ARCHIVUM MATHEMATICUM (BRNO)
Tomus 60 (2024), 61–99

METRIC ENRICHMENT, FINITE GENERATION, AND THE PATH COREFLECTION

Alexandru Chirvasitu

Abstract. We prove a number of results involving categories enriched over CMet, the category of complete metric spaces with possibly infinite distances. The category CPMet of path complete metric spaces is locally $\aleph_1$-presentable, closed monoidal, and coreflective in CMet. We also prove that the category CCMet of convex complete metric spaces is not closed monoidal and characterize the isometry-$\aleph_0$-generated objects in CMet, CPMet and CCMet, answering questions by Di Liberti and Rosický. Other results include the automatic completeness of a colimit of a diagram of bi-Lipschitz morphisms between complete metric spaces and a characterization of those pairs (metric space, unital C^*-algebra) that have a tensor product in the CMet-enriched category of unital C^*-algebras.

Introduction

Denote by CMet the category of complete metric spaces with contractive maps as morphisms, and distance functions allowed infinite values (see Convention 1.1). [3, Example 2.3 (2)] notes that CMet is *symmetric monoidal closed* [17, §§1.1, 1.4, 1.5], so it is a good candidate category for *enriching* over in the sense of [17].

Many categories of interest in functional analysis are CMet-*enriched* or CMet-*categories* in the sense of [17, §1.2]: for every two objects $x, y \in \mathcal{C}$ in the category of interest there is a *morphism object* $[x, y] \in$ CMet, there is an associative composition

$$[y, z] \otimes [x, y] \to [x, z]$$

for an appropriate monoidal structure on CMet, etc. Natural examples are in rich supply:

- CMet is self-enriched, the space of contractions between two complete metric spaces being metrized with the supremum distance;
- Ban, consisting of Banach spaces and linear maps of norm ≤ 1;

2020 *Mathematics Subject Classification*: primary 54E50; secondary 54E40, 51F30, 18A30, 18C35, 18D20, 18D15, 46L05, 46L09.

Key words and phrases: complete metric space, path metric, intrinsic metric, gluing, convex, monoidal closed, enriched, tensored, locally presentable, colimit, internal hom.

Received May 26, 2022, revised September 2023. Editor J. Rosický.

DOI: 10.5817/AM2024-2-61

- the category BanAlg_1 of (complex) unital Banach algebras or its variations BanAlg_1^* (unital complex Banach $*$-algebras), $\text{BanAlg}_{c,1}$ (*commutative* unital Banach algebras), etc.;
- $\mathcal{C}_1^*$, the category of unital C^*-algebras, or $\mathcal{C}_{c,1}^*$, that of *commutative* unital C^*-algebras.

Such metric-flavored category-theoretic considerations are by now pervasive in the literature: in discussing universal (Gurarii) Banach spaces [19, 20], or universal operators thereon [12], or more general issues of approximate embeddability [25, 3]; these are only a handful of examples, each with its own extensive cited literature.

The initial motivation for the present paper were a number of questions arising naturally in [10], in studying *local generation* in this enriched setting. Roughly speaking, an object x in a category $\mathcal{C}$ is κ-generated for a cardinal κ if $\hom_{\mathcal{C}}(x,-)$ preserves "sufficiently directed" colimits. Suppose, now, that $\mathcal{V}$ is what we will refer to as an *enriching category*: symmetric monoidal closed, complete and cocomplete (the assumptions of [17, §2.5 onward], for instance, or [10, §4]). When the category is $\mathcal{V}$-enriched one can instead consider

$$[x,-]\colon \mathcal{C} \to \mathcal{V},$$

leading to the notion studied in [10]. Formally, aggregating, say, [2, Definition 1.13] and [10, Definitions 2.1 and 4.1]:

Definition 0.1. Let κ be a regular cardinal.

- A poset $(I,\le)$ is *κ-directed* if every subset of I of cardinality $<\kappa$ has an upper bound.
- A *κ-directed colimit* in a category is a colimit of a functor defined on a κ-directed poset (regarded as a category, with an arrow $i\to j$ when $i\le j$).
- An object $x\in\mathcal{C}$ in a category is *$\mathcal{M}$-κ-generated* for a class of morphisms $\mathcal{M}$ if

$$\hom(x,-):\mathcal{C}\to \text{Set}$$

  preserves κ-directed colimits of morphisms in $\mathcal{M}$.
- Similarly, if $\mathcal{C}$ is $\mathcal{V}$-enriched, x is *$\mathcal{M}$-κ-generated in the enriched sense* (or *enriched $\mathcal{M}$-κ-generated*) if the above colimit-preservation condition holds for the enriched-hom functor

$$[x,-]\colon \mathcal{C}\to\mathcal{V}$$

  instead, where now the colimits in question are the *conical* ones of [17, §3.8] (cf. [10, §4, first paragraph] and [16, §1.1]).

Being κ-generated is a kind of smallness condition: in, say, categories of modules over rings, it literally means being generated by fewer than κ elements [2, Proposition 3.10]. For that reason, it is also customary to refer to $\aleph_0$-generated objects as *finitely generated*; this is the finite generation of the paper's title.

[10, Remark 6.9] briefly considers CCMet as another candidate to enrich over: this is the category of complete *convex* metric spaces, i.e. those for which pairs of points a finite distance apart can be connected by curves that realize that distance (this differs slightly from the definition adopted in [10]; see Definition 2.1 and surrounding discussion).

Given that the finite segments $[0,\ell] \in$ CCMet are in a sense the basic building blocks of CCMet, it is natural to ask, as [10, Remark 6.9] does, whether they are enriched-finitely-generated in the sense of Definition 0.1, with respect to the class of isometries. It turns out that not only is the answer negative, but finite generation is rather difficult to come by in any of the categories of interest. Summarizing Theorems 4.3 and 4.8 and Corollary 4.9:

Theorem. *In any of the categories*
- CMet *of complete metric spaces;*
- CPMet *of complete path metric spaces;*
- *or* CCMet *of complete convex metric spaces*

the isometry-$\aleph_0$-generated objects are precisely the finite discrete metric spaces, i.e. those with all pairwise distances infinite.

This also generalizes [3, Proposition 5.19], which proves that in CMet, the only isometry-$\aleph_0$-generated finite spaces are the discrete ones (i.e. in that statement finiteness is assumed).

Path or *intrinsic* metric spaces are recalled in Definition 2.3: they are those for which points a finite distance ℓ apart are connectable with curves of length arbitrarily close to ℓ; they thus intermediate between plain (complete) metric spaces and convex ones.

The appearance of CPMet in the discussion is at least in part motivated by another question asked in [10] (immediately preceding [10, Remark 6.10]): whether CCMet is monoidal closed. It is not (Example 2.25), but essentially because the right adjoint to the inclusion functor

$$\iota\colon \mathrm{CPM{\scriptsize ET}} \subset \mathrm{CM{\scriptsize ET}}$$

fails, in general, to produce convex spaces: see Proposition 2.9 and Corollary 2.10. CPMet, on the other hand, is much better behaved; coalescing Lemma 3.2, Corollary 3.3, and Theorems 3.4 and 3.5:

Theorem. *The full subcategory*

$$\mathrm{CPM{\scriptsize ET}} \subset \mathrm{CM{\scriptsize ET}}$$

of complete path metric spaces is
- *coreflective;*
- *locally $\aleph_1$-presentable (so in particular complete and cocomplete);*
- *and closed monoidal.*

We highlight a number of pathologies in otherwise well-behaved metric-enriched categories:
- the failure of CCMet to be monoidal closed in Example 2.25;
- the paucity of $\aleph_0$-generated objects in CPMet or CCMet (Theorem 4.8 and Corollary 4.9).

All of this requires piecing together metric spaces by the *gluing* process of [8, §3.1.2] (see §2.1 below). Gluing, say, metric spaces X_i, $i = 1, 2$ along a common subspace is nothing but a pushout in the category Met of (perhaps incomplete) metric spaces, and our examples need such colimits to have various desired properties (completeness, convexity, etc.). This is ensured by a number of auxiliary results I have not been able to locate in the literature.

To state a joint summary of Theorems 2.21 and 2.31, recall (e.g. [8, Definition 1.4.6]) that a map

$$f : (X, d_X) \to (Y, d_Y)$$

in Met is *bi-Lipschitz* if there are both bounds to how much it can scale distances, either up or down: for some $C, C' > 0$ we have

$$Cd_X(x, x') \le d_Y(fx, fx') \le C' d_X(x, x')\,,\ \forall x, x' \in X\,.$$

For added precision, we incorporate the constants into the term and call such maps (C, C')-*bi-Lipschitz*. The two aforementioned theorems then amalgamate to

Theorem. *Let Γ be an oriented forest (in the graph-theoretic sense) of finite diameter D and*

$$F : \Gamma \to \mathrm{CMet}$$

a functor consisting of $(C, 1)$-bi-Lipschitz morphisms.

(a) *The colimit $(X, d) := \varinjlim F$ of F in* Met *is then automatically complete, and hence also a colimit in* CMet.

(b) *And the canonical morphisms*

$$F(v) \to X,\ v \text{ a vertex of } \Gamma$$

are $(C', 1)$-bi-Lipschitz with C' depending only on C and the diameter D.

Gluing is also helpful in rendering a metric space convex. This produces not quite a reflection of CMet into CCMet, but rather a *weak* reflection (it will not, in general, have the requisite universality property requisite of a reflection functor). Nevertheless, Proposition 2.36 reads

Proposition. *For any complete metric space $(X, d) \in$ CMet, attaching intervals of length $d(x, x') < \infty$ with endpoints $x, x' \in X$ for any point pair not already connected by such an interval produces a complete convex metric space.*

As somewhat of a side-note, but in the same general circle of ideas, we identify in Section 4 those pairs

$$X \in \mathrm{CMet}, \quad C \in \mathcal{C}_1^* := \text{unital } C^*\text{-algebras}$$

that have a *tensor product* $X \otimes C$. This is by definition a unital C^*-algebra that represents the functor

$$[X, [C, -]] : \mathcal{C}_1^* \to \mathrm{CMet}\,,$$

and whether or not such tensor products always exist in an enriched category is yet another measure of how convenient it is to work with ($\mathcal{V}$-enriched categories admitting tensor products in this sense are called $\mathcal{V}$-*tensored* [17, §3.7]). In the context of metric enrichment, there is a discussion of the matter in [3, §4].

The earlier [9, Proposition 3.11] says that the category $\mathcal{C}^*_{c,1}$ of *commutative* unital C^*-algebras is CMet-tensored. On the other hand, Theorem 4.1 below negates the existence of tensors in $\mathcal{C}^*_1$ fairly strongly: in a sense, only the “obvious” ones exist.

Theorem. *For a complete metric space $X \in$ CMet and a unital C^*-algebra $B \in \mathcal{C}^*_1$ the tensor product $X \otimes B \in \mathcal{C}^*_1$ exists if and only if one of the following conditions holds:*
- *X has cardinality ≤ 1;*
- *or B has dimension ≤ 1.*

1. Preliminaries

[10, Remark 6.9] makes a number of observations on the category CCMet of *convex* complete generalized metric spaces, where
- ‘generalized’ means that distances are allowed infinite values;
- and convexity for a metric space (X, d) is as in, say, [18, §2.5]: for every $x \ne y \in X$ there is some $z \ne x, y$ *metrically between* x and y in the sense that

$$d(x, y) = d(x, z) + d(z, y)\,.$$

Convention 1.1. It is very natural, in the context of the present discussion, to work with possibly-infinite metrics; for that reason, we adopt the terminology of [8, Defiition 1.1.1]: the phrase ‘metric space’ allows for infinite distances. If, on occasion, we encounter $\mathbb{R}_{\ge 0}$-valued metrics and wish to emphasize the matter, we refer to these as *finite* distance functions or metrics.

Keeping this possible distance infinitude in mind, we write
- Met for the category of metric spaces;
- and CMet for that of *complete* metric spaces (following, say, [3, Example 2.3 (2)] and [10, §6]).

In both cases the morphisms are the *contractions* $f\colon (X, d_X) \to (Y, d_Y)$:

$$d_Y(fx, fx') \le d_X(x, x'),\ \forall x, x' \in X\,. \tag{1-1}$$

The contractions are also the *1-Lipschitz* maps of [14, Definition 1.1]: λ-Lipschitz, for positive λ, would mean (1-1) with the right-hand side scaled by λ.

Recalling the notion of κ-directedness from Definition 0.1, we remind the reader of [2, Definition 1.17]:

Definition 1.2. Let κ be a regular cardinal and $\mathcal{C}$ a category.
- An object $x \in \mathcal{C}$ is *κ-presentable* if $\hom_{\mathcal{C}}(x, -)$ preserves κ-directed colimits.
- $\mathcal{C}$ is *locally κ-presentable* if it is cocomplete and every object is a κ-directed colimit of κ-presentable objects.
- Finally, $\mathcal{C}$ is *locally presentable* if it is locally κ-presentable for some regular cardinal κ.

As observed in [3, Examples 2.3 (1) and (2)], Met and CMet are both locally $\aleph_1$-presentable.

We follow

- [10, §6] in writing $\mathbf{2}_\delta$ for the two-point space $\{x, x'\}$ with $d(x, x') = \delta \in \mathbb{R}_{>0} \cup \{\infty\}$;
- and [3, Example 2.3 (1)] in referring to metric spaces all of whose pairwise distances are infinite as *discrete*.

2. Convex metric spaces

In order to avoid some slightly bothersome corner cases (e.g. the issue of whether or not the two-point space $\mathbf{2}_\infty$ is convex) we depart from [10, §6] slightly in what is meant by 'convex':

Definition 2.1. A metric space $(X, d) \in \mathrm{CMet}$ is *convex* if for every $x \neq y \in X$ with $d(x, y) < \infty$ there is some $z \in X$ distinct from both x and y such that

$$d(x, y) = d(x, z) + d(z, y)\,.$$

In other words, we only require such "intermediate" points z for $x, y \in X$ a *finite* distance apart. This also conflicts slightly with the notion introduced in [8, Definition 3.6.5], where convexity automatically entails (by definition) the finiteness of the metric.

Per the discussion in [10, Remark 6.9], CCMet is symmetric monoidal with the tensor product $(X, d_X) \otimes (Y, d_Y)$ given by the Cartesian product $X \times Y$ as a set, together with the ℓ^1 metric:

$$d_{X\otimes Y}((x, y), (x', y')) := d_X(x, x') + d_Y(y, y')\,,\ \forall x, x' \in X\,,\ \forall y, y' \in Y\,.$$

It is a natural question (asked in passing in loc.cit.) whether this monoidal structure is closed. The existence of an internal hom object

$$[X, Y] \in \mathrm{CCMet}$$

makes sense for each pair of objects (X, d_X) and (Y, d_Y) in CCMet: by definition, it would be the object representing the contravariant functor

$$\mathrm{CCMet}(- \otimes X, Y)\colon \mathrm{CCMet}^{op} \to \mathrm{Set}\,.$$

Naturality in X or Y, when these objects exist, follows from this characterization and Yoneda (e.g. [1, Corollary 6.19]).

[10, §6] also considers the category CMet of complete metric spaces (i.e. CCMet sans convexity). It is monoidal closed, with

$$[X, Y]_{\mathrm{CMet}} \cong (\mathrm{CMet}(X, Y), d_{\sup})\ : \tag{2-1}$$

see [3, Remark 2.2 and Example 2.3 (2)]. Since CCMet is monoidal and full in CMet, the following simple remark tells us how the respective internal homs would relate to one another.

Lemma 2.2. *Let $\mathcal{V} \subseteq \mathcal{V}_0$ be a full monoidal subcategory of a monoidal closed category. For objects $X, Y \in \mathcal{V}$, the internal hom $[X, Y]_{\mathcal{V}}$ exists if and only if the object $[X, Y]_{\mathcal{V}_0} \in \mathcal{V}_0$ has a coreflection in $\mathcal{V}$, and in that case $[X, Y]_{\mathcal{V}}$ is that coreflection.*

Proof. Indeed, $[X, Y]_{\mathcal{V}}$ would have precisely the same universal property as the $\mathcal{V}$-coreflection of $[X, Y]_{\mathcal{V}_0}$: representing the contravariant SET-valued functor

$$\mathcal{V}(- \otimes X, Y) \cong \mathcal{V}_0(- \otimes X, Y) \cong \mathcal{V}_0(-, [X, Y]_{\mathcal{V}_0})$$

on $\mathcal{V}$. □

As in the above equation, we occasionally decorate the internal hom by the category where it is intended to live: $[X, Y]$ is also $[X, Y]_{\mathcal{V}}$.

We will need some more metric-geometry vocabulary, for which we refer to [14, Chapter 1] and [8, Chapter 2]. A small amount of care is needed in adapting some statements from the former source, where 'metric space' has the more conventional meaning allowing only for finite metrics [14, Introduction].

First, as we recall shortly, the distance of a complete convex metric space can, in a sense, be recovered from contractive paths in the space. The relevant notions follow ([14, Definitions 1.2 and 1.7] or [8, Definitions 2.1.6, 2.1.10 and 2.3.1]).

Definition 2.3. Let (X, d) be a metric space.

- The *length* $\ell(f)$ of a continuous curve $f\colon [a, b] \to X$ is

$$\ell(f) := \sup \sum_{i=0}^{n} d\big(f(t_i), f(t_{i+1})\big)\,,$$

where the supremum is taken over all selections of intermediate points

$$a = t_0 \leq t_1 \leq \cdots \leq t_n = t_{n+1} = b\,.$$

- A curve $f\colon [a, b] \to X$ is *rectifiable* if $\ell(f) < \infty$.
- The *path metric* d_ℓ attached to d is

$$d_\ell(x, y) := \inf \ell(f),\ \ f\colon [a, b] \to X,\ \ f(a) = x \text{ and } f(b) = y\,.$$

- (X, d) is a *path metric space* if $d = d_\ell$.
- A path metric space (X, d) is *strict* (and its metric is *strictly path*) if any two points x, x' with $d(x, x') < \infty$ can be connected by a path of length $d(x, x')$.

Remarks 2.4.

(1) It is immediate from the definition of d_ℓ that $d \leq d_\ell$, but in general the inequality is strict. Indeed, as observed in [14, Example 1.4 (a)], even the topologies induced by the two metrics are generally distinct: path components in the d-topology are *clopen* (both closed and open) in the d_ℓ-topology.

This same class of examples also shows that even when d takes only finite values, d_ℓ might not: $d_\ell(x, y) = \infty$ whenever x and y are in different path components.

(2) [14, Remark following Proposition 1.6] notes that for any (X, d), the resulting metric space (X, d_ℓ) is in fact a path metric space because the construction $d \mapsto d_\ell$ is idempotent:

$$(d_\ell)_\ell = d_\ell\,.$$

We take this for granted implicitly below.

(3) It is not difficult to see that if (X, d) is complete then so is (X, d_ℓ): see the proof of Proposition 2.9.

(4) Our notion of 'strictly path' is somewhat weaker than that of [8, Definition 2.1.10]; the latter automatically implies that all distances are finite.

Indeed, loc.cit. asks that *any* two points x and x' be the endpoints of a continuous map from an interval (of length precisely $d(x, x')$, but this is beside the point here). If $d(x, x') = \infty$ then x and x' lie in distinct clopen components in the topology induced by d, so this cannot happen.

Some of the results in [8] seem to ignore the issue of infinite distances, so that some care is required in applying them to generalized metric spaces: for [8, Theorem 2.4.16, part 1.] to hold, for instance,

- one must assume the metric is finite;
- or extend the notion of 'strictly path' to possibly ∞-valued metrics, as in the present definition;
- in which case, for [8, Lemma 2.4.8] to hold, one would have to also modify [8, Definition 2.4.7] of *midpoints* by requiring the defining constraint only for finite-distance pairs (as in Remark 2.8).

As we are working with categories of metric spaces where morphisms are *contractive*, it is perhaps worth noting the following alternative description of the $d \mapsto d_\ell$ construction.

Lemma 2.5. *For a metric space (X, d_X) and points $x,\ x' \in X$ the path metric $d_{X,\ell}$ can be recovered as*

$$\begin{aligned} \text{(2-2)}\quad & d_{X,\ell}(x, x') \\ &= \inf\{\ell \in \mathbb{R}_{\ge 0} \mid \exists \textit{ contraction } \varphi\colon [0, \ell] \to (X, d_X),\ \varphi(0) = x\,,\ \varphi(\ell) = x'\}\,. \end{aligned}$$

Proof. Consider a rectifiable curve

$$f\colon [a, b] \to X\,,\ a \mapsto x\,,\ b \mapsto x'\,.$$

By [8, Proposition 2.5.9] it decomposes as $f = \varphi \circ \alpha$ for non-decreasing $\alpha\colon [a, b] \to [0, \ell(f)]$ and an *arc-length-parametrized* ([8, Definition 2.5.7 and discussion following Remark 2.5.8])

$$\varphi\colon [0, \ell(f)] \to X\,,\ 0 \mapsto x\,,\ \ell(f) \mapsto x'\,.$$

We now have $\ell(f) = \ell(\varphi)$ (i.e. composition with a non-decreasing map makes no difference to the length), and φ is contractive. □

And an immediate consequence that we will take for granted repeatedly in the sequel:

Corollary 2.6. *A path metric space is strict in the sense of Definition 2.3 if and only if all finite infima* (2-2) *are achieved (i.e. are actual minima).*

It is a classical result of Menger's [22] that complete convex metric spaces are strict path metric spaces. Much more is true though; before stating the full result, recall ([18, §2.5], [6, Definition 14.2] and [14, Definition 1.9]):

Definition 2.7. Let (X, d) be a metric space.

A *metric segment* or *minimizing geodesic* in X is an isometry $f\colon [a, b] \to X$ from a finite interval (with its usual distance function) to X.

We refer to $f(a)$ and $f(b)$ as the *endpoints* of the segment or say that the segment (geodesic) *connects* them.

Menger's theorem, referred to above, says that not only are complete convex metric spaces strictly path, but in fact, for any two points x, y (with $d(x, y) < \infty$ in our present context of *generalized* metric spaces), there is a minimizing geodesic connecting x and y; see for instance [13, unnumbered Theorem preceding Lemma 2.1] or [6, Theorem 14.1] for proofs (the result also appears as [18, Theorem 2.16]).

Remark 2.8. We have now come full-circle back to path metrics: for a complete metric space (X, d), the following are equivalent:

(a) convexity;

(b) (X, d) is strictly path in the sense of Definition 2.3;

(c) any two $x, x' \in X$ with $d(x, x') < \infty$ have a *midpoint*: a point y with

$$d(x, y) = d(x', y) = \frac{d(x, x')}{2}\,.$$

Indeed, (a) implies (b) by Menger, while (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) are clear.

Lemma 2.2 suggests that we should study coreflections of CMet in CCMet. The following result describes the circumstances when these exist.

Proposition 2.9. *Consider an object* $(X, d_X) \in$ CMet.

(1) *The path metric* $d := d_{X,\ell}$ *of Definition 2.3 is a complete generalized metric on* X.

(2) X *has a coreflection in* CCMet *precisely when* (X, d) *is strict in the sense of Definition 2.3, in which case* $(X, d) \in$ CCMet *is the coreflection.*

Proof. We tackle the claims in turn.

(**1**) The triangle inequality follows from the fact that contractions

$$[0, \ell] \to X\,, \quad [0, \ell'] \to X$$

ending and respectively starting at the same point splice together to a contraction defined on $[0, \ell + \ell']$. The non-degeneracy condition

$$d(x, y) = 0 \Rightarrow x = y$$

being obvious (for instance because d dominates d_X), we do indeed have a generalized metric. As for completeness: note first that a d-Cauchy sequence $(x_n)_n$ is certainly d_X-Cauchy, because $d \geq d_X$. Such a sequence will thus converge to some $x \in X$ in the original d_X metric. We can now find positive integers

$$n_0 < n_1 < \cdots$$

such that $d(x_{n_{k-1}}, x_{n_k}) < \frac{1}{4^k}$ for $k \geq 1$. We thus have contractive curves

$$\left[\frac{1}{4} + \cdots + \frac{1}{4^{k-1}}\,, \quad \frac{1}{4} + \cdots + \frac{1}{4^k}\right] \to X$$

connecting $x_{n_{k-1}}$ and x_{n_k} respectively. For fixed k those with indices k and higher splice together to a contractive curve

$$\left[\frac{1}{4}+\cdots+\frac{1}{4^{k-1}}\,,\quad \frac{1}{3}\right]\to X$$

connecting $x_{n_{k-1}}$ and x, whence the conclusion that $x_{n_k}\to x$ in the d-topology.

(**2**) If $\mathbf{1}\in$ CCMET is the one-point space (and hence the monoidal unit of both CMET and CCMET) then the functors CCMET$(\mathbf{1},-)$ and CMET$(\mathbf{1},-)$ are both forgetful to SET. It follows from this that a coreflection of $(X,d_X)\in$ CMET in CCMET must be of the form

$$\mathrm{id}\colon (X,d')\to (X,d_X) \tag{2-3}$$

for some alternative distance $d'\ge d_X$, to be determined (when it exists). On to the two implications that constitute claim (2).

(⇐) We already know from part (1) that 2-2 is a complete generalized metric. Note furthermore that any contraction $f\colon (Y,d_Y)\to (X,d_X)$ with Y convex factors through a contraction to (X,d) with d as in (2-2): any two points $y_i\in Y$, $i=0,1$ are (by [18, Theorem 2.16]) the endpoints of a metric segment

$$\gamma\colon [0,d_Y(y_0,y_1)]\to Y\,,$$

so we have a contractive curve

$$\varphi:=f\circ\gamma\colon [0,d_Y(y_0,y_1)]\to X$$

with $\varphi(0)=x_0:=f(y_0)$ and $\varphi(d_Y(y_0,y_1))=x_1:=f(y_1)$. It follows, then, that

$$d_Y(y_0,y_1)\ge d(x_0,x_1) \tag{2-4}$$

for the distance d of (2-2). This shows that $\mathrm{id}\colon (X,d)\to (X,d_X)$ will indeed be a coreflection provided (X,d) is convex, which it is by Remark 2.8.

(⇒) If a coreflection exists, we have already noted it must be of the form (2-3) for some metric d'. It remains to argue that the infima (2-2) are achieved when finite and that (2-2) is the distance function on the coreflection.

To that end, let x, $x'\in X$ with $d(x,x')<\infty$ meaning simply that there are contractive curves connecting x and x'. Any such curve with domain $[0,\ell]$ will factor through (2-3) and hence $\ell\ge d'(x,x')$. But then the infimum (2-2) also dominates $d'(x,x')$; the opposite inequality was noted above, in the proof of (⇐) (see (2-4)), so that $d'=d$.

Finally, the fact that the infimum is in fact achieved then follows from Menger's [18, Theorem 2.16] again: every x, $x'\in X$ with $d(x,x')<\infty$ are the endpoints of a metric segment of length $d(x,x')$. □

We now have the following description of (potential) internal homs in CCMET.

Corollary 2.10. *Let (X,d_X) and (Y,d_Y) be two objects in* CCMET.

(1) *If $[X,Y]\in$ CCMET exists, then it must be* CCMET(X,Y) *equipped with the following metric:*

$$\begin{gathered} d(f,g):= \\ \min\{\ell\mid \exists\ \textit{contractive}\ \varphi\colon [0,\ell]\to (\mathrm{CCMET}(X,Y),d_{\sup}),\ \varphi(0)=f,\ \varphi(\ell)=g\}\,, \end{gathered} \tag{2-5}$$

where

$$d_{\sup}(f, g) := \sup_{x \in X} d_Y\big(f(x), g(x)\big) .$$

In particular, the existence of the internal hom requires that the minimum be achieved whenever the infimum is finite.

(2) *Conversely, if the minima* (2-5) *are achieved for arbitrary* $f,\ g \in \mathrm{CCM{\scriptsize ET}}(X, Y)$ *for which the respective infimum is finite, then* (2-5) *defines a generalized metric on* $\mathrm{CCM{\scriptsize ET}}(X, Y)$ *making it into the internal hom.*

Proof. This is an immediate application of Lemma 2.2, Proposition 2.9 and the description (2-1) of internal homs in CMet. □

Some preparatory remarks follow, aimed at giving sufficient conditions for the existence of internal homs in CCMet.

Proposition 2.11. *Let* $(X, d_X) \in \mathrm{CM{\scriptsize ET}}$ *be a complete generalized metric space. If the finite-radius closed balls of* X *are compact, then*

(a) *the same holds for the internal hom* $[Y, X]_{\mathrm{CM{\scriptsize ET}}}$ *for any compact* $(Y, d_Y) \in$ CMet*;*

(b) *and* (X, d_X) *satisfies the condition in Proposition 2.9 (2), and thus it has a coreflection in* CCMet*.*

Proof. The arguments are very similar, and both rely on *Ascoli's theorem* ([23, Theorem 47.1]) characterizing relatively compact spaces of maps in the *compact-open topology* [23, Definition preceding Theorem 46.8].

(a) As recalled in (2-1), the internal hom is simply the space of contractions with the supremum norm. Every family of contractions being *equicontinuous* [23, Definition preceding Lemma 45.2], this is the case in particular for the radius-r ball $B \subset [Y, X]_{\mathrm{CM{\scriptsize ET}}}$ ($r \in \mathbb{R}_{>0}$) around a contraction $f\colon Y \to X$. For each $y \in Y$ the set

$$\{f'(y) \mid f' \in B\}$$

is contained in a finite-radius ball of X and is thus relatively compact by assumption. The relative compactness of B in the compact-open (hence uniform, Y being compact) topology on

$$\mathrm{{\scriptsize CONT}}(Y \to X)$$

now follows from Ascoli's theorem. Clearly, though, closed balls in $[Y, X]_{\mathrm{CCM{\scriptsize ET}}}$ are also closed in the compact-open topology (indeed, even in the *point*-open topology of [23, Definition preceding Theorem 46.1], which is weaker).

(b) Writing $d := d_{X,\ell}$ for brevity, we have to argue that

$$d(x, x') := \inf\{\ell \mid \exists \text{ contraction } \varphi\colon [0, \ell] \to (X, d_X),\ \varphi(0) = x,\ \varphi(\ell) = x'\}$$

is achieved as an actual minimum whenever it is finite.

Suppose $d(x, x') = r \in \mathbb{R}_{>0}$, and hence we have contractive curves

$$[0, \lambda r] \to X , \quad 0 \mapsto x , \quad \lambda r \mapsto x'$$

for $\lambda > 1$ arbitrarily close to 1. Rescaling, this means λ-Lipschitz curves

$$\gamma_\lambda \colon [0,r] \to X\,,\quad \gamma_\lambda(0) = x\,,\quad \gamma_\lambda(r) = x'.$$

The family $\{\gamma_\lambda\}_\lambda$ is equicontinuous by the λ-Lipschitz condition, and each set

$$\{\gamma_\lambda(t) \mid \lambda\} \quad \text{for} \quad t \in [0,r]$$

is contained in a (compact, by assumption) finite-radius ball in X. It follows that $\{\gamma_\lambda\}_\lambda$ is relatively compact, and as $\lambda \searrow 1$ some subnet will converge to a contraction

$$[0,r] \to X,\ 0 \mapsto x,\ r \mapsto x'.$$

This finishes the proof, the coreflection claim being a consequence of Proposition 2.9 (2). □

Remark 2.12. The path space $(X, d_{X,\ell})$ of Definition 2.3 will not, in general, have compact finite-radius closed balls, even if X does; [14, Example 1.4 (b_+)] illustrates this phenomenon.

One first recovers the standard topology on $X := \mathbb{R}^n$ from the metric

$$d_X(x_1,x_2) := |t_1 - t_2| + \min(t_i)\|s_1 - s_2\|^{\frac{1}{2}}\,,$$

where

$$x_i = t_i s_i\,,\quad t_i \in \mathbb{R}_{\ge 0}\,,\quad s_i \in \mathbb{S}^{n-1}$$

are the respective polar-coordinate descriptions of x_i. Note also that d_X has the same bounded sets as the usual Euclidean metric, since the d_X-distance to the origin equals the usual distance (to the origin again).

As [14, Example 1.4 (b_+)] observes, the corresponding path distance $d_{X,\ell}$ induces on $\mathbb{R}^n$ the strongest topology for which the ray embeddings

$$\iota_s \colon \mathbb{R}_{\ge 0} \ni t \mapsto ts \in \mathbb{R}^n\,,\quad s \in \mathbb{S}^{n-1}$$

are continuous. In $(X, d_{X,\ell})$ the unit sphere $\mathbb{S}^{n-1}$ is discrete, infinite, and contained in the closed unit ball around the origin.

We can now state the aforementioned sufficiency result for internal-hom existence.

Theorem 2.13. *Let (X, d_X) and (Y, d_Y) be two objects in* CCMet *with Y compact. If either of the following equivalent conditions holds then the internal hom $[Y,X] \in$* CCMet *exists:*

- *X is locally compact;*
- *closed balls are compact in X.*

Proof. X is a path metric space in the sense of Definition 2.3 by [18, Theorem 2.16], so indeed local compactness is equivalent to closed balls being compact, by the *Hopf-Rinow theorem* ([14, following Definition 1.9] or [8, Theorem 2.5.28]). The two points of Proposition 2.11 show that $[Y,X]_{\mathrm{CMet}}$ has a coreflection in CCMet, and Lemma 2.2 finishes the proof. □

2.1. **On and around gluing.** In order to argue that (Theorem 2.13 notwithstanding) the category CCMET is *not* closed, we construct examples via the *gluing* procedure outlined in [8, §3.1.2].

Definition 2.14. (a) Let (X, d) be a metric space and '$\sim$' an equivalence relation on X. Define the *quotient semi-metric* ([8, Definition 3.1.12]) $d_\sim$ on $X/\sim$ by

$$d_\sim(x, x') := \inf \sum_{s=0}^{n} d(p_s, q_s)\,,$$

with the infimum taken over all tuples with

$$p_0 = x\,,\quad q_n = x' \quad \text{and} \quad q_s \sim p_{s+1} \quad \forall\, 0 \le s \le n-1\,.$$

The *quotient metric space* $(X/d_\sim, d_\sim)$ (or the metric space obtained by *gluing* (X, d) *along '$\sim$'*) is constructed from the semi-metric $d_\sim$, by identifying pairs of points with zero distance.

(b) Consider metric spaces (X_i, d_i), $i \in I$ and (X, d) equipped with contractions

$$\iota_i\colon (X, d) \to (X_i, d_i)\,.$$

The metric space

$$\coprod_X X_i \quad \text{or} \quad \coprod_{\iota_i} X_i$$

obtained by *gluing* X_i *along* X is constructed by

- first forming the disjoint union $\coprod_i (X_i, d_i)$ equipped with the original distances d_i on the individual X_i and infinite distances across distinct X_i (as in [8, Definition 3.1.15]);
- and then gluing that disjoint union as in item (a), along the relation with equivalence classes

$$\{\iota_i(x) \mid i \in I\} \quad \text{for} \quad x \in X\,.$$

We use points p_s and q_s as in Definition 2.14 frequently, so it will be handy to have a term for the notion.

Definition 2.15. For an equivalence relation '$\sim$' on a space X and points $x, x' \in X$ a *$\sim$-chain connecting x and x'* (or just 'chain' when the relation is understood) is a sequence of pairs $p_s, q_s \in X$, $0 \le s \le n$ with

$$p_0 = x,\ q_n = x',\ q_s \sim p_{s+1}\,, \quad \forall\, 1 \le i \le n-1\,.$$

We will have to construct certain pathological convex metric spaces by gluing, which requires that said glued spaces be strictly path (Remark 2.8). While [8, paragraph following Exercise 3.1.13] notes that the property of being path survives gluing, examples are easily produced of *strictly* path spaces which glue to non-strictly path quotients:

Example 2.16. Consider the family

$$(X_\varepsilon, d_\varepsilon) := [0, 1+\varepsilon]\,, \quad \varepsilon > 0$$

with their usual interval distances, glued along the embeddings

$$\iota_\varepsilon : \mathbf{2}_\infty \to X_\varepsilon$$

identifying the two-point space with the endpoints of said intervals. The result

$$(X,d) := \coprod_{\iota_\varepsilon} (X_\varepsilon, d_\varepsilon)$$

of the gluing procedure consists of length-$(1+\varepsilon)$ intervals with a common pair of endpoints a distance of 1 apart. Said points are not connected by any curves of length precisely 1 though, by construction: the curves $[0,1+\varepsilon] \to X$ have respective lengths $1+\varepsilon$.

In the category MET of (possibly non-complete) metric spaces with contractions the pushout of a pair

$$j_i : Y \to X_i\,, \quad i = 1,2$$

is nothing but the disjoint union $X_1 \coprod X_2$ glued along the relation identifying

$$j_1(y) \sim j_2(y)\,, \quad y \in Y\,.$$

By contrast, a pushout

(2-6)
$$\begin{tikzcd} & X_1 \arrow[dr, "\iota_1"] & \\ Y \arrow[ur, "j_1"] \arrow[dr, "j_2"'] & & X \\ & X_2 \arrow[ur, "\iota_2"'] & \end{tikzcd}$$

in CMET is the *completion* of that glued space. The qualification is crucial, as the gluing alone need not produce a complete space:

Example 2.17. Let

- X_1 be the disjoint union of intervals $[\ell_{2n}, r_{2n}]$ of respective lengths $\frac{1}{2^{2n}}$ for $n \in \mathbb{Z}_{\ge 0}$ (with infinite distances between points on distinct intervals);
- X_2 be the disjoint union of intervals $[\ell_{2n+1}, r_{2n+1}]$ of respective lengths $\frac{1}{2^{2n+1}}$ for $n \in \mathbb{Z}_{\ge 0}$;
- Y a countable discrete metric space;
- and j_i, $i = 1,2$, respectively, the identifications of Y with
  - the endpoints of the "even intervals", minus the leftmost:
    $$r_0,\ \ell_2,\ r_2,\ \ell_4,\ r_4,\ \dots$$
  - the endpoints of the "odd intervals":
    $$\ell_1,\ r_1,\ \ell_3,\ r_3,\ \dots$$

The glued space $X_1 \coprod_Y X_2$ is the splicing together of all of the intervals, consecutively, alternating between even and odd. It is, in short, a (non-complete) half-open interval of length 2.

It will be convenient to glue only under circumstances that avoid the issues exhibited by Example 2.17, in that completeness is automatic. This entails imposing some constraints on the maps j_i in a pushout diagram (2-6).

Definition 2.18. For a constant $C > 0$, a map $f\colon (X, d_X) \to (Y, d_Y)$ between two metric spaces is *C-expansive* if

$$d_Y(fx, fx') \ge C d_X(x, x')\,,\quad \forall x, x' \in X\,.$$

This is meant to hold for *all* pairs of points, including those with $d_X(x, x') = \infty$.

Remark 2.19. The Lipschitz maps (as most of ours are) that are also C-expansive are precisely the *bi-Lipschitz* maps of [8, Definition 1.4.6].

We refer to colimits of diagrams

$$j_i\colon Y \to X_i,\ \ i \in I \tag{2-7}$$

consisting of common-source arrows as 'pushouts', even when the family consists of more than two arrows. In the language of [1, Exercise 11L], say, these would be *multiple* pushouts, but context should serve as sufficient guard against ambiguity.

It will be convenient to set up some language and conventions for handling such diagrams and their colimits. Denote by

$$X := \coprod_{Y,i} X_i$$

the pushout in $\mathrm{M{\scriptsize ET}}$, i.e. the gluing of the disjoint union $\coprod_i X_i$ along the relation identifying, for each $y \in Y$, all $j_i(y) \in X_i$ (in the sense of Definition 2.14; so this entails identifying distance-0 pairs of points). The maps (2-7) will typically be one-to-one for us, as will the canonical contractions $\iota_i\colon X_i \to X$. For that reason, we occasionally identify X_i with its image $\iota_i(X_i) \subseteq X$.

By definition, for points x and x' in X the distance $d_X(x, x')$ is the infimum of the sums

$$\sum_{s=0}^{n} d(p_s, q_s) \tag{2-8}$$

for $\sim$-chains $(p_s, q_s)_s$ as in Definition 2.15, where

$$j_i(y) \sim j_{i'}(y)\,,\ \forall y \in Y\,,\ \forall i, i' \in I\,.$$

It is harmless to make a number of simplifying assumptions on the $\sim$-chains in question.

Definition 2.20. Let (2-7) be morphisms in $\mathrm{M{\scriptsize ET}}$ with the induced relation '$\sim$' on $\coprod_i X_i$. A $\sim$-chain $(p_s, q_s)_{s=0}^n$ connecting $p_0 = x$ and $q_n = x'$ is *streamlined* if

- for each s, the points p_s and q_s are a finite distance apart in $\coprod X_i$ (and in particular belong to the same X_i). For chains *not* satisfying this condition the sum (2-8) is infinite, so they contribute nothing to the infimum;
- no q_s is *equal* to the point p_{s+1} (to which it is supposed to be *equivalent*). Indeed, otherwise we can always collapse the chain to a shorter one without increasing (2-8): if $q_s = p_{s+1}$ then

$$d(p_s, q_s) + d(p_{s+1}, q_{s+1}) = d(p_s, q_s) + d(q_s, q_{s+1}) \ge d(p_s, q_{s+1})\,.$$

In particular, this second condition ensures that the “intermediate” points

$$q_0,\ p_1,\ q_1, \dots,\ p_n$$

all belong to images $j_i(Y) \subseteq X_i$: they must be equivalent to points they are not equal to.

The following result will be helpful in constructing various examples with desired properties by gluing, while making sure that completeness and the path property are preserved.

Theorem 2.21. *Let $1 \ge C > 0$ and consider a family*

$$j_i \colon Y \to X_i\,, \quad i \in I$$

of C-expansive morphisms in CMET*.*

(a) *The canonical maps*

$$\iota_i \colon X_i \to X := \coprod_{Y,j} X_j$$

into the pushout in MET *are C-expansive.*

(b) *The distance d_X between points in the image of Y through the canonical map*

$$\iota := \iota_i \circ j_i \colon Y \to X \quad \textit{(arbitrary } i\textit{)}$$

can be computed as

$$d_X(\iota y, \iota y') = \inf_{\mathbf{i},\mathbf{y}} \sum_{s=0}^{n} d_{X_{i_s}}(j_{i_s} y_s,\ j_{i_s} y_{s+1})\,, \quad y, y' \in Y\,, \tag{2-9}$$

where the infimum is taken over all

$$\mathbf{i} = (i_0,\ i_1, \dots,\ i_n) \subset I \quad \textit{and} \quad \mathbf{y} = (y = y_0,\ y_1, \dots,\ y_{n+1} = y') \subset Y\,.$$

In particular, $\iota \colon Y \to X$ is also C-expansive.

(c) *For $x \in X_i$ and $x' \in X_{i'}$ with $i \ne i'$ we have*

$$d_X(\iota_i x,\ \iota_{i'} x') = \inf_{y, y' \in Y} \left(d_{X_i}(x, j_i y) + d_X(\iota y, \iota y') + d_{X_{i'}}(j_{i'} y', x') \right)\,. \tag{2-10}$$

(d) *For $x,\ x' \in X_i$ the distance $d_X(\iota_i x, \iota_i x')$ is the smaller of* (2-10) *(with $i' = i$) and $d_{X_i}(x, x')$.*

(e) *The space X is automatically complete, and hence also the* CMET*-pushout of the j_i.*

Proof. We prove the statements in the order in which they were made.

(a) Consider two points x, x' in some X_{i_0}, for $i_0 \in I$ arbitrary but fixed throughout the proof, and a streamlined chain (Definition 2.20) $(p_s, q_s)_{s=0}^{n}$ connecting $\iota_{i_0} x$ and $\iota_{i_0} x'$.

The intermediate pair (p_s, q_s), $0 < s < n$ is contained in, say, X_i, with

$$p_s = j_i(y_s)\,, \quad q_s = j_i(y_{s+1})\,, \quad y_\bullet \in Y$$

(that there are such $y_\bullet$ follows from the assumption that the chain is streamlined). We then have

$$\begin{aligned} d(p_s, q_s) &= d_{X_i}(p_s, q_s) \\ &= d_{X_i}(j_i(y_s), j_i(y_{s+1})) \\ &\geq C d_Y(y_s, y_{s+1}) \quad \text{by } C\text{-expansivity} \\ &\geq C d_{X_{i_0}}(j_{i_0}(y_s), j_{i_0}(y_{s+1})) \quad j_\bullet \text{ are contractive.} \end{aligned}$$

The effect of this is that we can always move intermediate pairs from X_i back into X_{i_0}, at the cost of expanding distances but never by a larger factor than $\frac{1}{C}$. In other words, (2-8) dominates

$$C d_{X_{i_0}}(p_0, q_k) = C d_{X_{i_0}}(x, x')\,.$$

This, though, is precisely the C-expansivity claim for the map $\iota_{i_0} : X_{i_0} \to X$.

(b) The second claim, on C-expansivity, follows from (2-9) and the C-expansivity of the individual $j_i \colon Y \to X_i$. We thus focus on the first claim, to which end we write $\delta(y, y')$ for the right-hand side of (2-9):

$$\delta(y, y') := \inf_{\mathbf{i},\mathbf{y}} \sum_{s=0}^{n} d_{X_{i_s}}(j_{i_s} y_s,\ j_{i_s} y_{s+1})\,, \quad y, y' \in Y\,.$$

It satisfies the triangle inequality by construction and is bi-Lipschitz to the original metric d_Y, so is also a complete metric on Y. One inequality between the two distances is obvious:

$$d_X(\iota y, \iota y') \leq \delta(y, y')\,,\ \forall y, y' \in Y \tag{2-11}$$

simply because all maps in sight are contractive.

By Definition 2.20, in a streamlined chain (p_s, q_s) connecting $j_i y$ and $j_{i'} y'$, all points p_s and q_s belong to various images of Y through $j_\bullet \colon Y \to X_\bullet$:

$$p_s = j_{i_s}(y_s)\,, \quad q_s = j_{i_s}(y_{s+1})\,,$$

with $y_0 = y$ and $y_n = y'$. It follows that (2-8) dominates

$$\delta(y, y_1) + \delta(y_1, y_2) + \cdots + \delta(y_{n-1}, y') \geq \delta(y, y')\,,$$

and hence

$$d_X(\iota y, \iota y') \geq \delta(y, y')\,, \quad \forall y, y' \in Y\,.$$

Having noted the opposite inequality in (2-11), we are done.

(c) Consider a streamlined chain $(p_s, q_s)_{s=0}^n$ connecting x and x'. As in the proof of part (b), for intermediate values $0 < s < n$ (if any) we have

$$p_s = j_{i_s}(y_s)\,, \quad q_s = j_{i_s}(y_{s+1})\,, \quad y_\bullet \in Y\,.$$

This means that

$$\sum_{s=1}^{n-1} d(p_s, q_s) \geq \sum_{s=1}^{n-1} d_X(\iota y_s, \iota y_{s+1})\,,$$

so we may as well replace the original sum (2-8) with a (no-larger) three-term sum as in (2-10), with $y = y_1$ and $y' = y_n$.

If there are *no* intermediate values $0 < s < n$, i.e. the sum (2-8) has at most two terms, then at least one of x and x' belongs to the respective image of Y. If that is the case for x', say, then take $y' = j_{i'}^{-1}(x')$ in (2-10).

(d) This is very similar to the preceding argument, the only difference being that we also have to consider single-term sums (2-8).

(e) We take it for granted that Cauchy sequences with convergent subsequences are themselves convergent ([23, Proof of Lemma 43.1]).

For a Cauchy sequence $(x_n)_n$ in X there are two possibilities to consider: either some subsequence is contained in a single $\iota_i(X_i) \subseteq X$, or not. In the former case that subsequence is Cauchy, hence the image of a Cauchy sequence in X_i by the C-expansivity of $\iota_i\colon X \to X$ (part (a)), hence convergent by the completeness of X_i.

In the latter case the Cauchy property implies (via part (c), for instance) that we can find, for arbitrarily small ε, arbitrarily large n with x_n within ε of $\iota(Y) \subseteq X$. Or: a subsequence $(x_{n_k})_k$ with

$$d_X(x_{n_k}, \iota(Y)) \xrightarrow[k]{} 0\,.$$

This in turn gives a sequence $y_k \in Y$ with

$$d_X(x_{n_k}, \iota y_k) \xrightarrow[k]{} 0\,.$$

In particular $(\iota y_k)_k$ is Cauchy (because $(x_{n_k})_k$ is), and hence so is $(y_k)_k$ by the C-expansivity of ι (part (b)). Y being complete, (y_k), (ιy_k) and hence also (x_{n_k}) are all convergent. As observed, so, then, is $(x_n)_n$.

This concludes the proof. $\square$

Theorem 2.21 will be quite useful as-is, but it is perhaps worth noting that the proof provides more:

Theorem 2.22. *Consider a family*

$$j_i\colon Y \to X_i\,,\quad i \in I$$

of C_i-expansive morphisms in CMET *for $1 \ge C_i > 0$ such that*

$$C := \inf_i C_i > 0\,.$$

The canonical maps

$$\iota_i\colon X_i \to X := \coprod_{Y,i} X_i$$

into the pushout in MET *are ${}_iC$-expansive respectively, where*

$${}_iC := \inf_{j \ne i} C_j$$

is the infimum over all of the C_j indexed by indices other than i.

Proof. The claim follows from the proof of Theorem 2.21 (a): in transporting distances over from X_i, $i \ne i_0$ back to X_{i_0}, C-expansivity (in the earlier result's notation) was used only for

$$j_i\colon Y \to X_i$$

(*not* for i_0). For that reason, the argument delivers a lower bound of ${}_{i_0}C$ for the shrinkage of distances in X_{i_0}. $\square$

A sample application of Theorem 2.21:

Proposition 2.23. *Let* (2-7) *be a family of C-expansive morphisms in* CMet *for some* $1 \ge C > 0$ *and assume that*

- *the X_i are convex in the sense of Definition 2.1;*
- *the infima* (2-10) *are all achieved for arbitrary $i \ne i' \in I$ and $x \in X_i$, $x' \in X_{i'}$;*
- *as are the infima* (2-9), *for arbitrary y, $y' \in Y$.*

The pushout $X := \coprod_{Y,i} X_i$ is then convex.

Proof. This is a fairly straightforward consequence of Theorem 2.21. Consider, to fix ideas, $x \in X_i$ and $x' \in X_{i'}$ for $i \ne i'$ with finite distance in X (per Definition 2.1, there is nothing to check for infinite-distance pairs).

Theorem 2.21 (c) gives us the distance $d_X(\iota_i x, \iota_{i'} x')$ via (2-10). That infimum is by assumption achieved for two specific points y, $y' \in Y$, and in turn the middle term of that sum can be obtained (again by assumption) as

$$d_X(\iota y, \iota y') = d_{X_{i_0}}(j_{i_0} y,\ j_{i_0} y')$$

for some $i_0 \in I$. But now we can

- connect x to $j_i y$ in X_i with a minimizing geodesic (because X_i is assumed convex);
- similarly, connect $j_{i'} y'$ to x' in $X_{i'}$ with a minimizing geodesic;
- and also connect, once more minimally, $j_{i_0} y$ and $j_{i_0}(y')$ in X_{i_0}.

Splicing together these three metric segments will give one such segment connecting $\iota_i x$ and $\iota_{i'} x'$ in X, as desired.

The argument proceeds analogously for $i = i'$, etc. $\square$

A variant of Proposition 2.23:

Corollary 2.24. *Let* (2-7) *be a family of C-expansive morphisms in* CMet *for some* $1 \ge C > 0$ *and assume that*

- *the X_i are convex in the sense of Definition 2.1;*
- *Y is compact;*
- *and the infima* (2-9) *are achieved for arbitrary $y, y' \in Y$.*

The pushout $X := \coprod_{Y,i} X_i$ is then convex.

Proof. The compactness of Y automatically implies the second condition in the statement of Proposition 2.23, hence the conclusion by applying that earlier result. $\square$

The discussion thus far will help produce an example that shows CCMet not to be closed, answering a question posed in [10, discussion following Remark 5.9].

Example 2.25. Consider a closed connected *Riemannian manifold* X_0 equipped with its path metric d_0 [24, §7.2, Definition 2.4], together with

- a *closed geodesic* $\gamma_0\colon [0,1] \to X_0$ representing a non-trivial element of the fundamental group. *Any* non-trivial free homotopy loop class contains a closed geodesic, by a classical theorem of Cartan [24, §12.2, Theorem 2.2].

- and a point $p_0 \in X_0 \setminus \gamma_0$ in the same connected component as γ_0.

We can form an abstract object (Y, d_Y) of CMet as follows. First, fix some large L:

$$L > \max\left(1,\ \max\{d_0(p_0, \gamma_0(t)) \mid t \in [0,1]\}\right) . \tag{2-12}$$

Y will then consist of a closed length-1 geodesic and a point having distance $2L$ to all points of the geodesic. Embed Y into (X_0, d_0) in the obvious fashion: by identifying the closed geodesic with γ isometrically and the isolated point with p_0.

Next, we also embed (Y, d_Y) into closed Riemannian manifolds $(X_\varepsilon, d_\varepsilon)$, $\varepsilon \in [0,1]$ so that

- the closed geodesic in Y is again identified isometrically with a closed length-1 geodesic γ_ε in X_ε;
- while the isolated point of Y maps to a point $p_\varepsilon \in X_\varepsilon$, admitting a unique minimal-length geodesic connecting it to every point in γ_ε, with that length equal to $L + \varepsilon$.

This can be arranged by altering the usual Riemannian metric on a sphere, with γ_ε being an equator and p_ε a pole.

We now have the full package for gluing: embeddings $j_\varepsilon \colon (Y, d_Y) \to (X_\varepsilon, d_\varepsilon)$ for $\varepsilon \in \mathbb{R}_{\ge 0}$. That the glued space

$$(X, d_X) := \coprod_Y (X_\varepsilon, d_\varepsilon)$$

is complete convex (i.e. an object of CCMet) now follows from Theorem 2.21 (e) (completeness) and Corollary 2.24 (convexity): closed connected Riemannian manifolds are convex [24, §7.2, Corollary 2.7], Y is compact, and the infimum (2-9) is achieved by the embedding into X_0 by (2-12).

The claim now is that the internal hom $[\mathbb{S}^1, X]_{\text{CCMet}}$ does not exist, where $\mathbb{S}^1$ is the unit-length circle. Indeed, consider the embedding $\iota \colon \mathbb{S}^1 \to X$ identifying the circle with the closed geodesic to which all $\gamma_\varepsilon \subset X_\varepsilon$ collapse. That embedding can be homotoped to the constant map

$$\mathrm{ct} \colon \mathbb{S}^1 \to p_\varepsilon \in X_\varepsilon \to X \quad (\text{any } i \in [0,1])$$

along $X_\varepsilon \subset X$, $\varepsilon > 0$ by paths of respective length $L + \varepsilon$ (but no shorter), whereas in X_0 the closed geodesic is by assumption not homotopic to a constant map. It follows that the distance between

$$\iota \colon \mathbb{S}^1 \to X \quad \text{and} \quad \mathrm{ct} \colon \mathbb{S}^1 \to X$$

in $[\mathbb{S}^1, X]_{\text{CMet}}$ is L, but that infimum is not achievable by a path of length L. By Corollary 2.10, the internal hom in CCMet (rather than CMet) does not exist.

2.2. **Limits along trees and automatic completeness.** One handy procedure for producing metric spaces (employed below, a number of times) is to repeatedly glue metric segments of various lengths to the same initial space. This is a (perhaps infinite) iteration of the pushout construction discussed in §2.1, so results that ensure the automatic completeness of such glued spaces will be useful. Such results are the focus of the present subsection.

It will be convenient to consider colimits along *graphs* rather than categories; the former can be turned into the latter via the *free category* construction of [4, §1.7] or [21, §II.7]. Per those sources:

Definition 2.26. An *oriented graph* (or just plain 'graph', unless specified otherwise) is a quadruple

$$(E, V, \partial_i) = (E, V, \partial_0, \partial_1)$$

consisting of

- a set E of *edges*;
- a set V of *vertices*;
- and *source* and *target* maps $\partial_0 \colon E \to V$ and $\partial_1 \colon E \to V$ respectively.

We also refer to the source and target of $e \in E$ as the *extremities* or *vertices of* e.

The reader should picture edges as

$$\partial_1(e) \xleftarrow{\quad e \quad} \partial_0(e).$$

The usual graph-theoretic language (paths, cycles, etc.) applies, with the caveat that much of the combinatorial literature on graphs tends to assume an edge is uniquely determined by its source and target: see e.g. [7, §§I.1 and I.2] or [11, §§1.1-1.5].

Definition 2.27. Let $\Gamma := (E, V, \partial_i)$ be a graph.

- An *oriented path* (or just 'path') *of length* n is a sequence $(e_i)_{i=1}^n$ of edges with $\partial_1(e_i) = \partial_0(e_{i+1})$ for all $1 \le i \le n-1$. Pictorially:

  $$\bullet \xleftarrow{\quad e_n \quad} \bullet \quad \cdots \quad \bullet \xleftarrow{\quad e_2 \quad} \bullet \xleftarrow{\quad e_1 \quad} \bullet$$

  We will also occasionally refer to the *empty path* based at a vertex $v \in V$, consisting of no edges at all (and hence of length 0).
- An *oriented cycle* (or just 'cycle') is a path $(e_i)_{i=1}^n$ such that $\partial_1(e_n) = \partial_0(e_1)$ (i.e. the target of the last arrow is the source of the first; the path returns to its origin).
- *Unoriented* paths and cycles are defined as their oriented cousins, except that for each i only the weaker requirement

  $$\partial_\ell(e_i) = \partial_\kappa(e_{i+1}), \quad \kappa, \ell \in \{0, 1\}$$

  is made:

  $$\cdots \quad \bullet \xleftarrow{\quad e_3 \quad} \bullet \xrightarrow{\quad e_2 \quad} \bullet \xleftarrow{\quad e_1 \quad} \bullet$$

  is an unoriented path, for instance.
- Paths and cycles (oriented or not) are *simple* if there are no coincidences among their constituent edges.
- For a simple unoriented path $(e_i)_{i=1}^n$ of length $n \ge 2$ its *starting point* is the vertex v_0 of e_1 *not* shared by e_2 is there is such a vertex, or the single vertex of e_1 is the latter is a loop. Similarly, a simple unoriented path's *destination*

v_n is the vertex of e_n *not* shared by e_{n-1} if one exists, or the single vertex of the loop e_n otherwise.

We will then also refer to it as a path *from* v_0 *to* v_n, or say that it *starts at* v_0 and *ends at* v_n.

The terminology applies to length-1 paths too, in which case there is an ambiguity: a single edge might constitute an unoriented path from its source to its target, or vice versa.

- An edge e in a simple unoriented path $(e_i)_{i=1}^n$ from v_0 to v_n is *coherent (with the path)* if in listing the vertices
$$v_0,\ v_1,\ \ldots,\ v_{n-1},\ v_n$$
in order, so that
$$\{v_{i-1},\ v_i\} = \{\partial_0 e_i,\ \partial_1 e_i\}\,,\ \forall 1 \le i \le n\,,$$
the source $\partial_0 e$ is listed immediately *before* the target $\partial_1 e$ (rather than after).
- Γ is *connected* if every two vertices are extremities of edges belonging to a common unoriented path.
- Γ is a *forest* if it has no simple unoriented cycles;
- and a *tree* if it is in addition connected.

Recall from [4, §1.7] (or [21, §II.7, Theorem 1]):

Definition 2.28. The *free category* $\mathcal{C}(\Gamma)$ on a graph $\Gamma = (E, V, \partial_i)$ has

- V as its set of vertices;
- the (oriented) paths of Γ as morphisms;
- the empty paths as identity morphisms;
- and concatenation of paths as composition.

Via this construction, we transport terminology involving categories to graphs. Thus, Γ-*functors* are functors defined on $\mathcal{C}(\Gamma)$, Γ-*colimits* are colimits of such functors, etc.

In order to state Theorem 2.31 recall also ([11, discussion preceding Proposition 1.3.2])

Definition 2.29. The *distance* between two vertices x, $y \in V$ of a graph (E, V, ∂_i) is the length of a shortest unoriented path containing edges having x and y as endpoints; it is zero if $x = y$ and infinite if no such paths exist.

The *diameter* of a graph is the supremum of the distances between pairs of vertices.

Remark 2.30. It is easy to see that for a tree the diameter is also the supremum of the lengths of simple unoriented paths; this is implicit in the proof of Theorem 2.31.

Theorem 2.31. *Let* $\Gamma = (E, V, \partial_i)$ *be a forest consisting of finite-diameter trees,* $1 \ge C > 0$, *and*
$$V \ni v \xmapsto{\ F\ } (X_v, d_v) \in \mathrm{CM{\scriptsize ET}}$$
a Γ-*functor sending edges to* C-*expansive maps.*

(a) *For every vertex v of Γ, the canonical morphism from X_v to the colimit*

$$(X, d) := \varinjlim F \in \mathrm{M{\scriptsize ET}}$$

in Met *is $C^{N_{\Gamma,v}}$-expansive where*

$$N_{\Gamma,v} := \max_{\textit{simple paths } (e_i) \textit{ starting at } v} \sharp\{\textit{coherent edges in } (e_i)\}.$$

(b) *That colimit is automatically complete, and hence also a colimit in* CMet.

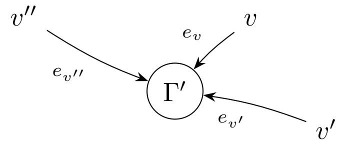
Proof. It is harmless to work with a *tree* Γ, since a colimit over a forest will be a coproduct of colimits over its connected components (i.e. constituent trees).

We proceed by induction on $\operatorname{diam}(\Gamma)$. When it is zero the graph is a vertex, and there is nothing to prove ($N = 0$ will do). A tree with diameter 1 is an edge e, so a functor thereon is a C-expansive contraction

$$(X_e, d_{X,e}) \longleftarrow (Y_e, d_{Y,e}) .$$

The colimit is X_e and we are once more done, taking $N = 1$. We now take for granted the claim for smaller diameters, assuming $\operatorname{diam}(\Gamma) \geq 2$.

(1) **Pruning leaf sources.** A *leaf* [11, §1.5] is a vertex attached to a single edge, and a *source* is a vertex that is not the target of any edges. Consider the tree $\Gamma' \subseteq \Gamma$ obtained by deleting the respective edges e_v adjacent to leaf sources v:



Assume for the moment that we have proven the claims for Γ' (on which more below). We have an isomorphism

$$\varinjlim (F|_{\Gamma'}) \cong \varinjlim F ,$$

so completeness follows (i.e. part (b)). Furthermore, for vertices v of Γ' we have

$$N_{\Gamma,v} = N_{\Gamma',v},$$

so for *them* claim (a) for Γ' entails it for Γ as well. On the other hand, for source leaves v_0 in the original Γ we have

$$N_{\Gamma,v_0} = N_{\Gamma,v_1} + 1 ,$$

where v_1 is the target of the unique Γ-edge incident to v_0. For that reason, claim (a) for $v_1 \in V(\Gamma')$ entails the analogous claim for $v \in V(\Gamma)$.

Now, consider the effect of pruning edges e_v whose sources are leaves. Should the deletion of e_v expose a new leaf source

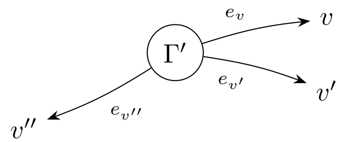
$$\bullet \xleftarrow{\ e_w\ } w$$

(that used to be the target of the now-absent e_v), e_v and e_w are composable to a path. This observation replicates, and since there is a bound of $\operatorname{diam}(\Gamma)$

on the length of an unoriented path the diameter also caps the number of times this pruning procedure can be repeated.

In conclusion, we can henceforth assume that all leaves are *sinks* (i.e. targets only) without affecting the desired conclusions.

(2) **All leaves are sinks.** Because every path of maximal length must contain (an edge adjacent to) a leaf, Γ consists of a tree Γ' of strictly smaller diameter connected to the leaves v via their unique respective incident edges e_v:



We have the induction hypothesis for Γ', and (X, d) can be obtained as the pushout of the morphisms

$$X' := \varinjlim (F|_{\Gamma'}) \xrightarrow{\iota_v} Y_{v_0}, \quad \text{leaves } v_0 ,$$

where

- we are suppressing metrics to keep the notation simple;
- Y_{v_0} is the two-arrow pushout of

$$X_{\partial_0 e_v} \xrightarrow{F(e_v)} X_v$$

and

$$X_{\partial_0 e_v} \xrightarrow{\text{canonical map}} X'; \tag{2-13}$$

- and ι_v is the resulting canonical map from X' into that pushout.

Now, because $F(e_v)$ is assumed C-expansive, Theorem 2.22 shows that all ι_v are also C-expansive (even though (2-13) will typically be only C^N-expansive for some possibly large N, by induction).

Part (b) (completeness) already follows from this and Theorem 2.21 (e) by induction.

As for part (a), we have two types of vertices to check it against:

- those belonging to the smaller tree: $w \in V(\Gamma')$. The conclusion holds for them, in Γ', with respective exponents $N_{\Gamma',w}$. But every unoriented path in Γ starting at w can always be prolonged until it eventually contains a leaf-incident edge e_v terminating at v, whence

$$N_{\Gamma,w} = N_{\Gamma',w} + 1 . \tag{2-14}$$

The conclusion follows from the induction hypothesis, together with the fact that since ι_v is C-expansive (as noted), so is

$$X' \to X := \varinjlim F ,$$

by Theorem 2.21 (a).

- the leaves v themselves. Unoriented paths in Γ starting at v must traverse the *in*coherent edge e_v terminating at v, so must pass through the vertex $w := \partial_0 e_v \in V(\Gamma')$.
  On the other hand, an unoriented Γ'-path starting at w (in particular, one that contains the maximal number $N_{\Gamma',w}$ of coherent Γ'-edges) can always be prolonged in Γ until it contains $e_{v'}$ for some *other* leaf v' (not being reduced to a single vertex, the tree contains at least two leaves).
  Because the new edge $e_{v'}$ is also coherent for the path, this argument shows that

$$N_{\Gamma,v} = N_{\Gamma',w} + 1\,. \tag{2-15}$$

  To finish, observe that
  – the canonical map (2-13) is $C^{N_{\Gamma',w}}$-expansive by induction;
  – hence so is the canonical map $X_v \to Y_v$ into the pushout, by Theorem 2.22;
  – while $Y_v \to X$ is C-expansive by yet another application of Theorem 2.22 and the noted C-expansivity of ι_v.
  Composing

$$X_v \xrightarrow{C^{N_{\Gamma',w}}\text{-expansive}} Y_v \xrightarrow{C\text{-expansive}} X$$

  produces an expansive map with constant

$$C^{N_{\Gamma',w}} \cdot C = C^{N_{\Gamma',w}+1} = C^{N_{\Gamma,v}}$$

  by (2-15), and we are done.

This concludes the argument. □

Theorem 2.31 can presumably be generalized in a number of ways, but not too cavalierly.

Examples 2.32. (1) Dropping the finite-diameter condition, even for tree colimits of isometries (i.e. $C = 1$ in Theorem 2.31), can invalidate completeness. Consider the graph

$$\begin{tikzcd}
\bullet & & \bullet & & \bullet & \cdots & \bullet & & \bullet & \cdots \\
& \bullet \arrow[ul] \arrow[ur] & & \bullet \arrow[ul] \arrow[ur] & & & & \bullet \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

The top row is assigned closed segments of respective lengths $\frac{1}{2^n}$, $n \geq 1$ decreasing rightward, the bottom row is assigned points (i.e. singletons, regarded as objects of CMET), and the arrows are identifications with endpoints, splicing together the segments.

The colimit in MET is then a length-1 half-open segment, hence not complete.

(2) Even finite graphs will not do, for either of the claims in Theorem 2.31, if loops are allowed (even if $C = 1$, i.e. the connecting morphisms are

isometries). Consider for instance a diagram in CMet of the form

$$\begin{tikzcd} (X,d) \arrow[loop right, "\varphi"] \end{tikzcd} \tag{2-16}$$

as we now describe.

- (X,d) is the portion of the first quadrant above a hyperbola:
$$(X,d) := \left\{(x,y)\in\mathbb{R}^2_{\ge 0}\ \mid\ xy\ge 1\right\},$$
with the usual Euclidean distance.
- the isometry φ shifts everything up by 1:
$$\varphi(x,y) := (x,y+1)\,.$$

The colimit in Met consists of the equivalence classes of *forward orbits*

$$O_x := \left\{\varphi^n x\ \mid\ n\in\mathbb{Z}_{\ge 0}\right\},\quad x\in X \tag{2-17}$$

under the action of the monoid generated by φ with the usual set-distance induced by d, i.e.

$$d(O_x,O_y) = \inf\{d(p,q)\mid p\in O_x,\ q\in O_y\}\,,$$

and two such orbits declared equivalent when the distance between them vanishes.

The (images of the) points $\left(\frac{1}{2^n},\ 2^n\right)$, $n\in\mathbb{Z}_{\ge 0}$ form a Cauchy sequence: the successive distances between their φ-orbits are, respectively, $\frac{1}{2^n}$.

Clearly though, that sequence has no limit in the quotient.

(3) The same effect can easily be replicated with finite graphs without single-edge loops:

$$\begin{tikzcd} X \arrow[r, bend left, "\varphi"] & X \arrow[l, bend left, "\mathrm{id}"] \end{tikzcd}$$

or

$$\begin{tikzcd} & X \arrow[dr, "\varphi"] & \\ X \arrow[ur, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & & X \\ & X \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

say.

(4) On the other hand, it was crucial that the isometry φ of (2-16) not be bijective. In other words, the colimit in Met of a diagram (2-16) with a metric *isomorphism* φ will automatically be complete if the original space (X,d) is.

This follows from the simple remark recorded as Lemma 2.33 below, given that for an isomorphism φ the diagram (2-16) can be regarded as a functor defined on a groupoid.

Lemma 2.33. *A colimit in* MET *of a functor*

$$F\colon \mathcal{G} \to \mathrm{CMET}$$

defined on a groupoid is automatically complete, and hence also a colimit in CMET.

Proof. Since the colimit in question is the coproduct of the restrictions $F|_{\mathcal{G}_i}$ to the connected components of $\mathcal{G}$, it is enough to assume $\mathcal{G}$ is connected to begin with. But then it will be equivalent (as a category) to a group Γ, so the colimit in MET is the coequalizer of the isometries constituting a group action

$$\Gamma \times (X, d) \to (X, d)\,.$$

Such a coequalizer is obtained by identifying the orbits (2-17) to single points and further identifying those that are distance zero apart. If $(O_{x_n})_n$ is a Cauchy sequence in that quotient then we can assume, perhaps after passing to a subsequence, that

$$d(O_{x_n},\ O_{x_{n+1}}) < \frac{1}{2^n}\,,\quad \forall n \geq 1\,.$$

Having fixed x_1, *some* $x \in O_{x_2}$ is less than $\frac{1}{2}$ from it by assumption, and upon translating x by some $g \in \Gamma$ we may as well assume that $x = x_2$. Similarly, we can assume $d(x_2, x_3) < \frac{1}{2^2}$, etc.

The limit of $(x_n)_n$ in the complete space (X, d) will map to a limit of $(O_{x_n})_n$. □

Remark 2.34. Lemma 2.33 only gives an analogue of part (b) of Theorem 2.31; the corresponding version of (a) does *not* hold in general, of course (since the identification of a Γ-orbit to a single point decreases positive distances to zero).

2.3. **The convex pseudo-reflection.** The gluing results above make it very easy to produce convex metric spaces by simply attaching the possibly-missing metric segments.

Definition 2.35. Let (X, d) be a metric space.

- For points $x, x' \in X$ with $d(x, x') < \infty$ the (x, x')-*convex completion* $\overline{X}^{x,x'}$ is the space obtained by gluing a length-$d(x, x')$ metric segment with endpoints x and x' to X.
- More generally, for a set $\mathcal{S}$ of finite-distance point-pairs in X the $\mathcal{S}$-*convex completion* $\overline{X}^{\mathcal{S}}$ is the space obtained by gluing a metric segment as above, for each pair in $\mathcal{S}$.
- Finally, the *convex completion* $\overline{X}^{\mathrm{CVX}}$ is obtained by gluing one metric segment connecting *every* finite-distance pair of distinct points.

The construction $X \mapsto \overline{X}^{\mathrm{CVX}}$ is a version of the *weak reflection* described in [10, Remark 6.9] (hence the title of the present subsection), with some differences. The following result, for instance, makes it clear that in order to produce a convex metric space one need not iterate the construction recursively.

Not only is $\overline{X}^{\mathrm{CVX}}$, as the name suggests, convex, but we can forego the redundant gluing.

Proposition 2.36. *Let (X,d) be a complete metric space and $\mathcal{S} \subset X^2$ a set of finite-distance pairs of points containing all pairs which are not connected in X by a metric segment.*

The $\mathcal{S}$-convex completion $\overline{X}^{\mathcal{S}}$ of Definition 2.35 is complete and convex.

Proof. Note first that every metric-segment gluing is a pushout (ordinary, not multiple) for a pair of arrows

$$\mathbf{2}_\delta \to \gamma \quad \text{and} \quad \mathbf{2}_\delta \to X :$$

the identification with the endpoints of a segment γ of length $\delta := d(x,x')$ and with the two points x, $x' \in X$. These embeddings are both isometries, so we can apply the above discussion on C-expansive maps with $C = 1$. This implies in particular, by Theorem 2.21 (a), that in every such gluing the original space (to which the segment is being glued) embeds isometrically into the glued space; we use this implicitly and repeatedly.

By construction, $\overline{X}^{\mathcal{S}}$ is the directed union of the partially-"convexified" $\overline{X}^{\mathcal{F}}$ for finite sets $\mathcal{F} \subseteq \mathcal{S}$ of point-pairs in X. We now turn to the claims.

(1): Completeness. This is a consequence of Theorem 2.31 (b): $\overline{X}^{\mathcal{S}}$ is the colimit in Met along the tree obtained by gluing the various sub-trees

$$\begin{tikzcd} X & & \gamma \\ & \mathbf{2}_\delta \arrow[ul, bend left] \arrow[ur, bend right] & \end{tikzcd}$$

along the common vertex X. This produces a tree of diameter ≤ 4, so Theorem 2.31 applies with $C = 1$.

(2): Convexity. As noted, X itself embeds into $\overline{X}^{\mathcal{S}}$ isometrically. It follows that, by construction, any two points therein are (if a finite distance apart) the endpoints of a metric segment.

Points on the same glued metric segment γ are connected by a portion of γ itself (which embeds isometrically into $\overline{X}^{\mathcal{S}}$ by Theorem 2.21 (a) with $C = 1$).

On the other hand, for a point $x \in X$ and one $y \in \gamma$ on one of the glued segments, $d(x,y) < \infty$ implies that the distances from x to the endpoints p and q of γ are also finite. Furthermore, Theorem 2.21 (c) shows that

$$d(x,y) = \min\left(d(x,p) + d(p,y),\ d(x,q) + d(q,y)\right) .$$

In either case we have the desired metric segments in $\overline{X}^{\mathcal{S}}$: for p, say, there is one connecting x to p (either originally in X or attached upon constructing $\overline{X}^{\mathcal{S}}$) and one connecting p to $y \in \gamma$ (a portion of γ itself).

Finally, for points on distinct glued segments $\gamma \neq \gamma'$ we can fall back on the preceding argument by first enlarging X with the addition of one segment, and then further gluing the other. ☐

In particular, taking for $\mathcal{S}$ the set of all finite-distance point-pairs, we have

Corollary 2.37. *For any complete metric space (X,d) the convex completion $\overline{X}^{\mathrm{cvx}}$ of Definition 2.35 is convex and complete.*

3. Path spaces and the path coreflection

Example 2.25 highlights one reason why CCMet is inadequate as an enriching category: lack of closure. As seen in Remark 2.8, convexity essentially means being strictly path; if we compromise on the stricture, coreflections are much easier to come by. Denoting by CPMet the category of complete path metric spaces (in the sense of Definition 2.3), we have the following version of Proposition 2.9;

Proposition 3.1. *For a complete metric space $(X, d_X) \in$ CMet the associated path metric space $(X, d_{X,\ell})$, equipped with the identity contraction*

$$\mathrm{id} : (X, d_{X,\ell}) \to (X, d_X)\,, \tag{3-1}$$

is a coreflection of (X, d_X) in CPMet.

Proof. The completeness of the metric $d_{X,\ell}$ is part of Proposition 2.9, the fact that it is indeed a path metric was observed in Remark 2.4 (2), and the universality property of (3-1) is not harder to prove than the analogous claim in Proposition 2.9. □

This affords an analogue of Corollary 2.10. Before stating it, we observe that the setup of Lemma 2.2 obtains for CPMet, just as it did for CCMet.

Lemma 3.2. *The full subcategory*

$$\mathrm{CPM{\scriptstyle ET}} \subset \mathrm{CM{\scriptstyle ET}}$$

of complete path metric spaces contains the monoidal unit and is closed under tensor products, so is a full monoidal subcategory.

Proof. The monoidal unit is the one-point space, whose metric is obviously path. As for closure under tensor products, consider complete path metric spaces (X, d_X) and (Y, d_Y) and points

$$(x, y),\ (x', y') \in X \times Y = X \otimes Y$$

with

$$\ell := d_{X\otimes Y}\big((x, y), (x', y')\big) = d_X(x, x') + d_Y(y, y')$$

finite. We can concatenate

- a path

$$[a, b] \to X \cong X \times \{y\}\,,\quad a \mapsto (x, y)\,,\quad b \mapsto (x', y)$$

  of approximate length $d_X(x, x')$;
- and a path

$$[b, c] \to Y \cong \{x'\} \times Y\,,\quad b \mapsto (x', y)\,,\quad b \mapsto (x', y')$$

  of approximate length $d_Y(y, y')$

to obtain a path in $X \otimes Y$ of length close to ℓ. In short: $(X \otimes Y,\ d_{X\otimes Y})$ is a path metric space. □

Finally, the CPMet-specific version of Corollary 2.10:

Corollary 3.3. *The monoidal category* $\mathrm{CPM{\small ET}}$ *of complete path metric spaces is closed: for objects* (X, d_X) *and* (Y, d_Y) *in* $\mathrm{CPM{\small ET}}$ *we have*

$$[X, Y]_{\mathrm{CPM{\small ET}}} = (\mathrm{CM{\small ET}}(X, Y), d_{\sup,\ell}) \, ,$$

the path space attached to the internal hom (2-1)*.*

Proof. A consequence of Lemma 2.2 (which applies by Lemma 3.2) and Proposition 3.1. ☐

The coreflection

$$\mathrm{CM{\small ET}} \to \mathrm{CPM{\small ET}}$$

of proposition 3.1 allows us to make sense of the $\mathrm{CPM{\small ET}}$-valued "internal hom" $[Y, X]_{\mathrm{CPM{\small ET}}}$ even for arbitrary X and Y in $\mathrm{CM{\small ET}}$ (not $\mathrm{CPM{\small ET}}$!): simply coreflect the original internal hom $[Y, X]_{\mathrm{CM{\small ET}}}$. We will use this notation without further comment in the sequel.

Theorem 3.4 summarizes the ways in which $\mathrm{CPM{\small ET}}$ is better behaved than $\mathrm{CCM{\small ET}}$ as a subcategory of $\mathrm{CM{\small ET}}$.

Theorem 3.4. *The full subcategory* $\iota : \mathrm{CPM{\small ET}} \subset \mathrm{CM{\small ET}}$ *is closed under colimits, and in particular cocomplete.*

Proof. The statement holds for arbitrary coreflective subcategories of cocomplete categories; since our inclusion is indeed coreflective by Proposition 3.1, we are done. ☐

Furthermore, we know from [3, Example 2.3 (2)] that $\mathrm{CM{\small ET}}$ is locally presentable; it turns out that so is $\mathrm{CPM{\small ET}}$.

Theorem 3.5. *The category* $\mathrm{CPM{\small ET}}$ *of complete path metric spaces is locally* $\aleph_1$*-presentable.*

Proof. We already know from Theorem 3.4 that $\mathcal{C} := \mathrm{CPM{\small ET}}$ is cocomplete; by [2, Theorem 1.20], it remains to show that it has a *strong generator* consisting of $\aleph_1$-presentable objects. This means [2, §§0.5 and 0.6]:

- a set $\mathcal{S}$ of $\aleph_1$-presentable objects;
- so that every object $X \in \mathcal{C}$ admits an *extremal epimorphism*

$$e\colon \coprod S \to X \tag{3-2}$$

  from a coproduct of objects $S \in \mathcal{S}$;
- in the sense that e is epic and for any factorization

$$e = m \circ - \tag{3-3}$$

  with monic m, the latter is an isomorphism.

The generator $\mathcal{S}$ consists of the finite segments $[0, \ell]$ for $\ell \in \mathbb{R}_{\geq 0}$. Clearly, these spaces are $\aleph_1$-presentable (as is every complete *separable* metric space). Moreover, every path metric space is a quotient of the disjoint union (i.e. coproduct) of its rectifiable curves, so we indeed have an epimorphism (3-2) (canonical, since we are surjecting from the disjoint union of *all* finite-length paths).

It remains to argue that (3-2) is extremal, for which purpose we fix a factorization (3-3) through a monomorphism m. CPMet is a *concrete* category (over Set, or a *construct* in the language of [1, Definition 5.1]) whose forgetful functor to Set is representable by the one-point space. This implies [1, Corollary 7.38] that its monomorphisms are precisely the injections.

It then follows that m is *bi*jective: surjectivity is immediate from (3-3) and the fact that e is itself onto. Naturally, m is also contractive, as are all maps in sight. On the other hand though, m cannot *strictly* decrease any distances: points $x, x' \in X$ a finite distance ℓ apart are connected by paths of lengths

$$\ell + \varepsilon\,,\ \text{arbitrarily small } \varepsilon > 0\,,$$

so their preimages through m are at most $\ell + \varepsilon$ apart no matter how small $\varepsilon > 0$ is. It follows that m is an isometry onto X, i.e. an isomorphism. ☐

Remark 3.6. It mattered, in the proof of Theorem 3.5, that the set $\mathcal{S}$ consisted of all segments of arbitrary finite lengths. Had we chosen a smaller $\mathcal{S}$, consisting, say, of only the singleton, the argument would have fallen through: every metric space (X, d) (path or not) admits a contractive bijection

$$(\text{discrete } X) \to (X, d)$$

from its own discrete version, with infinite distances between distinct points. Plainly, that morphism is epic but not extremally so. Having a rich supply of segments in $\mathcal{S}$ allowed the last part of the argument (wherein we connected points with "almost-metric-segments") to go through.

4. Complements and asides

4.1. **Tensors over CMet.** One natural question, given their CMet-enrichment, is whether the various categories mentioned above are *tensored* over CMet in the sense of [17, §3.7]: whether, in other words, for each $X \in$ CMet and $B \in \mathcal{C}$ (the category of interest) the functor

$$\hom_{\mathrm{CMet}}\big(X, \hom_{\mathcal{C}}(B, -)\big) : \mathcal{C} \to \mathrm{CMet} \tag{4-1}$$

is representable (note that we are regarding hom spaces in both CMet and $\mathcal{C}$ as objects in CMet; so enrichment is used repeatedly to make sense of the concept). If that is the case, we denote the representing object by $X \otimes B$. The aim here is to note that this cannot be the case for $\mathcal{C}^*_1$; in fact, a fairly strong negation of tensor-existence holds:

Theorem 4.1. *For a unital C^*-algebra $B \in \mathcal{C}^*_1$ and a complete metric space $(X, d) \in$ CMet the tensor product $X \otimes B$ over CMet exists if and only if one of the following conditions holds:*

- *X is empty, in which case $X \otimes B = \mathbb{C}$ (initial object in $\mathcal{C}^*_1$);*
- *X is a singleton, whence $X \otimes B \cong B$;*
- *or X is non-empty and B is at most 1-dimensional (i.e. $\{0\}$ or $\mathbb{C}$), so that $X \otimes B \cong B$.*

Naturally, this implies

Corollary 4.2. *The category $\mathcal{C}_1^*$ of unital C^*-algebras is not tensored over* CMET.

Some auxiliary notation will be useful in handling (possible) tensors over CMET, both in $\mathcal{C}_1^*$ and, later, in $\mathcal{C}_{c,1}^*$. Note that the SET-valued version of (4-1) is always representable, with $\mathcal{C}$ either $\mathcal{C}_1^*$ or $\mathcal{C}_{c,1}^*$. In both cases the representing object, which we denote by

$$X \otimes_{\mathrm{SET}} B \in \mathcal{C} = \mathcal{C}_1^* \text{ or } \mathcal{C}_{c,1}^* ,$$

can be constructed as follows:

- first form the coproduct of copies B_x of B indexed by $x \in X$;
- and then impose the additional constraints

  $$\|a_x - a_{x'}\| \le d(x,x') , \ \ \forall a \in B_1 := \text{unit ball of } B$$

  where $a_x \in B_x$ denotes the copy of $a \in B$.

We thus always have a canonical identification

$$\hom_{\mathcal{C}}(X \otimes_{\mathrm{SET}} B, -) \cong \hom_{\mathrm{CMET}}\big(X, \hom_{\mathcal{C}}(B,-)\big) \tag{4-2}$$

of SET-valued functors. Both sides are metric spaces, and the metric tensor product $X \otimes B \in$ CMET will exist precisely when that canonical morphism is isometric. In that case, of course, we will have

$$X \otimes B \cong X \otimes_{\mathrm{SET}} B . \tag{4-3}$$

We take this discussion for granted in the sequel.

Proof of Theorem 4.1. That the tensor products are as described in the three listed cases is an easy check, so we focus on proving the converse: that as soon as

$$\dim B \ge 2 \quad \text{and} \quad |X| \ge 2$$

the metric tensor product (4-3) (which would otherwise have to coincide with the corresponding SET-tensor product) does not exist. To that end we will construct

- a net

  $$(f_\alpha)_\alpha = (f_{x,\alpha},\ x \in X)_\alpha$$

  of X-tuples of morphisms

  $$f_{x,\alpha} \colon B \to A$$

  such that

  $$\|f_{x,\alpha} - f_{x',\alpha}\|_\infty \le d(x,x') \quad \text{on the unit ball } B_1 \subset B ;$$

- with each $(f_{x,\alpha})_\alpha$ converging, respectively, to some $f_x \colon B \to A$ uniformly in x and on the unit ball B_1;
- but such that the morphisms

  $$f_\alpha \colon X \otimes_{\mathrm{SET}} B \to A$$

  respectively induced by $f_{x,\alpha}$ do *not* converge to the corresponding morphism

  $$f \colon X \otimes_{\mathrm{SET}} B \to A$$

  induced by f_x uniformly on the unit ball of $X \otimes_{\mathrm{SET}} B$.

This will then show that the identification (4-2) (evaluated at A) is only one of sets, and not a homeomorphism.

We start with the $f_x\colon B \to A$ (which will later be the limits, respectively, of $(f_{x,\alpha})_\alpha$); they will all be equal to a fixed (unital) embedding

$$\iota\colon B \to A := B(\mathcal{H}) = \text{bounded operators on a Hilbert space } \mathcal{H};$$

one always exists, by the *Gelfand-Naimark theorem* ([5, Corollary II.6.4.10]). We will typically suppress ι and identify B with its realization inside $A = B(\mathcal{H})$.

Because B is assumed at least 2-dimensional, there must be a unitary $u \in B$ that fails to commute with some unitary $v_\alpha \in B(\mathcal{H})$. Furthermore, we can choose v_α to be arbitrarily close to 1, so that

$$\|v_\alpha - 1\| \xrightarrow[\alpha]{} 0$$

(the αs are the elements of an otherwise unspecified directed poset).

Fix distinct elements $x_{0,1} \in X$ (assumed to exist: $|X| \geq 2$). If v_α is sufficiently close to 1 then it is connectable to 1 by a short path of unitaries ([26, Proposition 4.2.4 and its proof]), so in particular we can assume those paths are shorter than $\ell := d(x_0, x_1)$. We can now

- map $\{x_0, x_1\}$ contractively to the endpoints of a segment I_α of length $\|v_\alpha - 1\|$;
- extend that map to a contraction from X to I_α, given that the latter is an *injective* object in the category MET [18, Theorem 4.7];
- and further map I_α contractively onto a path γ_α of unitaries connecting 1 and v_α, so as to obtain a contraction

$$\varphi_\alpha\colon X \to \gamma_\alpha \subset B(\mathcal{H}), \quad \varphi_\alpha(x_0) = 1,\ \varphi_\alpha(x_1) = v_\alpha\,. \tag{4-4}$$

Finally, define

$$f_{x,\alpha}\colon B \to B(\mathcal{H}),\ f_{x,\alpha} := \varphi_\alpha(x)\ \cdot\ \varphi_\alpha(x)^*.$$

In words, this is conjugation by the unitary $\varphi_\alpha(x)$; rescaling the metrics involved slightly if necessary we can assume that (4-4) were in fact C-contractive for small $C > 0$, so that

$$X \ni x \mapsto f_{x,\alpha} \in \hom_{\mathcal{C}_1^*}\big(B, B(\mathcal{H})\big)$$

is contractive for each α (the right-hand side being metrized as usual, uniformly on the unit ball of C).

The φ_α take values close to 1 for large α by construction, so the X-uniform convergence

$$(f_{x,\alpha})_\alpha \to f_x = \iota\colon B \to B(\mathcal{H})$$

follows. On the other hand though, consider the unitaries

$$1 \neq w_\alpha := v_\alpha u v_\alpha^* u^* = f_\alpha(u_{x_1} u_{x_0}^*) \in f_\alpha(X \otimes_{\mathrm{SET}} B).$$

They converge to 1 in norm, so by the norm-continuity of the spectrum for normal operators (e.g. [15, Problem 105]) their spectra will be contained in a small but non-trivial arc around $1 \in \mathbb{S}^1 \subset \mathbb{C}$. It follows that no matter how large α_0 and n_0 are, we can find

$$\alpha \geq \alpha_0\,,\quad n \geq n_0\,,\quad w_\alpha^n \text{ uniformly far from 1.}$$

In other words, we cannot have convergence

$$w_\alpha^n = f_\alpha\left((u_{x_1}u_{x_0}^*)^n\right) \longrightarrow f\left((u_{x_1}u_{x_0}^*)^n\right) = 1$$

uniformly in n; or again:

$$f_\alpha \not\rightarrow f$$

uniformly on the unit ball of $X \otimes_{\mathrm{SET}} B$. □

Contrast Theorem 4.1 with its commutative version: [9, Proposition 3.11] states that unlike $\mathcal{C}_1^*$, the category of commutative (unital) C^*-algebras *is* tensored over CMET.

4.2. **Finite presentability.** [3, Proposition 5.19] classifies those finite metric spaces that are (enriched) $\aleph_0$-generated in CMET with respect to the class of isometries: they are precisely the discrete ones (i.e. those with infinite pairwise distances; see Section 1). As we will soon see, *assuming* finiteness is not necessary:

Theorem 4.3. *The objects in* CMET *isometry-$\aleph_0$-generated in the enriched sense are precisely the finite discrete metric spaces.*

A number of preliminary observations will simplify the main line of attack. The following terminology will be useful.

Definition 4.4. The *finite-metric components* (or just plain 'components', when context permits it) of a metric space are the maximal subspaces on which the metric takes finite value.

Clearly, an object of CMET is the coproduct of its finite-distance components ([8, Exercise 1.1.2 and discussion preceding it]); the same goes for MET.

Lemma 4.5. *An isometry-$\aleph_0$-generated object in* CMET *has finitely many components.*

Proof. Every space (X, d) is the directed union of its subspaces $X_{\mathcal{F}}$, unions of finite families $\mathcal{F}$ of X-components. $\aleph_0$-generation then requires that X be equal to one of the $X_{\mathcal{F}}$, for otherwise the identity $X \to X$ would not be approximable by a morphism $X \to X_{\mathcal{F}}$. □

Lemma 4.5 reduces the problem to finite coproducts. Note next that it is enough to consider the individual components themselves.

Lemma 4.6. *Let $X_i \in$ CMET, $1 \le i \le n$ be a finite family of objects. The coproduct*

$$X := \coprod_{i=1}^{n} X_i$$

is isometry-$\aleph_0$-generated in the enriched sense if and only if the X_i are.

Proof. This is almost entirely formal. If

$$Y = \varinjlim_{\alpha} Y_\alpha$$

is a directed colimit of isometries, then the canonical morphism

$$\varinjlim_{\alpha} \hom(X, Y_\alpha) \to \hom(X, Y) \quad \text{in} \quad \mathrm{CM{\small ET}}$$

is

$$\begin{aligned}
\varinjlim_{\alpha} \hom(X, Y_\alpha) &\cong \varinjlim_{\alpha} \prod_{i=1}^{n} \hom(X_i, Y_\alpha) \\
&\cong \prod_{i=1}^{n} \varinjlim_{\alpha} \hom(X_i, Y_\alpha) \quad \text{(see below)} \\
&\to \prod_{i=1}^{n} \hom(X_i, Y) \\
&\cong \hom(X, Y)\,.
\end{aligned}$$

The second isomorphism is the commutation of finite products and directed colimits of isometries in CMet, which is vary similar to the analogous statement in the category of sets ([21, §IX.2, Theorem 1]), and admits a parallel proof.

This is an isomorphism if and only if the individual components

$$\varinjlim_{\alpha} \hom(X_i, Y_\alpha) \to \hom(X_i, Y)$$

are, hence the conclusion. □

Consequently:

Corollary 4.7. *The isometry-$\aleph_0$-generated objects in* CMet *are those that*

- *have finitely many finite-distance components;*
- *all of which are themselves isometry-$\aleph_0$-generated.*

Proof. Immediate from Lemmas 4.5 and 4.6. □

Proof of Theorem 4.3. Corollary 4.7 reduces the problem to showing that a non-empty finite-distance $\aleph_0$-generated object (X, d) (fixed throughout the sequel) must be a singleton. The proof proceeds in a number of stages.

(1) X **is compact.** Given that it is complete by assumption, this will follow from [23, Theorem 45.1] as soon as we show that X is *totally bounded* [23, §45, Definition preceding Example 1]: for every ε, X can be covered with finitely many ε-radius balls. To see this, note that X is the directed union of its finite subspaces (equipped with the restricted metric)

$$(X, d) = \varinjlim (F, d), \ \ F \subseteq X \text{ finite.}$$

Finite generation in the present (enriched) context means that for every $\varepsilon > 0$ the identity $X \to X$ is uniformly ε-approximable by a contraction $X \to F$, whence every point is within ε of one of the finitely many elements of F.

(2) X **is a singleton.** Suppose not. We can then fix $\varepsilon > 0$ sufficiently small that (by finite generation) the identity $X \to X$ is uniformly ε-close to a contraction

$$X \xrightarrow{\quad\pi\quad} F \subseteq X$$

factoring through some non-singleton finite $F \subseteq X$. The fibers

$$X_p := \pi^{-1}(p)\,, \quad p \in F$$

are *clopen* (i.e. closed and open) and partition X; denote by $\ell > 0$ the smallest distance between two of them, say X_p and X_q. By compactness that distance is actually achieved:

$$\ell = d(X_p, X_q) := \inf\{d(x,y) \mid x \in X_p,\ y \in X_q\}$$

is in fact a minimum:

$$d(x,y) = \ell \quad \text{for some} \quad x \in X_p\,,\ y \in X_q\,.$$

The embedding $\iota\colon \mathbf{2}_\ell \to X$ sending the two points to x and y now admits a contractive retraction (i.e. left inverse) $r\colon X \to \mathbf{2}_\ell$ sending X_p to one of the points and everything else to the other point.

That the $\aleph_0$-generation property survives under retractions is a simple exercise, so $\mathbf{2}_\ell$ must be isometry-$\aleph_0$-generated. This, though, contradicts [3, Proposition 5.19].

This finishes the proof. □

The precise analogue of Theorem 4.3 holds in CPMet:

Theorem 4.8. *The objects in* CPMet *isometry-$\aleph_0$-generated in the enriched sense are precisely the finite discrete metric spaces.*

Proof. The CPMet version of Corollary 4.7 goes through just as easily, so we are again reduced to showing that a non-empty finite-distance $(X, d_X) \in$ CPMet is isometry-$\aleph_0$-generated only if it is a singleton (the 'if' implication being obvious).

The proof strategy for this last claim will be very different, as most devices employed in the proof of Theorem 4.3 are absent here (we cannot work with finite spaces, etc., since every metric in sight must be path). We instead proceed to construct a non-approximable morphism

$$\iota\colon (X, d_X) \to (Y, d := d_Y) \cong \varinjlim (Y_n, d_Y|_{Y_n}) \tag{4-5}$$

for (isometrically embedded) $Y_n \subset (Y, d)$ as follows.

- First, attach a metric segment Γ_x of length 1 to every point $x \in X$. This attachment occurs only at a single endpoint of Γ_x (which then becomes identified with x); the other endpoint is, say, p_x.

  Denote this (intermediate) space by (Z, d_Z). It is complete by Theorem 2.31, as in the proof of Proposition 2.36. It is also of course a path metric space, being a gluing of such [8, discussion following Exercise 3.1.13].

- Next, connect any two newly-added points
$$p \in \Gamma_x\ ,\quad q \in \Gamma_y\ ,\quad x \ne y \in X$$
on *distinct* segments Γ_x and Γ_y with a metric segment $\Gamma_{p,q}$ of length $d_Z(p,q)$. The result (once more a complete path metric space) will be our (Y, d_Y).
- This all falls within the scope of Theorem 2.21 with $C = 1$: first gluing along points to produce Z, and then gluing along isometrically embedded two-point spaces. It follows from the selfsame Theorem 2.21 that the various component spaces (the original (X, d), the segments Γ_x and the $\Gamma_{p,q}$) all embed isometrically into the end result (Y, d_Y).

  We henceforth refer to the single ambient distance d_Y as d.
- As just noted, the initial space (X, d_X) embeds isometrically into (Y, d); that identification is the map (4-5).
- It remains to identify the $Y_n \subseteq Y$, which we index by positive integers $n \in \mathbb{Z}_{>0}$. By definition, Y_n consists of
  - the points on the partial segments
$$\Gamma_{x,n} := \left\{ p \in \Gamma_x \mid d(p, X) = d(p, x) \ge \frac{1}{n} \right\},\ x \in X;$$
  - and the points on those $\Gamma_{p,q}$ that connect these:
$$\Gamma_{p,q} \quad \text{for} \quad p \in \Gamma_{x,n}\ ,\ q \in \Gamma_{y,n}\ ,\ x \ne y\ .$$

By construction, the Y_n are path metric spaces: points on the same $\Gamma_{x,n}$ are already on a metric segment ($\Gamma_{x,n}$ itself), while those on distinct $\gamma_{n,x_{0,1}}$, $x_0 \ne x_1$ are connected by their own dedicated metric segment (one of the $\Gamma_{p,q}$). It is also clear that Y is the directed colimit (in Met, or CMet, or CPMet) of the Y_n, as the near endpoints of the $\Gamma_{x,n}$ respectively approach $x \in X$ uniformly in n.

It remains to argue that if X has at least two points, then $\iota : X \to Y$ is not arbitrarily approximable in the path-metric sense by morphisms into the Y_n.

Consider two points $x_0 \ne x_1 \in X$. Since the latter is path, there is a path
$$\gamma\colon\ (I := [0,1]) \to X$$
connecting x_0 and x_1 (not necessarily a contraction). To flesh out what it would take to prove the claim, suppose ι *were* approximable by morphisms $X \to Y_n$, which would then have to be connectable to ι by short paths in $\mathrm{CPMet}(X, Y_n)$. Composing with $\gamma\colon I \to X$, this means that the latter would be connectable to paths $\gamma_n : I \to Y_n$ by a path in
$$\mathrm{Cont}(I \to Y) := \text{continuous maps } I \to Y$$
of arbitrarily small length. To reach a contradiction, suppose $\gamma_n\colon I \to Y_n$ is uniformly close to γ. The endpoints of γ_n must then lie on $\Gamma_{p,q}$ for p and q on $\Gamma_{x_0',n}$ and $\Gamma_{x_1',n}$ respectively, for small
$$d(x_0', p)\ ,\ d(x_1', q)\ ,\ d(x_0, x_0') \text{ and } d(x_1, x_1')\ .$$

In particular, this imposes a uniform positive lower bound on the distance $d(p, q)$, and hence a uniform lower bound $\ell > 0$ (dependent only on $d(x_0, x_1) > 0$ and nothing else mentioned here) on the lengths of the segments $\Gamma_{p,q}$ that such a γ_n must traverse. But then any path connecting, say, the midpoint of $\Gamma_{p,q}$ with a point on $X \subseteq Y$ must have length at least $\frac{\ell}{2}$, and hence cannot be arbitrarily small.

This concludes the proof. □

The following consequence of (the proof of) Theorem 4.8 answers (negatively) the question of whether metric segments are isometry-$\aleph_0$-generated, asked in [10, Remark 6.9].

Corollary 4.9. *The objects in* CCMET *isometry-$\aleph_0$-generated in the enriched sense are precisely the finite discrete metric spaces.*

Proof. If (X, d) is convex then the spaces Y and Y_n constructed in the proof of Theorem 4.8 are also convex, for instance by Proposition 2.36. □

Acknowledgement. I am very grateful for the anonymous referee's numerous valuable suggestions and apparently endless patience, as well as J. Rosický's many insightful comments on earlier drafts of the paper.

This work is partially supported by NSF grant DMS-2001128.

References

[1] Adámek, J., Herrlich, H., Strecker, G.E., *Abstract and concrete categories: the joy of cats*, Repr. Theory Appl. Categ. **17** (2006), 1–507, Reprint of the 1990 original [Wiley, New York; MR1051419].

[2] Adámek, J., Rosický, J., *Locally presentable and accessible categories*, London Math. Soc. Lecture Note Ser., vol. 189, Cambridge University Press, Cambridge, 1994.

[3] Adámek, J., Rosický, J., *Approximate injectivity and smallness in metric-enriched categories*, J. Pure Appl. Algebra **226** (6) (2022), 30 pp., Paper No. 106974.

[4] Awodey, S., *Category theory*, second ed., Oxford Logic Guides, vol. 52, Oxford University Press, Oxford, 2010.

[5] Blackadar, B., *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006.

[6] Blumenthal, L.M., *Theory and applications of distance geometry*, Chelsea Publishing Co., New York, 1970.

[7] Bollobás, B., *Modern graph theory*, Graduate Texts in Mathematics, vol. 184, Springer-Verlag, New York, 1998.

[8] Burago, D., Burago, Y., Ivanov, S., *A course in metric geometry*, Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2001.

[9] Chirvasitu, A., Ko, J., *Monadic forgetful functors and (non-)presentability for C^*- and W^*-algebras*, `http://arxiv.org/abs/2203.12087v2`, 2022.

[10] Di Liberti, I., Rosický, J., *Enriched locally generated categories*, Theory Appl. Categ. **38** (2022), 661–683, Paper No. 17.

[11] Diestel, R., *Graph theory*, Grad. Texts in Math., vol. 173, Springer, Berlin, fifth ed., 2018.

[12] Garbulińska-Węgrzyn, J., Kubiś, W., *A universal operator on the Gurariĭ space*, J. Operator Theory **73** (1) (2015), 143–158.

[13] Goebel, K., Kirk, W.A., *Topics in metric fixed point theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1990.

[14] Gromov, M., *Metric structures for Riemannian and non-Riemannian spaces*, english ed., Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2007, Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[15] Halmos, P.R., *A Hilbert space problem book*, Encyclopedia of Mathematics and its Applications, vol. 17, Springer-Verlag, New York-Berlin, second ed., 1982.

[16] Kelly, G.M., *Structures defined by finite limits in the enriched context, I*, Cah. Topol. Géom. Dffér. Catég. **23** (1) (1982), 3–42, Third Colloquium on Categories, Part VI (Amiens, 1980.

[17] Kelly, G.M., *Basic concepts of enriched category theory*, Repr. Theory Appl. Categ., vol. 10, Cambridge Univ. Press, Cambridge, 2005, pp. vi+137.

[18] Khamsi, M.A., Kirk, W.A., *An introduction to metric spaces and fixed point theory*, Pure Appl. Math. (N. Y.), Wiley-Interscience, New York, 2001.

[19] Kubiś, W., *Metric-enriched categories and approximate Fraïssé limits*, 2012, `http://arxiv.org/abs/1210.6506v3`.

[20] Lupini, M., *Fraïssé limits in functional analysis*, Adv. Math. **338** (2018), 93–174.

[21] Mac Lane, S., *Categories for the working mathematician*, Grad. Texts in Math., Springer-Verlag, New York, second ed., 1998.

[22] Menger, K., *Untersuchungen über allgemeine Metrik*, Math. Ann. **100** (1) (1928), 75–163.

[23] Munkres, J.R., *Topology*, Prentice Hall, Inc., Upper Saddle River, NJ, 2000, Second edition of [MR0464128].

[24] Perdigão do Carmo, M., *Mathematics: Theory & Applications*, Birkhäuser Boston, Inc., Boston, MA, 1992, Translated from the second Portuguese edition by Francis Flaherty.

[25] Rosický, J., Tholen, W., *Approximate injectivity*, Appl. Categ. Structures **26** (4) (2018), 699–716.

[26] Wegge-Olsen, N.E., *K-theory and C^*-algebras*, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993, A friendly approach.

Department of Mathematics,
University at Buffalo,
Buffalo, NY 14260-2900, USA
E-mail: `achirvas@buffalo.edu`