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# FIXED POINTS WITH RESPECT TO THE L-SLICE HOMOMORPHISM $\sigma_a$

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Abstract. Given a locale $L$ and a join semilattice $J$ with bottom element $0_J$, a new concept $(\sigma, J)$ called $L$-slice is defined,where $\sigma$ is as an action of the locale $L$ on the join semilattice $J$. The $L$-slice $(\sigma, J)$ adopts topological properties of the locale $L$ through the action $\sigma$. It is shown that for each $a \in L$, $\sigma_a$ is an interior operator on $(\sigma, J)$.The collection $M = \{\sigma_a ; a \in L\}$ is a Priestly space and a subslice of $L$-Hom$(J, J)$. If the locale $L$ is spatial we establish an isomorphism between the $L$-slices $(\sigma, L)$ and $(\delta, M)$. We have shown that the fixed set of $\sigma_a$, $a \in L$ is a subslice of $(\sigma, J)$ and prove some equivalent properties.

## 1. Introduction

It is well known that a topological space is a lattice of open sets. The interrelation between topology and lattice was first studied by Marshall Stone. Johnstone, in his paper 'The point of pointless topology' [6] expressed the complete lattice satisfying infinite distributive law as pointless topology. Afterwards most of the topological ideas have been studied in a localic background. Dual to the notion of theory of locales, we have theory of frames. The study using frame theory are more algebraic and those in localic background are topological.

At the same time among many introductions to topology, a particular view that has arisen in Theoretical Computer Science starts with the theory of domains as defined by Scott and Strachey [10] to provide a mathematical foundation for semantics of programming languages, establishing that domains could be put into a topological setting. Duality between frames and topological spaces have been utilized to make a connection between syntactical and semantical approach to logic. But the application of Stone duality in modal logic require a duality for Boolean algebras or distributive lattices endowed with additional operations. This has inspired the concept of action of a locale on a join semilattice introduced in this paper.

In this paper we have taken up the following study which is relevant in the above context. Given a locale $L$ and a join semilattice $J$ with bottom element $0_J$, we have introduced a new concept called $L$-slice denoted by $(\sigma, J)$, where $\sigma$ is an action of

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the locale $L$ on the join semilattice $J$ together with a set of conditions. The $L$-slice though algebraic in nature adopts topological properties such as compactness of $L$ through the action $\sigma$. We prove the following results

1. For each $a \in L$, $\sigma_a$ is an interior operator on $(\sigma, J)$ and is an $L$-slice homomorphism from the $L$-slice $(\sigma, J)$ to itself.

2. The collection $M = \{\sigma_a; a \in L\}$ is a bounded distributive lattice and so by Priestly duality gives a topological space $P$. Also $M$ is a subslice of $L$-Hom$(J, J)$. If the locale $L$ is spatial we establish an isomorphism between the $L$-slices $(\sigma, L)$ and $(\delta, M)$.

3. The fixed set of $\sigma_a$ is a subslice of $(\sigma, J)$ and is an ideal if $\sigma_a$ has the property that whenever $x \leq y \leq z$ with $\sigma_a(x) = x$ and $\sigma_a(z) = z$, then $\sigma_a(y) = y$. Also If $\sigma_b$ or $\sigma_c$ is one-one for all $b, c$ with $\sigma(b \sqcap c, x) \in Fix_{\sigma_a}$, then $Fix_{\sigma_a}$ is a prime ideal.

The locale $L$ can be viewed as an $L$-slice in which case the action $\sigma$ is the meet.

**Notation.** Meet and join in the locale $L$ are denoted by $\sqcap$ and $\sqcup$ respectively.

**Discussion.** In the existing context of theory of topological semigroups, topological groups, topological vector spaces and so on, the development of the theories pertain to points, their neighbourhoods and their local behaviour. In the set up of locale theory(generalized topological spaces or point free topology) we could develop the action of a locale on a join semilattice to form the entity $(\sigma, J)$ that has properties which could be studied algebraically as well as topologically. Viewing topology as theory of information, the properties of $L$-slice could be used as an effective tool in cryptography, image processing and mathematical morphology. In particular it serve as a foundation for semantics of programming languages as suggested by Samson Abramsky [1].

## 2. Preliminaries

A frame (or a locale) is a complete lattice L satisfying the infinite to distributive law $a \wedge \bigvee B = \bigvee\{a \wedge b; b \in B\}$ for all $a \in L$ and $B \subseteq L$, [9]. Given the frames $L$, $M$ a frame homomorphism is a map $h\colon L \to M$ preserving all finite meets (including the top 1) and all joins (including the bottom 0). The category of frames is denoted by **Frm**. The opposite category of **Frm** is the category **Loc** of locales. We can represent the morphism in **Loc** as the infima-preserving $f\colon L \to M$ such that the corresponding left adjoint $f^*\colon M \to L$ preserves finite meet. When we consider the objects the category **Frm** and **Loc** are same. But it behaves differently when we consider morphisms. We can define a functor $\Omega$ from the category **Top** of topological spaces into the category **Frm** as follows. $\Omega$ sends each object $X$ of **Top** to the frame $\Omega(X)$ of its open sets. If $f\colon X \to Y$ is a morphism in **Top**, that is $f$ a continuous function, then define $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ by $\Omega(f)(V) = f^{-1}(V)$. Then $\Omega$ is a contravarient functor from the category **Top** to the category **Frm**.

**Examples 2.1** ([9]).
i. The lattice of open subsets of topological space.

ii. The Boolean algebra $B$ of all open subsets $U$ of real line $R$ such that
$U = \text{int}(\text{cl}(U))$.

**Definition 2.2** ([9]). A subset $I$ of a locale $L$ is said to be an ideal if

i) $I$ is a sub-join-semilattice of $L$; that is $0 \in I$ and $a \in I$, $b \in I$ implies $a \vee b \in I$; and

ii) $I$ is a lower set; that is $a \in I$ and $b \leq a$ imply $b \in I$.

If $a \in L$, the set $\downarrow(a) = \{x \in L; x \leq a\}$ is an ideal of $L$. $\downarrow(a)$ is the smallest ideal containing $a$ and is called the principal ideal generated by $a$.

**Definition 2.3** ([9]). A subset $F$ of locale $L$ is said to be a filter if

i) $F$ is a sub-meet-semi lattice of $L$; that is $1 \in F$ and $a \in F$, $b \in F$ imply $a \wedge b \in F$.

ii) $F$ is an upper set; that is $a \in F$ and $a \leq b$ imply $b \in F$.

**Definition 2.4** ([9]). A filter $F$ is proper if $F \neq L$, that is if $0 \notin F$.

A proper filter $F$ in a locale $L$ is prime if $a_1 \vee a_2 \in F$ implies that $a_1 \in F$ or $a_2 \in F$.

**Definition 2.5** ([9]). A proper filter $F$ in a locale $L$ is a completely prime filter if for any $J$ and $a_i \in L$, $i \in J$, $\bigvee a_i \in F \Rightarrow \exists i \in J$ such that $a_i \in F$.

A proper filter $F$ in a locale $L$ is partially completely prime filter if for any $J$ and $a_i \in L$, $i \in J$, $\bigvee a_i \in F \Rightarrow \exists a_1, a_2 \dots a_n$ such that $a_1 \vee a_2 \vee \dots \vee a_n \in F$.

**Definition 2.6** ([9]). A subset $S \subseteq L$ is a sublocale of $L$ if

1) $S$ is closed under all meets

2) For every $s \in S$ and every $x \in L$,$x \rightarrow s \in S$.

**Example 2.7** ([9]). Let $L$ be a locale. For each $a \in L$, the closed sublocales are given by $c(a) = \{x \in L : a \leq x\}$ and open sublocales are given by
$o(a) = \{a \rightarrow x : x \in L\}$.

**Proposition 2.8** ([9]). *Let L be a locale. A subset $S \subseteq L$ is a sublocale if and only if it is a locale in the induced order and the embedding map $j\colon S \subseteq L$ is a localic map.*

**Definition 2.9** ([5]). An element $a$ in a lattice $L$ is said to be join irreducible if and only if $a$ is not a bottom element, and, whenever $a = b \vee c$, then $a = b$ or $a = c$.

**Definition 2.10.** Let $(X, \leq)$ be a poset. A map $f\colon X \rightarrow X$ is called interior operator if

1) $f$ is order preserving

2) $f(x) \leq x$ for all $x \in X$

3) $f \circ f = f$.

**Definition 2.11.** Let $L$ be bounded distributive law, and let $X$ denote the set of prime filters of $L$. For each $a \in L$, let $\phi_+(a) = \{x \in X : a \in x\}$. Then $(X, \tau_+)$ is a spectral space, where the topology $\tau_+$ on $X$ is generated by $\{\phi_+(a); a \in L\}$. The spectral space $(X, \tau_+)$ is called the prime spectrum of $L$.

The map $\phi_+$ is a lattice isomorphism from $L$ onto the lattice of all compact open subsets of $(X, \tau_+)$. Similarly, if $\phi_-(a) = \{x \in X : a \notin x\}$ and $\tau_-$ denotes the topology generated by $\{\phi_-(a); a \in L\}$, then $(X, \tau_-)$ is also spectral space. Let $\leq$ be set-theoretic inclusion on the set of prime filters of $L$ and let $\tau = \tau_+ \vee \tau_-$. Then $(X, \tau, \leq)$ is a Priestley space.

## 3. Main results

In this section, the concept of $L$-slice $(\sigma, J)$ is introduced, where $\sigma$ is an action of a locale $L$ on a join semilattice $J$ with bottom element and studies some of its properties. For each $a \in L$, the properties of the $L$-slice homomorphism $\sigma_a$ have studied and it has shown that $M = \{\sigma_a : a \in L\}$ is an $L$-slice.

**Definition 3.1.** Let $L$ be a locale and $J$ be join semilattice with bottom element $0_J$. By the "action of $L$ on $J$" we mean a function $\sigma \colon L \times J \to J$ such that the following conditions are satisfied.
1) $\sigma(a, x_1 \vee x_2) = \sigma(a, x_1) \vee \sigma(a, x_2)$ for all $a \in L$, $x_1$, $x_2 \in J$.
2) $\sigma(a, 0_J) = 0_J$ for all $a \in L$.
3) $\sigma(a \sqcap b, x) = \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x))$ for all $a$, $b \in L$, $x \in J$.
4) $\sigma(1_L, x) = x$ and $\sigma(0_L, x) = 0_J$ for all $x \in J$.
5) $\sigma(a \sqcup b, x) = \sigma(a, x) \vee \sigma(b, x)$ for $a$, $b \in L$, $x \in J$.

If $\sigma$ is an action of the locale $L$ on a join semilattice $J$, then we call $(\sigma, J)$ as $L$-slice.

**Proposition 3.2.** *Let $L$ be a locale, and let $S$ be a set of order preserving maps $L \to L$ such that:*
(a) *The constant map $\boldsymbol{0} \in S$ ($\boldsymbol{0}$ takes everything to 0).*
(b) *If $f$, $g \in S$, then $f \vee g \in S$.*
(c) *For all $a \in L$ and for all $f \in S$, the meet of the constant map $\boldsymbol{a}$ and $f$ is in $S$ (i.e. $f \wedge \boldsymbol{a} \in S$).*
*Then the map $\sigma \colon L \times S \to S$ defined by $\sigma(a, f)(x) = f(x) \sqcap a$ is an action of $L$ on $S$ and $(\sigma, S)$ is an $L$-slice.*

**Proof.** By the hypothesis, $S$ is a join semilattice with bottom element $\mathbf{0}$ and the map $\sigma$ is well defined.
1. $\sigma(a, f \vee g)(x) = (f \vee g)(x) \sqcap a = (f(x) \sqcup g(x)) \sqcap a = (f(x) \sqcap a) \sqcup (g(x) \sqcap a)$
$= \sigma(a, f)(x) \sqcup \sigma(a, g)(x) = (\sigma(a, f) \vee \sigma(a, g))(x)$.
2. $\sigma(a, \mathbf{0})(x) = \mathbf{0}(x) \sqcap a = 0 \sqcap a = 0 = \mathbf{0}(x)$.
3. $\sigma(a \sqcap b, f)(x) = f(x) \sqcap (a \sqcap b) = a \sqcap (f(x) \sqcap b) = a \sqcap \sigma(b, f)(x)$
$= \sigma(a, \sigma(b, f))(x) = \sigma(b, \sigma(a, f))(x)$.
4. $\sigma(1_L, f)(x) = f(x) \sqcap 1_L = f(x)$ and $\sigma(0_L, f)(x) = f(x) \sqcap 0_L = 0 = \mathbf{0}(x)$.
5. $\sigma(a \sqcup b, f)(x) = f(x) \sqcap (a \sqcup b) = (f(x) \sqcap a) \sqcup (f(x) \sqcap b)$
$= \sigma(a, f)(x) \sqcup \sigma(b, f)(x) = (\sigma(a, f) \vee \sigma(b, f))(x)$.
Hence $(\sigma, S)$ is an $L$-slice. □

**Examples 3.3.**
1. Let $L$ be a locale and $I$ be any ideal of $L$. Consider each $x \in I$ as constant map $\mathbf{x}\colon L \to L$. Then by Proposition 3.2, $(\sigma, I)$ is an $L$-slice. In particular $(\sigma, L)$ is an $L$-slice.

2. Let the locale $L$ be a chain with Top and Bottom elements and $J$ be any join semilattice with bottom element. Define $\sigma\colon L \times J \to J$ by $\sigma(a, j) = j \forall a \neq 0$ and $\sigma(0_L, j) = 0_J$.
Then $\sigma$ is an action of $L$ on $J$ and $(\sigma, J)$ is an $L$-slice.

**Definition 3.4.** Let $(\sigma, J)$ be an $L$-slice. A subjoin semilattice $J'$ of $J$ is said to be $L$-subslice of $J$ if $J'$ is closed under action by elements of $L$.

**Example 3.5.** Let $L$ be a locale and $O(L)$ denotes the collection of all order preserving maps on $L$. Then $(\sigma, O(L))$ is a $L$-slice, where $\sigma\colon L \times O(L) \to O(L)$ is defined by $\sigma(a, f) = f_a$, where $f_a\colon L \to L$ is defined by $f_a(x) = f(x) \sqcap a$. Let $K = \{f \in O(L) : f(x) \leq x, \forall x \in L\}$. Then $(\sigma, K)$ is a $L$-subslice of the slice $(\sigma, O(L))$.

**Definition 3.6.** A subslice $(\sigma, I)$ of an $L$-slice $(\sigma, J)$ is said to be ideal of $J$ if $x \in I$ and $y \in J$ are such that $y \leq x$, then $y \in I$.

**Definition 3.7.** An ideal $(\sigma, I)$ of an $L$-slice $(\sigma, J)$ is a prime ideal if it has the following properties:
1. If $a$ and $b$ are any two elements of $L$ such that $\sigma(a \sqcap b, x) \in I$, then either $\sigma(a, x) \in I$ or $\sigma(b, x) \in I$.
2. $(\sigma, I)$ is not equal to the whole slice $(\sigma, J)$.

**Definition 3.8.** Let $(\sigma, J)$,$(\mu, K)$ be $L$-slices. A map $f : (\sigma, J) \to (\mu, K)$ is said to be $L$-slice homomorphism if
1. $f(x_1 \vee x_2) = f(x_1) \vee' f(x_2)$ for all $x_1$, $x_2 \in J$ .
2. $f(\sigma(a, x)) = \mu(a, f(x))$ for all $a \in L$ and all $x \in J$.

**Definition 3.9.** An $L$-slice homomorphism from an $L$-slice $(\sigma, J)$ to itself is called $L$-slice endomorphism.

**Examples 3.10.**
i. Let $(\sigma, J)$ be an $L$-slice and $(\sigma, J')$ be an $L$-subslice of $(\sigma, J)$. Then the inclusion map $i\colon (\sigma, J') \to (\sigma, J)$ is an $L$-slice homomorphism.
ii. Let $I = \downarrow(a), K = \downarrow(b)$ be principal ideals of the locale $L$. Then $(\sigma, I)$,$(\sigma, K)$ are $L$-slices. Then the map $f\colon (\sigma, I) \to (\sigma, K)$ defined by $f(x) = x \sqcap b$ is an $L$-slice homomorphism.

**Proposition 3.11.** *The composition of two $L$-slice homomorphisms is an $L$-slice homomorphism.*

**Definition 3.12.** Let $(\sigma, J)$, $(\mu, K)$ be $L$-slices. A map $f\colon (\sigma, J) \to (\mu, K)$ is said to be isomorphism if
1. $f$ is one-one
2. $f$ is onto
3. $f$ is a $L$-slice homomorphism.

**Proposition 3.13.** *Let $(\sigma, J)$, $(\mu, K)$ be $L$-slices and $f\colon (\sigma, J) \to (\mu, K)$ be $L$-slice homomorphism. Then*
(1) $\ker f = \{x \in J/f(x) = 0_K\}$ *is an ideal of* $(\sigma, J)$.
(2) $\operatorname{im} f = \{y \in K/y = f(x) \quad for \quad some \quad x \quad \in J\}$ *is an $L$-subslice of* $(\mu, K)$.

**Proposition 3.14.** *Let $(\sigma, J)$, $(\mu, K)$ be $L$-slices. Then the collection* $L$-Hom$(J, K)$ *of all $L$-slice homomorphisms from $(\sigma, J)$ to $(\mu, K)$ is an $L$-slice.*

**Proof.** Define a map $\delta\colon L\times$ $L$-Hom$(J, K) \to L$-Hom$(J, K)$ as follows.
For each $a \in L$ and $f \in L$-Hom$(J, K)$ define $\delta(a, f)\colon J \to K$ by $\delta(a, f)(x) = \mu(a, f(x))$. Then $(\delta, L$-Hom$(J, K))$ is an $L$-slice. □

**Definition 3.15.** Let $(\sigma, J)$ be an $L$-slice. For each $a \in L$, define $\sigma_a\colon (\sigma, J) \to (\sigma, J)$ by $\sigma_a(x) = \sigma(a, x)$.

**Proposition 3.16.** *Let $(\sigma, J)$ be an $L$-slice. For each $a \in L$, $\sigma_a\colon (\sigma, J) \to (\sigma, J)$ is an $L$-slice homomorphism.*

**Proof.** $\sigma_a(x \vee y) = \sigma(a, x \vee y) = \sigma(a, x) \vee \sigma(a, y) = \sigma_a(x) \vee \sigma_a(y)$. For $b \in L$,$\sigma_a(\sigma(b, x)) = \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x)) = \sigma(b, \sigma_a(x))$. Hence $\sigma_a$ is an $L$-slice homomorphism. □

**Proposition 3.17.** *Let $(\sigma, J)$ be an $L$-slice and $a \in L$:*
1. $\sigma_a(x) \leq x$ *for all* $x \in (\sigma, J)$.
2. *If $(\sigma, I)$ is an ideal in $(\sigma, J)$, then $(\sigma, \sigma_a(I))$ is a subslice of $(\sigma, J)$ and $\sigma_a(I) \subseteq I$.*

**Proof.**
1. $x = \sigma(1, x) = \sigma(a \sqcup 1, x) = \sigma(a, x) \vee \sigma(1, x) = \sigma_a(x) \vee x$. Thus $\sigma_a(x) \leq x$ for all $x \in (\sigma, J)$.
2. Let $(\sigma, I)$ be any ideal in $(\sigma, J)$. Let $\sigma_a(x), \sigma_a(y) \in \sigma_a(I)$. Then $x$, $y \in I$.
$\sigma_a(x) \vee \sigma_a(y) = \sigma_a(x \vee y) \in \sigma_a(I)$. Thus $\sigma_a(I)$ is a sub join semi lattice of $(\sigma, J)$. For $b \in L$, and $\sigma_a(x) \in \sigma_a(I)$, $\sigma(b, \sigma_a(x)) = \sigma(b, \sigma(a, x)) = \sigma(a, \sigma(b, x)) = \sigma_a(\sigma(b, x)) \in \sigma_a(I)$. Hence $\sigma_a(I)$ is a subslice of $(\sigma, J)$.

For each $x \in I$, since $\sigma_a(x) \leq x$, $\sigma_a(x) \in I$. Hence $\sigma_a(I) \subseteq I$. □

**Proposition 3.18.** *Let $(\sigma, J)$ be an $L$-slice and $a$, $b \in L$:*
1. $\sigma_0$ *is the zero map and* $\sigma_1$ *is the identity map on* $(\sigma, J)$
2. $\sigma_{a\sqcup b} = \sigma_a \vee \sigma_b$ *and* $\sigma_{a\sqcap b} = \sigma_a \circ \sigma_b = \sigma_b \circ \sigma_a$.

**Proof.**
1. $\sigma_0(x) = \sigma(0, x) = 0_J$ for all $x \in J$. Hence $\sigma_0$ is the zero map on $(\sigma, J)$. $\sigma_1(x) = \sigma(1, x) = x$ for all $x \in J$. Hence $\sigma_1$ is the identity map on $(\sigma, J)$.
2. $\sigma_{a\sqcup b}(x) = \sigma(a \sqcup b, x) = \sigma(a, x) \vee \sigma(b, x) = \sigma_a(x) \vee \sigma_b(x) = (\sigma_a \vee \sigma_b)(x)$ for all $x \in (\sigma, J)$. Hence $\sigma_{a\sqcup b} = \sigma_a \vee \sigma_b$.
$\sigma_{a\sqcap b}(x) = \sigma(a \sqcap b, x) = \sigma(a, \sigma(b, x)) = \sigma(a, \sigma_b(x)) = \sigma_a(\sigma_b(x)) = (\sigma_a \circ \sigma_b)(x) = (\sigma_b \circ \sigma_a)(x)$ for all $x \in (\sigma, J)$. Hence $\sigma_{a\sqcap b} = \sigma_a \circ \sigma_b = \sigma_b \circ \sigma_a$. □

**Proposition 3.19.** *Let $(\sigma, J)$ be an $L$-slice. Then for each $a \in L, \sigma_a$ is an interior operator on $(\sigma, J)$.*

**Proof.** Since for each $a \in L$, $\sigma_a$ is an $L$-slice homomorphism, $\sigma_a$ is order preserving. By 3.17 Proposition, $\sigma_a(x) \leq x$ for all $x \in L$ and by 3.18 Proposition $\sigma_a \circ \sigma_a = \sigma_a$. Hence $\sigma_a$ is an interior operator on $(\sigma, J)$. $\square$

**Proposition 3.20.** *The collection* $M = \{\sigma_a : a \in L\}$ *is a bounded distributive lattice and a subslice of* $(\delta, L\text{-Hom}(J, J))$.

**Proof.** $(M, \vee, \sigma_0)$ is a join semilattice with bottom element $\sigma_0$ and $(M, \circ, \sigma_1)$ is a meet semilattice with top element $\sigma_1$. $\sigma_a \circ (\sigma_a \vee \sigma_b) = \sigma_a \circ \sigma_{a \sqcup b} = \sigma_{a \sqcap (a \sqcup b)} = \sigma_a$ and $\sigma_a \vee (\sigma_a \circ \sigma_b) = \sigma_a \vee \sigma_{a \sqcap b} = \sigma_{a \sqcup (a \sqcap b)} = \sigma_a$. Thus absorption laws are satisfied and so M is a bounded lattice with top $\sigma_1$ and bottom $\sigma_0$. Also $\sigma_a \circ (\sigma_b \vee \sigma_c) = \sigma_a \circ (\sigma_{b \sqcup c}) = \sigma_{a \sqcap (b \sqcup c)} = \sigma_{(a \sqcap b) \sqcup (a \sqcap c)} = \sigma_{a \sqcap b} \vee \sigma_{a \sqcap c} = (\sigma_a \circ \sigma_b) \vee (\sigma_a \circ \sigma_c)$ and $\sigma_a \vee (\sigma_b \circ \sigma_c) = \sigma_a \vee \sigma_{b \sqcap c} = \sigma_{a \sqcup (b \sqcap c)} = \sigma_{(a \sqcup b) \sqcap (a \sqcup c)} = \sigma_{a \sqcup b} \circ \sigma_{a \sqcup c} = (\sigma_a \vee \sigma_b) \circ (\sigma_a \vee \sigma_c)$. Hence $M$ is a bounded distributive lattice. Clearly $M \subseteq L\text{-Hom}(J, J)$.

Let $b \in L$ and $\sigma_a \in M$. Then $\delta(b, \sigma_a)(x) = \sigma(b, \sigma_a(x)) = \sigma(b, \sigma(a, x)) = \sigma(b \sqcap a, x) = \sigma_{b \sqcap a}(x)$. Thus $M$ is closed under action by elements of $L$. Hence $(\delta, M)$ is a $L$-subslice of $(\delta, L\text{-Hom}(J, J))$. $\square$

**Proposition 3.21.** *There is an onto $L$-slice homomorphism from* $(\sqcap, L)$ *to* $(\delta, M)$.

**Proof.** Define $\phi \colon (\sqcap, L) \to (\delta, M)$ by $\phi(a) = \sigma_a$. $\phi(a \sqcup b) = \sigma_{a \sqcup b} = \sigma_a \vee \sigma_b = \phi(a) \vee \phi(b)$ and $\phi(\sigma(a, b)) = \phi(a \sqcap b) = \sigma_{a \sqcap b} = \sigma_a \circ \sigma_b = \sigma(a, \sigma_b) = \sigma(a, \phi(b))$. Surjection of $\phi$ is clear from the definition. Hence $\phi$ is an onto $L$-slice homomorphism from $(\sqcap, L)$ to $(\delta, M)$.

Let $\psi$ be natural frame homomorphism from $L$ to $\Omega(\mathrm{Sp}(L))$ and $\phi \colon (\sqcap, L) \to (\delta, M)$ is the $L$-slice homomorphism of 3.21 Proposition. Then there is a lattice homomorphism $f$ from $M$ to $\Omega(\mathrm{Sp}(L))$ such that the following triangle commutes. The map $f$ is defined by $f(\sigma_a) = \Sigma_a$

$$
\begin{tikzcd}
L \arrow[r, "\phi"] \arrow[d, "\psi"'] & M \arrow[dl, "f"] \\
\Omega(\mathrm{Sp}(L)) &
\end{tikzcd}
$$

If $L$ is a spatial locale, then $\psi$ is one-one and so $\phi$ is one-one. Thus if $L$ is spatial locale, $L$-slices $(\sigma, L), (\delta, M)$ are isomorphic. $\square$

Since $M$ is a distributive lattice, by Priestly duality, the distributive lattice $M$ is dual to a topological space $P$. Then there is a frame homomorphism f from $\Omega(\mathrm{Sp}(L))$ to $\Omega(P)$ such that the following rectangle commutes. Then $f_* \colon \Omega(P) \to \Omega(\mathrm{Sp}(L))$

is a localic map.

$$\begin{tikzcd} L \arrow[r, "\phi"] \arrow[d, "\psi"'] & M \arrow[d, "g"] \\ \Omega(\mathrm{Sp}(L)) \arrow[r, "f"] & \Omega(P) \end{tikzcd}$$

Now we discuss some properties of the set $\mathrm{Fix}_{\sigma_a} = \{x \in J : \sigma_a(x) = x\}$. We will show that $N = \{\mathrm{Fix}_{\sigma_a}; a \in L\}$ together with an action $\gamma$ is an $L$-slice.

**Proposition 3.22.** *For each $a \in L$, $\mathrm{Fix}_{\sigma_a} = \{x \in J : \sigma_a(x) = x\}$ is a subslice of $(\sigma, J)$.*

**Proof.** Let $x, y \in \mathrm{Fix}_{\sigma_a}$. Then $\sigma_a(x) = x, \sigma_a(y) = y$. Then $\sigma_a(x \vee y) = \sigma(a, x \vee y) = \sigma(a, x) \vee \sigma(a, y) = \sigma_a(x) \vee \sigma_a(y) = x \vee y$. So $\mathrm{Fix}_{\sigma_a}$ is a subjoin semilattice of $(\sigma, J)$. Let $x \in \mathrm{Fix}_{\sigma_a}$ and $b \in L$. Then $\sigma_a(\sigma(b, x)) = \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x)) = \sigma(b, x)$. So $\sigma(b, x) \in \mathrm{Fix}_{\sigma_a}$. Hence $(\sigma, \mathrm{Fix}_{\sigma_a})$ is an $L$-subslice of $(\sigma, J)$. $\square$

**Proposition 3.23.** *Let $(\sigma, J)$ be an $L$-slice and $a \in L$. Then the following statements are equivalent.*
1. *$\sigma_a$ has the property that if $x \leq y \leq z$ with $\sigma_a(x) = x$ and $\sigma_a(z) = z$, then $\sigma_a(y) = y$.*
2. *$(\sigma, \mathrm{Fix}_{\sigma_a})$ is an ideal of $(\sigma, J)$.*

**Proof.** First assume statement 1. By above theorem $(\sigma, \mathrm{Fix}_{\sigma_a})$ is a subslice of $(\sigma, J)$. Let $x \in \mathrm{Fix}_{\sigma_a}$ and $y \in J$ such that $y \leq x$. We have $\sigma_a(y) \leq y \leq x$ and $\sigma_a(\sigma_a(y)) = \sigma_a(y), \sigma_a(x) = x$. Then by assumption $\sigma_a(y) = y$. So $y \in \mathrm{Fix}_{\sigma_a}$ and hence $(\sigma, \mathrm{Fix}_{\sigma_a})$ is an ideal in $(\sigma, J)$.
(2) implies (1) follows directly from the definition of ideal of an $L$-slice. $\square$

**Proposition 3.24.** *Let $(\sigma, \mathrm{Fix}_{\sigma_a})$ is an ideal for some fixed $a \in L$. If $\sigma_b$ or $\sigma_c$ is one-one for all $b$, $c$ with $\sigma(b \sqcap c, x) \in \mathrm{Fix}_{\sigma_a}$, then $(\sigma, \mathrm{Fix}_{\sigma_a})$ is a prime ideal.*

**Proof.** Let $(\sigma, \mathrm{Fix}_{\sigma_a})$ is an ideal and $\sigma(b \sqcap c, x) \in \mathrm{Fix}_{\sigma_a}$. Then $\sigma_a(\sigma(b \sqcap c, x)) = \sigma(b \sqcap c, x)$ or $\sigma_a(\sigma_{b \sqcap c})(x) = \sigma_{b \sqcap c}(x)$. Equivalently $(\sigma_a \circ \sigma_b \circ \sigma_c)(x) = (\sigma_b \circ \sigma_c)(x)$. Suppose $\sigma_c$ is one-one and $\sigma(b, x) \notin Fix_{\sigma_a}$. Then $\sigma_a(\sigma(b, x)) \neq \sigma(b, x)$. That is $(\sigma_a \circ \sigma_b)(x) \neq \sigma_b(x)$. Since $\sigma_c$ is one-one, $(\sigma_c \circ \sigma_a \circ \sigma_b)(x) \neq (\sigma_c \circ \sigma_b)(x)$ or $(\sigma_a \circ \sigma_b \circ \sigma_c)(x) \neq (\sigma_b \circ \sigma_c)(x)$, which is a contradiction. Hence $\sigma(b, x) \in \mathrm{Fix}_{\sigma_a}$. Similarly if $\sigma_b$ is one-one, then $\sigma(c, x) \in \mathrm{Fix}_{\sigma_a}$. Hence $(\sigma, \mathrm{Fix}_{\sigma_a})$ is a prime ideal. $\square$

**Proposition 3.25.** *Consider the locale $L$ as an $L$-slice. Then for each $a \in L$, $\mathrm{Fix}_{\sigma_a}$ is a principal ideal.*

**Proof.** $\mathrm{Fix}_{\sigma_a} = \{x \in L : \sigma_a(x) = x\} = \{x \in L : a \sqcap x = x\} = \{x \in L : x \leq a\} = \downarrow a$. Hence $\mathrm{Fix}_{\sigma_a}$ is a principal ideal. $\square$

**Proposition 3.26.** *Let $(\sigma, J)$ be an $L$-slice and $a$, $b \in L$*
1. *If $a \leq b$, then $\mathrm{Fix}_{\sigma_a} \subseteq \mathrm{Fix}_{\sigma_b}$.*

2. $\mathrm{Fix}_{\sigma_0} = \{0_J\}$ *and* $\mathrm{Fix}_{\sigma_1} = J$.
3. $\mathrm{Fix}_{\sigma_{a \sqcap b}} = \mathrm{Fix}_{\sigma_a} \cap \mathrm{Fix}_{\sigma_a}$.

**Proof.** Let $a \leq b$ and $x \in \mathrm{Fix}_{\sigma_a}$, then $a \sqcap b = a$ and $\sigma_a(x) = x$. $\sigma_a(x) = x$ implies that $\sigma(a, x) = \sigma(a \sqcap b, x) = x$. Then we have $\sigma(b, \sigma(a, x)) = x$. Equivalently $\sigma_b(x) = x$ and so $x \in \mathrm{Fix}_{\sigma_b}$. Thus $\mathrm{Fix}_{\sigma_a} \subseteq \mathrm{Fix}_{\sigma_b}$.
2. Since $\sigma_0(x) = 0_J$ for all $x \in J$, $\mathrm{Fix}_{\sigma_0} = \{0_J\}$.
$\mathrm{Fix}_{\sigma_1} = \{x \in J : \sigma_1(x) = \sigma(1, x) = x\} = J$.
3. By part 1, $\mathrm{Fix}_{\sigma_{a \sqcap b}} \subseteq \mathrm{Fix}_{\sigma_a} \cap \mathrm{Fix}_{\sigma_b}$.
Now let $x \in \mathrm{Fix}_{\sigma_a} \cap \mathrm{Fix}_{\sigma_b}$. Then $\sigma_a(x) = \sigma_b(x) = x$. $\sigma_{a \sqcap b}(x) = \sigma(a \sqcap b, x) = \sigma(a, \sigma(b, x)) = \sigma(a, \sigma_b(x)) = \sigma(a, x) = \sigma_a(x) = x$. Thus $x \in \mathrm{Fix}_{\sigma_{a \sqcap b}}$ and hence $\mathrm{Fix}_{\sigma_{a \sqcap b}} = Fix_{\sigma_a} \cap \mathrm{Fix}_{\sigma_a}$. □

**Proposition 3.27.** *Let $(\sigma, J), (\mu, J)$ be two $L$-slices and let $\sigma_a$ or $\mu_a$ is onto for some $a \in L$. Then $\sigma_a = \mu_a$ if and only if $Fix_{\sigma_a} = \mathrm{Fix}_{\mu_a}$.*

**Proof.** Suppose $\sigma_a$ is onto and $Fix_{\sigma_a} = \mathrm{Fix}_{\mu_a}$. Let $y \in J$. Then there exist $x \in J$ such that $\sigma_a(x) = y$. Now $\sigma_a(y) = \sigma(a, y) = \sigma(a, \sigma_a(x)) = \sigma(a, \sigma(a, x)) = \sigma(a, x) = y$. Hence $y \in \mathrm{Fix}_{\sigma_a} = \mathrm{Fix}_{\mu_a}$. So $\mu_a(y) = y$. Hence $\sigma_a(y) = \mu_a(y)$. Converse is simple. □

**Proposition 3.28.** *Let $(\sigma, J)$ be an $L$-slice. Denote $N = \{\mathrm{Fix}_{\sigma_a} : a \in L\}$. Define $\mathrm{Fix}_{\sigma_a} \vee \mathrm{Fix}_{\sigma_b} = \mathrm{Fix}_{\sigma_{a \sqcup b}}, \mathrm{Fix}_{\sigma_a} \wedge \mathrm{Fix}_{\sigma_b} = \mathrm{Fix}_{\sigma_{a \sqcap b}}$. Then $(N, \vee, \wedge)$ is a distributive lattice and an $L$-slice.*

**Proof.** It is easy to show that $(N, \vee, \mathrm{Fix}_{\sigma_0}), (N, \wedge, \mathrm{Fix}_{\sigma_1})$ are semilattices. Also $\mathrm{Fix}_{\sigma_a} \vee (\mathrm{Fix}_{\sigma_a} \wedge \mathrm{Fix}_{\sigma_b}) = \mathrm{Fix}_{\sigma_a} \vee \mathrm{Fix}_{\sigma_{a \sqcap b}} = \mathrm{Fix}_{\sigma_{a \sqcup (a \sqcap b)}} = \mathrm{Fix}_{\sigma_a}$ and
$\mathrm{Fix}_{\sigma_a} \wedge (\mathrm{Fix}_{\sigma_a} \vee \mathrm{Fix}_{\sigma_b}) = \mathrm{Fix}_{\sigma_a} \wedge \mathrm{Fix}_{\sigma_{a \sqcup b}} = \mathrm{Fix}_{\sigma_{a \sqcap (a \sqcup b)}} = \mathrm{Fix}_{\sigma_a}$. Hence absorption laws are satisfied and so $(N, \vee, \wedge)$ is a lattice. Also we can verify distributive law easily.

Define $\gamma \colon L \times N \to N$ by $\gamma(b, \mathrm{Fix}_{\sigma_a}) = \mathrm{Fix}_{\sigma_{a \sqcap b}}$.
1. $\gamma(b, Fix_{\sigma_a} \vee \mathrm{Fix}_{\sigma_c}) = \gamma(b, \mathrm{Fix}_{\sigma_{a \sqcup c}}) = \mathrm{Fix}_{\sigma_{b \sqcap (a \sqcup c)}} = \mathrm{Fix}_{\sigma_{(b \sqcap a) \sqcup (b \sqcap c)}}$
$= \mathrm{Fix}_{\sigma_{b \sqcap a}} \vee \mathrm{Fix}_{\sigma_{b \sqcap c}} = \gamma(b, \mathrm{Fix}_{\sigma_a}) \vee \gamma(b, \mathrm{Fix}_{\sigma_c})$.
2. $\gamma(b, \mathrm{Fix}_{\sigma_0}) = \mathrm{Fix}_{\sigma_{b \sqcap 0}} = \mathrm{Fix}_{\sigma_0}$.
3. $\gamma(b \sqcap c, \mathrm{Fix}_{\sigma_a}) = \mathrm{Fix}_{\sigma_{(b \sqcap c) \sqcap a}} = Fix_{\sigma_{b \sqcap (c \sqcap a)}} = \gamma(b, \mathrm{Fix}_{\sigma_{c \sqcap a}}) = \gamma(b, \gamma(c, \mathrm{Fix}_{\sigma_a}))$.
4. $\gamma(1, \mathrm{Fix}_{\sigma_a}) = \mathrm{Fix}_{\sigma_{1 \sqcap a}} = \mathrm{Fix}_{\sigma_a}$ and $\gamma(0, \mathrm{Fix}_{\sigma_a}) = \mathrm{Fix}_{\sigma_{0 \sqcap a}} = Fix_{\sigma_0}$
5. $\gamma(b \sqcup c, \mathrm{Fix}_{\sigma_a}) = \mathrm{Fix}_{\sigma_{(b \sqcup c) \sqcap a}} = \mathrm{Fix}_{\sigma_{(b \sqcap a) \sqcup (c \sqcap a)}} = \mathrm{Fix}_{\sigma_{b \sqcap a}} \vee \mathrm{Fix}_{\sigma_{c \sqcap a}}$
$= \gamma(b, \mathrm{Fix}_{\sigma_a}) \vee \gamma(c, \mathrm{Fix}_{\sigma_a})$.
Hence $(\gamma, N)$ is an $L$-slice. □

**Proposition 3.29.** *There is an onto $L$-slice homomorphism from $(\delta, M)$ to $(\gamma, N)$.*

**Proof.** The map $g \colon (\delta, M) \to (\gamma, N)$ defined by $g(\sigma_a) = \mathrm{Fix}_{\sigma_a}$ is an onto $L$-slice homomorphism from $(\delta, M)$ to $(\gamma, N)$.

Since the composition of two $L$-slice homomorphism is again an $L$-slice homomorphism, $g \circ \phi$ is an $L$-slice homomorphism from $L$ to $N$. □

## 4. Conclusions

In this paper we investigate the properties of $L$-slice $(\sigma, J)$ of a locale $L$. We show that for each $a \in L$, $\sigma_a\colon (\sigma, J) \to (\sigma, J)$is an interior operator on $(\sigma, J)$ and is an $L$-slice endomorphism on $(\sigma, J)$. Moreover, the collection $M = \{\sigma_a; a \in L\}$ is a bounded distributive lattice and so by Priestly duality gives a topological space $P$. Also $(\delta, M)$ is an $L$-subslice of the $L$-slice $(\delta, L - \mathrm{Hom}(J, J))$ of all $L$-slice endomorphisms on the $(\sigma, J)$. If the locale $L$ is spatial we establish an isomorphism between the $L$-slices $(\sqcap, L)$ and $(\delta, M)$. We introduce fixed set $\mathrm{Fix}_{\sigma_a} = \{x \in J : \sigma_a(x) = x\}$ of $L$-slice homomorphism $\sigma_a$ and prove that $(\sigma, \mathrm{Fix}_{\sigma_a})$ is an $L$-subslice of the $L$-slice $(\sigma, J)$ and is an ideal of $(\sigma, J)$ if the action $\sigma$ satisfies certain properties. We show that $N = \{\mathrm{Fix}_{\sigma_a}; a \in L\}$ together with an action $\gamma$ is an $L$-slice and there is an $L$-slice homomorphism from $(\sqcap, L)$ to $(\gamma, N)$.

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**Author's Contributions.** The problem of defining the concept of $L$-slice for a locale $L$ is motivated by the theory of topological vector spaces, topological vector lattices and theory of modules. The authors developed the theory as suitable foundation for semantics of programming languages as suggested by Samson Abramsky [1]. Key exchange protocol in public key cryptography, mathematical morphology and image processing are other areas where these results should have major contributions.

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