_{i=0}^n \alpha_i = 0.$$

We continue with the same steps without forgetting that

$$\sum_{i=0}^n \alpha_i = 0.$$

In the proof of the previous theorem, following equation (3.16), we concluded that

$$a(z) = 0 \quad \text{or} \quad \sum_{i=0}^n \alpha_i - 1 = 0,$$

but since

$$\sum_{i=0}^n \alpha_i = 0,$$

we conclude that

$$a(z) = 0.$$

From this, we continue with the same steps until the end of the previous proof. Finally, we conclude that

$$f(z) = h(z)e^{\frac{\beta}{c}z}, \quad \text{and} \quad a(z) = 0,$$

where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c . \square

Proof of Corollary 1.2. Suppose that $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^n f(z+c)$ share $a(z)$ CM. Then

$$(3.25) \quad \frac{\Delta_c^n f(z) - a(z)}{f(z) - a(z)} = e^{p(z)},$$

and

$$(3.26) \quad \frac{\Delta_c^n f(z+c) - a(z)}{f(z+c) - a(z)} = e^{q(z)},$$

where p and q are polynomials. By using equation (3.25), we deduce that

$$(3.27) \quad \frac{\Delta_c^n f(z+c) - a(z)}{f(z+c) - a(z)} = e^{p(z+c)},$$

By equations (3.26) and (3.27), we get the following result

$$(3.28) \quad \frac{f(z+c) - a(z)}{f(z) - a(z)} = e^{q(z)-p(z+c)},$$

and finally, using equations (3.25) and (3.28), we can deduce $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$ ($n \geq 1$) share $a(z)$ CM, then by Theorem 1.2 we conclude that

$$f(z) = h(z)e^{\frac{\beta}{c}z}, \quad \text{and} \quad a(z) = 0,$$

where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c . \square

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