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LOGARITHMICALLY IMPROVED BLOW-UP CRITERION FOR SMOOTH SOLUTIONS TO THE LERAY-α-MAGNETOHYDRODYNAMIC EQUATIONS

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Abstract. In this paper, the Cauchy problem for the $3D$ Leray-α-MHD model is investigated. We obtain the logarithmically improved blow-up criterion of smooth solutions for the Leray-α-MHD model in terms of the magnetic field B only in the framework of homogeneous Besov space with negative index.

1. Introduction

In this paper we consider the following incompressible Leray-α-MHD model in $\mathbb{R}^3$ take the form (see e.g. [5] and references therein):

$$\left\{\begin{array}{c} \partial_t v + (u\cdot\nabla)v - \Delta v + \nabla\pi + \frac{1}{2}\nabla|B|^2 = (B\cdot\nabla)B, \\ \partial_t B + (u\cdot\nabla)B - (B\cdot\nabla)v - \Delta B = 0, \\ v = (1-\alpha^2\Delta)u, \quad \alpha > 0, \\ \operatorname{div} u = \operatorname{div} v = \operatorname{div} B = 0, \\ (v,B)|_{t=0} = (v_0,B_0), \operatorname{div} u_0 = \operatorname{div} v_0 = \operatorname{div} B_0 = 0 \ \text{ in } \ \mathbb{R}^3, \end{array}\right. \tag{1.1}$$

where v : the fluid velocity field, u : "the filtered" fluid velocity, B : the magnetic field and π : the pressure, are the unknowns; α is the lengthscale parameter that represents the width of the filter. Note that the magnetic field is not regularized.

It has lately received significant attention in mathematical fluid dynamics due to its connection to three-dimensional incompressible flows. When $\alpha \to 0$, the model (1.1) reduce to the following MHD equations:

$$\left\{\begin{array}{c} \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla\pi + \frac{1}{2}\nabla|B|^2 = (B\cdot\nabla)B, \\ \partial_t B + (u\cdot\nabla)B - (B\cdot\nabla)u - \Delta B = 0, \\ \operatorname{div} u = \operatorname{div} B = 0, \\ (u,B)|_{t=0} = (u_0,B_0), \operatorname{div} u_0 = \operatorname{div} B_0 = 0 \ \text{ in } \ \mathbb{R}^3, \end{array}\right. \tag{1.2}$$

(1.1) is smoother than (1.2). It is currently unknown whether solutions of the initial value problem of the 3D Navier-stokes equations or the 3D MHD equations can develop finite time singularities even if the initial data is sufficiently smooth. Thus

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it is easier to prove that the problem (1.1) has a unique local smooth solution. However, it is still open to prove whether the local solution is global or not. For simplicity, without loss of generality, we assume $\alpha = 1$. When $B = 0$, the system (1.2) becomes the well-known Navier-Stokes-α (also known as the Lagrangian averaged Navier-Stokes equations-α or the viscous Camassa-Holm equations) as a closure model of turbulence in infinite channels and pipes, whose solutions give an excellent agreement with empirical data for a wide range of large Reynolds numbers, the alpha subgrid scale models of turbulence have been extensively studied (see [1, 2, 3, 4, 5, 6, 11]).

An extension of the Navier-Stokes-α model to the nondissipative MHD is given, e.g., in [10]. The model was obtained from variational principles by modifying the Hamiltonian associated with the ideal MHD equations subject to the incompressibility constraint. Then the dissipation is introduced in an ad hoc fashion in analogy to the Navier-Stokes-α model following (see [5] and the references therein).

The existence of weak solutions to the problem (1.1) has been established by Linshiz and Titi [14]. Also, it is easy to prove the existence and uniqueness of local smooth solutions to the problem (1.1) with initial data $(v_0, B_0) \in H^1(\mathbb{R}^3)$ and $\operatorname{div} u_0 = \operatorname{div} v_0 = \operatorname{div} B_0 = 0$ in $\mathbb{R}^3$. However, the regularity of weak solutions to the problem (1.1)is still open. In [19], the authors established various regularity criteria in terms of the velocity field, which implies that the velocity field plays a dominant role in the regularity theorem. It is reasonable and similar to the case for the standard MHD equations (for example, see [17, 18, 20, 21, 23]).

We recall a solution pair (v, B) is a local smooth solution of system (1.1) in the interval $[0, T]$ for $(v_0, B_0) \in H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$ provided that

$$(v, B) \in C\big(0, T; H^3(\mathbb{R}^3)\big) \times C\big(0, T; H^3(\mathbb{R}^3)\big) .$$

Since the solutions to the Leray-α-MHD model are smoother than that of the original MHD equations, Fan and Ozawa [5] consider the blow-up criterion of smooth solution in terms of the magnetic field B only by using the Fourier localization technique and Bony's paraproduct decomposition. More precisely, they proved that (v, u, π, B) is smooth at time $t = T$ provided that

$$\int_0^T \|B(\cdot,t)\|^2_{\dot{B}^0_{\infty,\infty}} \, dt < \infty ,$$

where $\dot{B}^0_{\infty,\infty}$ denotes the homogeneous Besov space. There are other types of blow up criteria of smooth solutions to the Leray-α-MHD model, (see for example [5, 19]).

Here, motivated by the results in [5] and [19], our aim is to establish the logarithmically improved blow-up criterion to (1.1) in the framework homogeneous Besov space with negative index $\dot{B}^{-1}_{\infty,\infty}$ by means of only magnetic field B. More precisely, we will prove the following.

Theorem 1.1. *Suppose that* $v_0, B_0 \in H^3(\mathbb{R}^3)$ *with* $\operatorname{div} v_0 = \operatorname{div} B_0 = 0$ *and* (v, u, B) *is a local smooth solution to the system* (1.1) *for* $0 \le t < T$. *If* B *satisfies*

$$\int_0^T \frac{\|\nabla B(\cdot,t)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\ln(e + \|\nabla B(\cdot,t)\|_{\dot{B}^{-1}_{\infty,\infty}})} dt < \infty\,, \tag{1.3}$$

then the solution (v, u, B) *can be extended beyond* $t = T$.

Remark 1.1. Usually, regularity criteria are established in terms of the velocity field for the MHD equations. But here the magnetic field plays a dominant role. As a corollary, we reprove the solution to the density-dependent Leray-α-Navier-Stokes equations ($B = 0$ in (1.2) exists globally in time).

We have the following corollary immediately.

Corollary 1.2. *Assume that* $v_0, B_0 \in H^3(\mathbb{R}^3)$ *with* $\operatorname{div} v_0 = \operatorname{div} B_0 = 0$. *Let* (v, u, B) *be a local smooth solution to the system* (1.1) *for* $0 \le t < T$. *Suppose that* T *is the maximal existence time. Then*

$$\int_0^T \frac{\|\nabla B(\cdot,t)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\ln(e + \|\nabla B(\cdot,t)\|_{\dot{B}^{-1}_{\infty,\infty}})} dt = \infty\,.$$

It is well known that $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ is the biggest critical homogeneous space of degree -1, and as shown by Frazier, Jawerth and Weiss [7] any critical homogeneous space continuously embedded in $S'(\mathbb{R}^3)$ is also continuously embedded into $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$.

As a consequence of the fact $\|\nabla B\|_{\dot{B}^{-1}_{\infty,\infty}} \approx \|B\|_{\dot{B}^{0}_{\infty,\infty}}$, from Theorem 1.1, we obtain immediately the following result

Corollary 1.3. *Assume that* $v_0, B_0 \in H^3(\mathbb{R}^3)$ *with* $\operatorname{div} v_0 = \operatorname{div} B_0 = 0$. *Let* (v, u, B) *be a local smooth solution to the system* (1.1) *for* $0 \le t < T$. *If* B *satisfies*

$$\int_0^T \frac{\|B(\cdot,t)\|^2_{\dot{B}^{0}_{\infty,\infty}}}{\ln(e + \|B(\cdot,t)\|_{\dot{B}^{0}_{\infty,\infty}})} dt < \infty\,,$$

then the solution (v, u, B) *remains smooth on* $[0, T]$.

Thanks to

$$\int_0^T \frac{\|B(\cdot,t)\|^2_{\dot{B}^{0}_{\infty,\infty}}}{\ln(e + \|B(\cdot,t)\|_{\dot{B}^{0}_{\infty,\infty}})} dt \le \int_0^T \|B(\cdot,t)\|^2_{\dot{B}^{0}_{\infty,\infty}}\, dt,$$

it is easy to deduce that our criterion (1.3) can be viewed as a generalization of the result of Fan and Ozawa [5]. Moreover, thanks to the fact that the system (1.1) with $\alpha \to 0$ and $B = 0$ reduce to the 3D Navier-Stokes equations, we notice that our criterion becomes the recent reults of Gala-Guo [8] for the Navier-Stokes equations (see also [22]).

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Since we deal with the regularity condition of the smooth solutions, we only need to prove the a priori estimates for smooth solutions. Throughout the rest of the paper, C denotes various positive and finite constants whose exact values are unimportant and may vary from line to line.

To prove the theorem we will use the following bilinear commutator estimate. We can find the detailed proof in [13] for example.

Lemma 2.1. *suppose that $1 < p < \infty$ and $s > 0$. Let f and g be two smooth functions such that $\nabla f \in L^{q_1}$, $\Lambda^s f \in L^{r_2}$, $\Lambda^{s-1} g \in L^{r_1}$ and $g \in L^{q_2}$. Then there exists an abstract constant C such that*

$$\left\| \Lambda^s (fg) - f\Lambda^s g \right\|_{L^p} \le C \left(\|\nabla f\|_{L^{q_1}} \left\|\Lambda^{s-1} g\right\|_{L^{r_1}} + \|\Lambda^s f\|_{L^{r_2}} \|g\|_{L^{q_2}} \right) , \tag{2.1}$$

with $1 < q_1, q_2 \le \infty$, $1 < r_1, r_2 < \infty$ such that $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{r_1} = \frac{1}{q_2} + \frac{1}{r_2}$, where $\Lambda = (-\Delta)^{\frac{1}{2}}$.

The following Gagliardo-Nirenberg inequality (see [9]) will be frequently used later.

Lemma 2.2. *Let j, m, k be any integers satisfying $k \le j \le m$ and let $1 \le q, r \le \infty$ and $p \in \mathbb{R}$, $0 \le \theta \le 1$ such that*

$$\frac{1}{p} - \frac{j}{3} = \theta\Big(\frac{1}{q} - \frac{m}{3}\Big) + (1-\theta)\Big(\frac{1}{q} - \frac{k}{3}\Big) .$$

Then for all $f \in W^{k,q}(\mathbb{R}^3) \cap W^{m,r}(\mathbb{R}^3)$, there is a positive constant C dependending only on m, j, k, q, r, θ such that the following inequality holds

$$\left\|\Lambda^j f\right\|_{L^p} \le C \left\|\Lambda^k f\right\|_{L^q}^{1-\theta} \|\Lambda^m f\|_{L^r}^{\theta} . \tag{2.2}$$

In order to prove our main result, we need the following interpolation inequality which may be found in ([15, 16]):

$$\|f\|_{L^4}^2 \le C \|f\|_{\dot{B}^{-1}_{\infty,\infty}} \|\nabla f\|_{L^2} . \tag{2.3}$$

Recall also that for $0 \le s < \frac{3}{2}$, we have

$$H^s(\mathbb{R}^3) \subset L^p(\mathbb{R}^3) \quad \forall p \in \left[2, \frac{6}{3-2s}\right) .$$

Now let us proceed to prove Theorem 1.1.

Proof. Owing to (1.3), we know that that for any small constant $\epsilon > 0$, there exists $T_0 = T_0(\epsilon) < T$ such that

$$\int_{T_0}^{T} \frac{\|\nabla B(\cdot,t)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\ln(e + \|\nabla B(\cdot,t)\|_{\dot{B}^{-1}_{\infty,\infty}})} dt \le \epsilon . \tag{2.4}$$

Consequently, the main goal of this section is to establish the following a priori estimate

$$\lim_{t\to T^-}\sup\left(\left\|\Lambda^3 v(\cdot,t)\right\|_{L^2}^2+\left\|\Lambda^3 B(\cdot,t)\right\|_{L^2}^2\right)<\infty\,.$$

Step 1. L^2 and H^1 energy estimates.

To begin with, we multiply the first, second equation of (1.1) by v, B, respectively and integrate them over $\mathbb{R}^3$ with respect to x, then add the resulting equation, it yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left(v^2+B^2\right)dx+\int_{\mathbb{R}^3}|\nabla v|^2+|\nabla B|^2\,dx\\
&=\int_{\mathbb{R}^3}(B\cdot\nabla B)\cdot v dx-\int_{\mathbb{R}^3}(u\cdot\nabla v)\cdot v dx-\int_{\mathbb{R}^3}\nabla\pi\cdot v dx+\int_{\mathbb{R}^3}(B\cdot\nabla v)\cdot B dx-\int_{\mathbb{R}^3}(u\cdot\nabla B)\cdot B dx\\
&=\int_{\mathbb{R}^3}B\cdot\nabla(Bv)dx-\frac{1}{2}\int_{\mathbb{R}^3}u\cdot\nabla(v^2)dx-\frac{1}{2}\int_{\mathbb{R}^3}u\cdot\nabla(B^2)dx=0\,,
\end{aligned}$$

where we have used $\nabla\cdot u=\nabla\cdot v=\nabla\cdot B=0$.

Integrating above equality with respect to time yields

$$\begin{cases}\|v\|_{L^\infty(0,T;L^2)}+\|v\|_{L^2(0,T;H^1)}\le C\,,\\ \|B\|_{L^\infty(0,T;L^2)}+\|B\|_{L^2(0,T;H^1)}\le C\,.\end{cases}\tag{2.5}$$

As a consequence, the relation between v and u allows us to show

$$\|u\|_{L^\infty(0,T;H^2)}+\|u\|_{L^2(0,T;H^3)}\le C\,.\tag{2.6}$$

We apply ∇ to $(1.1)_1$ and multiplying the resulting equation by ∇v, and integrating with respect to x on $\mathbb{R}^3$, using integration by parts, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla v(\cdot,t)\|_{L^2}^2+\left\|\nabla^2 v(\cdot,t)\right\|_{L^2}^2\\
&\quad=-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla)v\cdot\nabla v dx+\int_{\mathbb{R}^3}\nabla(B\cdot\nabla)B\cdot\nabla v dx\\
&\quad=\int_{\mathbb{R}^3}(u\cdot\nabla)v\cdot\Delta v dx-\int_{\mathbb{R}^3}(B\cdot\nabla)B\cdot\Delta v dx\\
&\quad=\int_{\mathbb{R}^3}(u\cdot\nabla)v\cdot\Delta v dx+\sum_{j,k=1}^3\int_{\mathbb{R}^3}\left(\partial_k B_j\cdot\partial_j B+B_j\partial_j\partial_k B\right)\partial_k v dx\,,
\end{aligned}\tag{2.7}$$

where we have used the fact

$$-\int_{\mathbb{R}^3}(B\cdot\nabla)B\cdot\Delta v dx=-\sum_{j,k=1}^3\int_{\mathbb{R}^3}(B_j\cdot\partial_j B)\partial_k^2 v dx=\sum_{j,k=1}^3\int_{\mathbb{R}^3}(\partial_k B_j\cdot\partial_j B+B_j\partial_j\partial_k B)\partial_k v dx\,.$$

Similarly, applying ∇ to $(1.1)_2$ and multiplying the resulting equation by ∇B, and integrating with respect to x on $\mathbb{R}^3$, using integration by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla B(\cdot,t)\|_{L^2}^2+\left\|\nabla^2 B(\cdot,t)\right\|_{L^2}^2\\
&\quad=-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla)B\cdot\nabla B dx+\int_{\mathbb{R}^3}\nabla(B\cdot\nabla)v\cdot\nabla B dx\\
&\quad=\int_{\mathbb{R}^3}(u\cdot\nabla)B\cdot\Delta B dx-\int_{\mathbb{R}^3}(B\cdot\nabla)v\cdot\Delta B dx\\
&\quad=\int_{\mathbb{R}^3}(u\cdot\nabla)B\cdot\Delta B dx+\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}\left(\partial_k B_j\cdot\partial_k B\partial_j v-B_j\partial_j\partial_k B\cdot\partial_k v\right)dx\,,
\end{aligned}\tag{2.8}$$

where we have used the fact

$$\begin{aligned}
-\int_{\mathbb{R}^3}(B\cdot\nabla)v\cdot\Delta B dx&=-\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(B_j\partial_j v)\cdot\partial_k^2 B dx=-\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k(B_j v)\partial_k\partial_j B dx\\
&=-\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_k B_j v+B_j\partial_k v)\partial_j\partial_k B dx\\
&=\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}\left(\partial_k B_j\cdot\partial_k B\cdot\partial_j v-B_j\partial_j\partial_k B\cdot\partial_k v\right)dx\,.
\end{aligned}$$

Summing up (2.7) and (2.8), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla v(\cdot,t)\|_{L^2}^2+\|\nabla B(\cdot,t)\|_{L^2}^2)+\left\|\nabla^2 v(\cdot,t)\right\|_{L^2}^2+\left\|\nabla^2 B(\cdot,t)\right\|_{L^2}^2\\
&\quad=\int_{\mathbb{R}^3}\left[(u\cdot\nabla)v\cdot\Delta v+(u\cdot\nabla)B\cdot\Delta B\right]dx\\
&\qquad+\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k B_j\cdot\left(\partial_j B\cdot\partial_k v+\partial_k B\cdot\partial_j v\right)dx\,.
\end{aligned}\tag{2.9}$$

Using Hölder inequality, (2.6) and then due to the inequality $a^{1-\beta}b^{\beta}\le a+b$, $0<\beta<1$, $a,b>0$, one has

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left[(u\cdot\nabla)v\cdot\Delta v+(u\cdot\nabla)B\cdot\Delta B\right]dx\\
&\quad\le\|u\|_{L^\infty}\|\nabla v\|_{L^2}\left\|\nabla^2 v\right\|_{L^2}+\|u\|_{L^\infty}\|\nabla B\|_{L^2}\left\|\nabla^2 B\right\|_{L^2}\\
&\quad\le C\|\nabla v\|_{L^2}\left\|\nabla^2 v\right\|_{L^2}+C\|\nabla B\|_{L^2}\left\|\nabla^2 B\right\|_{L^2}\\
&\quad\le\frac{1}{2}\left\|\nabla^2 v\right\|_{L^2}^2+\frac{1}{4}\left\|\nabla^2 B\right\|_{L^2}^2+C\left(\|\nabla v\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right)\,,
\end{aligned}\tag{2.10}$$

where the Sobolev embedding $H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$ is applied.

Thanks to (2.3) and Young inequality, we can obtain

$$
\begin{aligned}
&\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k B_j\cdot(\partial_j B\cdot\partial_k v+\partial_k B\cdot\partial_j v)\,dx\\
&\qquad\le 2\,\|\nabla B\|_{L^4}^2\,\|\nabla v\|_{L^2}\le C\,\|\nabla B\|_{\dot{B}^{-1}_{\infty,\infty}}\,\left\|\nabla^2 B\right\|_{L^2}\|\nabla v\|_{L^2}\\
&\qquad\le C\,\|\nabla B\|^2_{\dot{B}^{-1}_{\infty,\infty}}\,\|\nabla v\|^2_{L^2}+\frac{1}{8}\left\|\nabla^2 B\right\|^2_{L^2}\\
&\qquad\le C\,\|\nabla B\|^2_{\dot{B}^{-1}_{\infty,\infty}}\left(\|\nabla v\|^2_{L^2}+\|\nabla B\|^2_{L^2}\right)+\frac{1}{4}\left\|\nabla^2 B\right\|^2_{L^2}\,. \qquad (2.11)
\end{aligned}
$$

Substituting (2.10) and (2.11) into (2.9), we arrive at

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla v(\cdot,t)\|^2_{L^2}+\|\nabla B(\cdot,t)\|^2_{L^2})+\left\|\nabla^2 v(\cdot,t)\right\|^2_{L^2}+\left\|\nabla^2 B(\cdot,t)\right\|^2_{L^2}\\
&\qquad\le C(1+\|\nabla B\|^2_{\dot{B}^{-1}_{\infty,\infty}})(\|\nabla v\|^2_{L^2}+\|\nabla B\|^2_{L^2})\,. \qquad (2.12)
\end{aligned}
$$

For any $T_0\le t<T$, we denote

$$
F(t)=\max_{\tau\in[T_0,t]}\left(\|v(\cdot,\tau)\|^2_{H^3}+\|B(\cdot,\tau)\|^2_{H^3}\right).
$$

Integrating above inequality (2.12) over interval $[T_0,t)$ and observing that $F(t)$ is a monotonically increasing function of t, we thus obtain

$$
\begin{aligned}
&\|\nabla v(\cdot,t)\|^2_{L^2}+\|\nabla B(\cdot,t)\|^2_{L^2}+\int_{T_0}^{t}\left(\left\|\nabla^2 v(\cdot,\tau)\right\|^2_{L^2}+\left\|\nabla^2 B(\cdot,\tau)\right\|^2_{L^2}\right)d\tau\\
&\le\left(\|\nabla v(\cdot,T_0)\|^2_{L^2}+\|\nabla B(\cdot,T_0)\|^2_{L^2}\right)\exp\left(C\int_{T_0}^{t}\left(1+\|\nabla B(\cdot,\tau)\|^2_{\dot{B}^{-1}_{\infty,\infty}}\right)d\tau\right)\\
&\le M\exp\left(C\int_{T_0}^{t}\|\nabla B(\cdot,\tau)\|^2_{\dot{B}^{-1}_{\infty,\infty}}\,d\tau\right)\\
&= M\exp\left(C\int_{T_0}^{t}\frac{\|\nabla B(\cdot,\tau)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\ln(e+\|\nabla B(\cdot,\tau)\|_{\dot{B}^{-1}_{\infty,\infty}})}\ln(e+\|\nabla B(\cdot,\tau)\|_{\dot{B}^{-1}_{\infty,\infty}})d\tau\right)\\
&\le M\exp\left(C\int_{T_0}^{t}\frac{\|\nabla B(\cdot,\tau)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\ln(e+\|\nabla B(\cdot,\tau)\|_{\dot{B}^{-1}_{\infty,\infty}})}\ln(e+\|B(\cdot,\tau)\|_{\dot{B}^{0}_{\infty,\infty}})d\tau\right)\\
&\le M\exp\left(C\int_{T_0}^{t}\frac{\|\nabla B(\cdot,\tau)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\ln(e+\|\nabla B(\cdot,\tau)\|_{\dot{B}^{-1}_{\infty,\infty}})}\ln(e+\|B(\cdot,\tau)\|_{L^\infty})d\tau\right)
\end{aligned}
$$

$$\leq M\exp\left(C\int_{T_0}^{t}\frac{\|\nabla B(\cdot,\tau)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\ln(e+\|\nabla B(\cdot,\tau)\|_{\dot{B}^{-1}_{\infty,\infty}})}\ln(e+\|B(\cdot,\tau)\|_{H^3})d\tau\right)$$
$$\leq M\exp\left(C\int_{T_0}^{t}\frac{\|\nabla B(\cdot,\tau)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\ln(e+\|\nabla B(\cdot,\tau)\|_{\dot{B}^{-1}_{\infty,\infty}})}\ln(e+F(\tau))d\tau\right)$$
$$\leq M\exp\left(C\int_{T_0}^{t}\frac{\|\nabla B(\cdot,\tau)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\ln(e+\|\nabla B(\cdot,\tau)\|_{\dot{B}^{-1}_{\infty,\infty}})}d\tau\ln(e+F(t))\right),$$

where the Sobolev embedding $H^3(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$ is applied. We want to state here that from the above observation C is an absolute constant and M depends on $\|\nabla v(\cdot,T_0)\|_{L^2}, \|\nabla B(\cdot,T_0)\|_{L^2}, T_0$ and T.

By Gronwall's inequality, we conclude that

$$\|\nabla v(\cdot,t)\|^2_{L^2}+\|\nabla B(\cdot,t)\|^2_{L^2}+\int_{T_0}^{t}\left(\left\|\nabla^2 v(\cdot,\tau)\right\|^2_{L^2}+\left\|\nabla^2 B(\cdot,\tau)\right\|^2_{L^2}\right)d\tau$$
$$\leq C_0(e+F(t))^{C\epsilon}.$$

Step 2. We go to the estimate for the H^3-norm. Taking the operation $\Lambda^3=(-\Delta)^{\frac{3}{2}}$ on both sides of $(1.1)_1$, then multiplying them by $\Lambda^3 v$ and integrate with respect to x on $\mathbb{R}^3$, using integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^3 v(\cdot,t)\right\|^2_{L^2}+\left\|\Lambda^3\nabla v(\cdot,t)\right\|^2_{L^2}=-\int_{\mathbb{R}^3}\Lambda^3(u\cdot\nabla)v\cdot\Lambda^3 v dx$$
$$+\int_{\mathbb{R}^3}\Lambda^3(B\cdot\nabla)B\cdot\Lambda^3 v dx. \tag{2.13}$$

Likewise, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^3 B(\cdot,t)\right\|^2_{L^2}+\left\|\Lambda^3\nabla B(\cdot,t)\right\|^2_{L^2}=-\int_{\mathbb{R}^3}\Lambda^3(u\cdot\nabla)B\cdot\Lambda^3 B dx$$
$$+\int_{\mathbb{R}^3}\Lambda^3(B\cdot\nabla)v\cdot\Lambda^3 B dx. \tag{2.14}$$

Summing up (2.13) and (2.14) and using $\nabla\cdot u=\nabla\cdot v=\nabla\cdot B=0$, we have

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^3 v(\cdot,t)\right\|^2_{L^2}+\left\|\Lambda^3 B(\cdot,t)\right\|^2_{L^2}+\left\|\Lambda^3\nabla v(\cdot,t)\right\|^2_{L^2}+\left\|\Lambda^3\nabla B(\cdot,t)\right\|^2_{L^2}$$
$$=-\int_{\mathbb{R}^3}[\Lambda^3(u\cdot\nabla v)-u\cdot\nabla\Lambda^3 v]\cdot\Lambda^3 v dx$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^3}[\Lambda^3(u\cdot\nabla B)-u\cdot\nabla\Lambda^3 B]\cdot\Lambda^3 B dx\\
&+\int_{\mathbb{R}^3}[\Lambda^3(B\cdot\nabla B)-B\cdot\nabla\Lambda^3 B]\cdot\Lambda^3 v dx\\
&+\int_{\mathbb{R}^3}[\Lambda^3(B\cdot\nabla v)-B\cdot\nabla\Lambda^3 v]\cdot\Lambda^3 B dx\\
&=J_1+J_2+J_3+J_4\,, \qquad (2.15)
\end{aligned}
$$

where we have used the divergence free condition $\nabla\cdot u=\nabla\cdot v=\nabla\cdot B=0$, that is,

$$
\int_{\mathbb{R}^3}(u\cdot\nabla)\Lambda^3 v\cdot\Lambda^3 v dx=\int_{\mathbb{R}^3}(u\cdot\nabla)\Lambda^3 B\cdot\Lambda^3 B dx=0\,,
$$

$$
\int_{\mathbb{R}^3}(B\cdot\nabla)\Lambda^3 B\cdot\Lambda^3 v dx+\int_{\mathbb{R}^3}(B\cdot\nabla)\Lambda^3 v\cdot\Lambda^3 B dx=0\,.
$$

According to the fact that $v=\big(I-\alpha^2\Delta\big)\,u$ and using (2.6), we easily get

$$
u\in L^\infty(0,T;H^2)
$$

whence

$$
\nabla u\in L^\infty(0,T;H^1)\subset L^\infty(0,T;L^2)\,.
$$

Note the following fact

$$
\begin{aligned}
\left\|\Lambda^{s+2}u\right\|_{L^2}^2=\left\|\widehat{\Lambda^{s+2}u}(\omega)\right\|_{L^2}^2&=\int_{\mathbb{R}^3}|\omega|^{2(s+2)}\left|\widehat{u}(\omega)\right|^2 d\omega\\
&\le\int_{\mathbb{R}^3}|\omega|^{2s}\left(1+\alpha^2|\omega|^2\right)^2\left|\widehat{u}(\omega)\right|^2 d\omega\\
&=\int_{\mathbb{R}^3}|\omega|^{2s}\left|\widehat{v}(\omega)\right|^2 d\omega=\|\Lambda^s v\|_{L^2}^2\,,\quad\text{for all}\quad s\ge 0\,.
\end{aligned}
$$

In what follows, we will use the following Gagliardo-Nirenberg inequalities which follows from (2.2):

$$
\|\Lambda^s v\|_{L^4}\le C\,\|\Lambda^s v\|_{L^2}^{\frac14}\left\|\Lambda^{s+1}v\right\|_{L^2}^{\frac34}\,, \qquad (2.16)
$$

$$
\|\Lambda^s u\|_{L^4}\le C\,\|\nabla u\|_{L^2}^{\frac{5}{4(s+1)}}\left\|\Lambda^{s+2}u\right\|_{L^2}^{\frac{4s-1}{4(s+1)}}\,. \qquad (2.17)
$$

Now we start to estimate each term of (2.15). To estimate the first term J_1, we use Hölder inequality, (2.1), (2.5), (2.16), (2.17) and Young inequality to obtain

$$\begin{aligned}
J_1 &\leq \left\|\Lambda^3 v\right\|_{L^4} \left\|[\Lambda^3(u\cdot\nabla)v - u\cdot\Lambda^3\nabla v]\right\|_{L^{\frac{4}{3}}} \\
&\leq C\left\|\Lambda^3 v\right\|_{L^4}\left(\|\nabla u\|_{L^2}\left\|\Lambda^3 v\right\|_{L^4} + \left\|\Lambda^3 u\right\|_{L^4}\|\nabla v\|_{L^2}\right) \\
&\leq C\left\|\Lambda^3 v\right\|_{L^4}^2\|\nabla u\|_{L^2} + C\|\nabla v\|_{L^2}\left\|\Lambda^3 v\right\|_{L^4}\left\|\Lambda^3 u\right\|_{L^4} \\
&\leq C\|\nabla u\|_{L^2}\left\|\Lambda^3 v\right\|_{L^2}^{\frac{1}{2}}\left\|\Lambda^4 v\right\|_{L^2}^{\frac{3}{2}} \\
&\quad + C\|\nabla v\|_{L^2}\left\|\Lambda^3 v\right\|_{L^2}^{\frac{1}{4}}\left\|\Lambda^4 v\right\|_{L^2}^{\frac{3}{4}}\|\nabla u\|_{L^2}^{\frac{5}{16}}\left\|\Lambda^5 u\right\|_{L^2}^{\frac{11}{16}} \\
&\leq C\|v\|_{L^2}\left\|\Lambda^3 v\right\|_{L^2}^{\frac{1}{2}}\left\|\Lambda^4 v\right\|_{L^2}^{\frac{3}{2}} \\
&\quad + C\|\nabla v\|_{L^2}\left\|\Lambda^4 v\right\|_{L^2}^{\frac{3}{4}}\|v\|_{L^2}^{\frac{5}{16}}\left\|\Lambda^3 v\right\|_{L^2}^{\frac{15}{16}} \\
&= \left(C\|v\|_{L^2}^4\left\|\Lambda^3 v\right\|_{L^2}^2\right)^{\frac{1}{4}}\left(\left\|\Lambda^4 v\right\|_{L^2}^2\right)^{\frac{3}{4}} \\
&\quad + \left(C\|\nabla v\|_{L^2}^{\frac{8}{5}}\left\|\Lambda^3 v\right\|_{L^2}^{\frac{3}{2}}\|v\|_{L^2}^{\frac{1}{2}}\right)^{\frac{5}{8}}\left(\left\|\Lambda^4 v\right\|_{L^2}^2\right)^{\frac{3}{8}} \\
&\leq C\|v\|_{L^2}^4\left\|\Lambda^3 v\right\|_{L^2}^2 + \frac{1}{16}\left\|\Lambda^4 v\right\|_{L^2}^2 \\
&\quad + C\|\nabla v\|_{L^2}^{\frac{8}{5}}\left\|\Lambda^3 v\right\|_{L^2}^{\frac{3}{2}}\|v\|_{L^2}^{\frac{1}{2}} + \frac{1}{16}\left\|\Lambda^4 v\right\|_{L^2}^2 \\
&\leq \frac{1}{8}\left\|\Lambda^4 v\right\|_{L^2}^2 + C\|v\|_{L^2}^4\left\|\Lambda^3 v\right\|_{L^2}^2 + C\|\nabla v\|_{L^2}^{\frac{32}{5}}\|v\|_{L^2}^2 + \left\|\Lambda^3 v\right\|_{L^2}^2 \\
&\leq \frac{1}{8}\left\|\Lambda^4 v\right\|_{L^2}^2 + \left(1 + C\|v\|_{L^2}^4\right)\left\|\Lambda^3 v\right\|_{L^2}^2 + \left(C\|\nabla v\|_{L^2}^2\|v\|_{L^2}^2\right)^{\frac{5}{32}} \\
&\leq \frac{1}{8}\left\|\Lambda^4 v\right\|_{L^2}^2 + C\left\|\Lambda^3 v\right\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2 \\
&\leq \frac{1}{8}\left\|\nabla^4 v\right\|_{L^2}^2 + C\left(\|\nabla v\|_{L^2}^8 + \|\nabla B\|_{L^2}^8\right)\left\|\nabla^2 v\right\|_{L^2}^2 + C\left\|\nabla^2 v\right\|_{L^2}^2 .
\end{aligned}$$

Arguing similarly as above J_1, we can obtain

$$\begin{aligned}
J_2 &\leq \left|\int_{\mathbb{R}^3}[\Lambda^3, u\cdot\nabla]B\cdot\Lambda^3 B dx\right| \leq \left\|\Lambda^3 B\right\|_{L^4}\left\|[\Lambda^3, u\cdot\nabla]B\right\|_{L^{\frac{4}{3}}} \\
&\leq C\left\|\nabla^3 B\right\|_{L^4}^2\|\nabla u\|_{L^2} + C\|\nabla u\|_{L^2}\left\|\nabla^3 B\right\|_{L^4}\left\|\nabla^3 u\right\|_{L^4} \\
&\leq C\|v\|_{L^2}\left\|\nabla^2 B\right\|_{L^2}^{\frac{1}{4}}\left\|\nabla^4 B\right\|_{L^2}^{\frac{7}{4}} + C\|v\|_{L^2}\left\|\nabla^2 B\right\|_{L^2}^{\frac{1}{8}}\left\|\nabla^4 B\right\|_{L^2}^{\frac{7}{8}}\|\nabla u\|_{L^2}^{\frac{3}{8}}\left\|\nabla^6 u\right\|_{L^2}^{\frac{5}{8}} \\
&\leq \frac{1}{8}\left\|\nabla^4 B\right\|_{L^2}^2 + C\|v\|_{L^2}^8\left\|\nabla^2 B\right\|_{L^2}^2 + C\|v\|_{L^2}^{\frac{22}{3}} + \left\|\nabla^2 B\right\|_{L^2}^2 \\
&\quad + \frac{1}{8}\left\|\nabla^4 B\right\|_{L^2}^2 + \frac{1}{8}\left\|\nabla^4 v\right\|_{L^2}^2 \\
&\leq \frac{1}{8}\left\|\nabla^4 v\right\|_{L^2}^2 + \frac{1}{4}\left\|\nabla^4 B\right\|_{L^2}^2 + C\left\|\nabla^2 B\right\|_{L^2}^2 \\
&\leq \frac{1}{8}\left\|\nabla^4 v\right\|_{L^2}^2 + \frac{1}{4}\left\|\nabla^4 B\right\|_{L^2}^2 + C\left\|\nabla^2 B\right\|_{L^2}^2\left(\|\nabla v\|_{L^2}^8 + \|\nabla B\|_{L^2}^8\right) + C\|\nabla^2 B\|_{L^2}^2 .
\end{aligned}$$

The Gagliardo-Nirenberg inequality (2.16) as well as inequality (2.1) allow as to show that

$$
\begin{aligned}
J_3 &= \int_{\mathbb{R}^3} [\nabla^3, B\cdot\nabla]B\cdot\nabla^3 v dx \le \left\|\nabla^3 v\right\|_{L^4}\left\|[\nabla^3, B\cdot\nabla]B\right\|_{L^{\frac{4}{3}}} \\
&\le \left\|\nabla^3 v\right\|_{L^4}\|\nabla B\|_{L^2}\left\|\nabla^3 B\right\|_{L^4} \\
&\le C\|\nabla B\|_{L^2}\left\|\nabla^2 v\right\|_{L^2}^{\frac{1}{8}}\left\|\nabla^4 v\right\|_{L^2}^{\frac{7}{8}}\left\|\nabla^2 B\right\|_{L^2}^{\frac{1}{8}}\left\|\nabla^4 B\right\|_{L^2}^{\frac{7}{8}} \\
&\le \frac{1}{8}\left\|\nabla^4 v\right\|_{L^2}^2 + \frac{1}{4}\left\|\nabla^4 B\right\|_{L^2}^2 + C\|\nabla B\|_{L^2}^8\left(\left\|\nabla^2 v\right\|_{L^2}^2 + \left\|\nabla^2 B\right\|_{L^2}^2\right).
\end{aligned}
$$

By using the same estimates as above, we have

$$
\begin{aligned}
J_4 &= \int_{\mathbb{R}^3} [\nabla^3, B\cdot\nabla]v\cdot\nabla^3 B dx \le \left\|\nabla^3 B\right\|_{L^4}\left\|[\nabla^3, B\cdot\nabla]v\right\|_{L^{\frac{4}{3}}} \\
&\le C\left\|\nabla^3 B\right\|_{L^4}\|\nabla B\|_{L^2}\left\|\nabla^3 v\right\|_{L^4} + C\left\|\nabla^3 B\right\|_{L^4}^2\|\nabla v\|_{L^2} \\
&\le C\|\nabla B\|_{L^2}\left\|\nabla^2 B\right\|_{L^2}^{\frac{1}{8}}\left\|\nabla^4 B\right\|_{L^2}^{\frac{7}{8}}\left\|\nabla^2 v\right\|_{L^2}^{\frac{1}{8}}\left\|\nabla^4 v\right\|_{L^2}^{\frac{7}{8}} \\
&\quad + C\|\nabla v\|_{L^2}\left\|\nabla^2 B\right\|_{L^2}^{\frac{1}{4}}\left\|\nabla^4 B\right\|_{L^2}^{\frac{7}{4}} \\
&\le C\|\nabla B\|_{L^2}^8\left(\left\|\nabla^2 B\right\|_{L^2}^2 + \left\|\nabla^2 v\right\|_{L^2}^2\right) + \frac{1}{8}\left\|\nabla^4 B\right\|_{L^2}^2 + \frac{1}{8}\left\|\nabla^4 v\right\|_{L^2}^2 \\
&\quad + C\|\nabla v\|_{L^2}^8\left\|\nabla^2 B\right\|_{L^2}^2 + \frac{1}{8}\left\|\nabla^4 B\right\|_{L^2}^2 \le \frac{1}{8}\left\|\nabla^4 v\right\|_{L^2}^2 + \frac{1}{4}\left\|\nabla^4 B\right\|_{L^2}^2 \\
&\quad + C\left(\|\nabla v\|_{L^2}^8 + \|\nabla B\|_{L^2}^8\right)\left(\left\|\nabla^2 B\right\|_{L^2}^2 + \left\|\nabla^2 v\right\|_{L^2}^2\right).
\end{aligned}
$$

Therefore, combining the estimates of J_1, J_2, J_3 and J_4, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|\nabla^3 v(\cdot,t)\right\|_{L^2}^2 + \left\|\nabla^3 B(\cdot,t)\right\|_{L^2}^2\right) \\
&\quad \le C\left(\|\nabla v\|_{L^2}^8 + \|\nabla B\|_{L^2}^8\right)\left(\left\|\nabla^2 B\right\|_{L^2}^2 + \left\|\nabla^2 v\right\|_{L^2}^2\right) \\
&\qquad + C\left(\left\|\nabla^2 v\right\|_{L^2}^2 + C\left\|\nabla^2 B\right\|_{L^2}^2\right).
\end{aligned}
$$

Integrating the above inequality over (T_0, t), we infer that

$$
\begin{aligned}
&\|\nabla^3 v(\cdot,t)\|_{L^2}^2 + \left\|\nabla^3 B(\cdot,t)\right\|_{L^2}^2 - \left(\left\|\nabla^3 v(\cdot,T_0)\right\|_{L^2}^2 + \left\|\nabla^3 B(\cdot,T_0)\right\|_{L^2}^2\right) \\
&\le C\int_{T_0}^{t}\left(\left\|\nabla^2 v(\cdot,\tau)\right\|_{L^2}^2 + \left\|\nabla^2 B(\cdot,\tau)\right\|_{L^2}^2\right)d\tau \\
&\quad + C\int_{T_0}^{t}\left(\|\nabla v(\cdot,\tau)\|_{L^2}^8 + \|\nabla B(\cdot,\tau)\|_{L^2}^8\right)\left(\left\|\nabla^2 v(\cdot,\tau)\right\|_{L^2}^2 + \left\|\nabla^2 B(\cdot,\tau)\right\|_{L^2}^2\right)d\tau
\end{aligned}
$$

$$\leq C_0\big(e+F(t)\big)^{C\epsilon}+C_0\int_{T_0}^{t}\big(e+F(\tau)\big)^{4C\epsilon}\big(\left\|\nabla^2 v(\cdot,\tau)\right\|_{L^2}^2+\left\|\nabla^2 B(\cdot,\tau)\right\|_{L^2}^2\big)d\tau$$

$$\leq C_0\big(e+F(t)\big)^{C\epsilon}+C_0\big(e+F(t)\big)^{4C\epsilon}\int_{T_0}^{t}\big(\left\|\nabla^2 v(\cdot,\tau)\right\|_{L^2}^2+\left\|\nabla^2 B(\cdot,\tau)\right\|_{L^2}^2\big)d\tau$$

$$\leq C_0\big(e+F(t)\big)^{C\epsilon}+C_0\big(e+F(t)\big)^{5C\epsilon}$$

$$\leq C_0\big(e+F(t)\big)^{5C\epsilon},$$

which leads to

$$e+F(t)\leq C_{T_0}+C_0\big(e+F(t)\big)^{5C\epsilon}$$

where $C_{T_0}=\left\|\nabla^3 v(\cdot,T_0)\right\|_{L^2}^2+\left\|\nabla^3 B(\cdot,T_0)\right\|_{L^2}^2$. Now we choose ϵ small enough so that $5C\epsilon\leq 1$, to conclude

$$F(t)\leq C\Big(\left\|\nabla^3 v(\cdot,T_0)\right\|_{L^2}^2+\left\|\nabla^3 B(\cdot,T_0)\right\|_{L^2}^2,T_0,T\Big)<\infty \quad \text{for all} \quad t\in[T_0,T].$$

As a consequence, we get the boundedness of $H^3\times H^3$-norm of (v,B) for all $t\in[0,T]$. This completes the proof. □

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